Multi-dimensional quasi simple waves

in weakly dissipative flows

Basic equation

1- $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$ Continuity equation

2- $$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{c^2}{\rho}\vec{\nabla}\rho + \frac{\eta}{\rho}\nabla^2\vec{u}$$ Motion equation

3- $$\frac{\partial s}{\partial t} + (\vec{u} \cdot \vec{\nabla})s = \frac{\kappa}{\rho T}\nabla^2 T$$ Entropy equation

4- $$T = T(P, s)$$ Equation of state

Continuity equation +Entropy equation+Equation of state=Pressure equation

$$\nabla^2 T = -T\left(\frac{1}{c_V} - \frac{1}{c_P}\right)\nabla^2 p$$

$$\frac{\partial P}{\partial t} + (\vec{u}\cdot\vec{\nabla})P + \rho c^2 \vec{\nabla}\cdot\vec{u} = \frac{\kappa}{\rho}\left(\frac{1}{c_V} - \frac{1}{c_P}\right)\nabla^2 p$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p + \frac{\eta}{\rho}\nabla^2\vec{u}$$

$$\vec{u} = \vec{u}^{\,0}(\varphi) + \vec{\sigma}(\vec{r}, t)$$

$$P = P^{0}(\varphi) + \psi(\vec{r}, t)$$

$$\rho = \rho^{0}(\varphi) + \xi(\vec{r}, t)$$

$$\frac{\partial \psi}{\partial t} + c_0 \vec{n}_0 \cdot \vec{\nabla}\, \psi = 0$$

$$\frac{\partial \xi}{\partial t} + c_0 \vec{n}_0 \cdot \vec{\nabla}\, \xi = 0$$

$$\frac{\partial(\vec{\sigma} \cdot \vec{n}_0)}{\partial t} + c_0 \vec{\nabla} \cdot \vec{\sigma} = 0$$

$$\frac{d\vec{u}}{d\varphi}\left(\frac{\partial\varphi}{\partial t}+\vec{u}\cdot\vec{\nabla}\varphi\right)+\frac{\partial\vec{\sigma}}{\partial t}=-c\left(\frac{d\vec{u}^{0}}{d\varphi}\cdot\vec{n}\right)\vec{\nabla}\varphi-\frac{1}{\rho_0}\vec{\nabla}\psi+\frac{\eta}{\rho_0}\frac{d\vec{u}^{0}}{d\varphi}\nabla^2\varphi$$

$$\frac{d\vec{u}^{0}}{d\varphi}\cdot\vec{n}\left(\frac{\partial\varphi}{\partial t}+\vec{u}\cdot\vec{\nabla}\varphi\right)+\frac{1}{\rho_0 c_0}\frac{\partial\psi}{\partial t}+c\left(\frac{d\vec{u}^{0}}{d\varphi}\cdot\vec{\nabla}\varphi\right)+c_0\vec{\nabla}\cdot\vec{\sigma}=\frac{\kappa}{\rho_0}\left(\frac{1}{c_V}-\frac{1}{c_P}\right)\left(\frac{d\vec{u}^{0}}{d\varphi}\cdot\vec{n}\right)\nabla^2\varphi$$

$$\frac{d\rho^{0}}{d\varphi}=\frac{\rho^{0}}{c}\frac{d\vec{u}^{0}}{d\varphi}\cdot\vec{n}$$

$$\frac{\partial\varphi}{\partial t}+(\vec{u}^{0}+c\,\vec{n})\cdot\vec{\nabla}\varphi-\mu\nabla^2\varphi=0$$

$$\mu=\frac{1}{2}\left[\frac{\kappa}{\rho_0}\left(\frac{1}{c_V}-\frac{1}{c_P}\right)+\frac{\eta}{\rho_0}\right]$$

$$\vec{n} = (-\sin\varphi, \cos\varphi, 0)$$

$$u_x^0 = 2\alpha c_0 \tilde{\varphi} \sin^2\varphi_0 \cos\varphi_0 \qquad \alpha = \frac{2}{\gamma - 1}$$

$$u_y^0 = -2\alpha c_0 \tilde{\varphi} \sin\varphi_0 \cos^2\varphi_0$$

$$A = -c_0 \sin\varphi_0 \qquad B = c_0 \cos\varphi_0$$

$$\bar{x} = \frac{x - At}{\sqrt{\mu t_0}}, \quad \bar{y} = \frac{y - Bt}{\sqrt{\mu t_0}}, \quad \phi = c_0 \sqrt{\frac{t_0}{\mu}} (\varphi - \varphi_0), \quad \bar{t} = \frac{t}{t_0}$$

$$m = 2(\alpha \cos^2\varphi_0 + 1) \frac{\sin^2\varphi_0}{\cos\varphi_0} - \cos\varphi_0$$

$$n = -(2\alpha \cos^2\varphi_0 + 3) \sin\varphi_0$$

$$\frac{\partial \phi}{\partial \overline{t}} + m\phi \frac{\partial \phi}{\partial \overline{x}} + n\phi \frac{\partial \phi}{\partial \overline{y}} - \left(\frac{\partial^2 \phi}{\partial \overline{x}^2} + \frac{\partial^2 \phi}{\partial \overline{y}^2}\right) = 0$$

1- A stable traveling waves solutions

2-Diffusing Gaussian Solutions

3- Reduction to the 1-D Burgers equation

1- A stable traveling waves solutions

$$\phi = \beta \tanh(\bar{x} + a\bar{y} + b\bar{t} + c) + \delta$$

$$\begin{cases} b + (m + na)\delta = 0 \\ (m + na)\beta + 2(1 + a^2) = 0 \end{cases}$$

$$\vec{V}_{ph} = b'\vec{d}, \qquad \vec{d} = \frac{1}{\sqrt{1 + a^2}}(1, a)$$

$$b' \equiv \frac{1}{\sqrt{1 + a^2}}[c_0(a\cos\varphi_0 - \sin\varphi_0) - \sqrt{\frac{\mu}{t_0}}b]$$

$$m + na = 2(\alpha \cos^2 \varphi_0 - \sin \varphi_0)\frac{\sin^2 \varphi_0}{\cos \varphi_0} - (2\alpha \cos^2 \varphi_0 + 3)a \sin \varphi_0 - \cos \varphi_0$$

In a polytropic one atomic gas:

$$\gamma = \frac{5}{3} \qquad \alpha = 3 \qquad \varphi_0 = \frac{\pi}{4}$$

$$m + na = \sqrt{2}(2 - 3a)$$

The first state

$$\begin{cases} m+na<0 \quad \Rightarrow \quad \beta>0 \\ \delta>0 \quad \Rightarrow \quad b>0 \end{cases}$$

$$\vec{r}\cdot\vec{d}-b't+c', \qquad b'=\frac{1}{\sqrt{1+a^2}}[\frac{c_0}{\sqrt{2}}(a-1)-\sqrt{\frac{\mu}{t_0}}b]$$

If $b'<0$ then

$$\vec{r}\cdot\vec{d}\to-\infty, \quad t\to-\infty, \qquad \phi\to\delta-\beta$$

$$\vec{r}\cdot\vec{d}\to+\infty, \quad t\to+\infty, \qquad \phi\to\delta+\beta$$

If $b' > 0$ Then

$t \rightarrow -\infty,$ $\phi \rightarrow \delta + \beta$

$t \rightarrow +\infty,$ $\phi \rightarrow \delta - \beta$

The second state $\begin{cases} m+na>0 \quad \Rightarrow \quad \beta<0 \\ \delta>0 \quad \Rightarrow \quad b>0 \end{cases}$

$$|\phi| = |\delta| + |\beta| \quad or \quad |\phi| = \left||\delta| - |\beta|\right|$$

$$u_x = \alpha c_0^2 \sin\varphi_0 [2\sqrt{\frac{t_0}{\mu}} \sin\varphi_0 \cos\varphi_0 + c_0 \frac{t_0}{\mu}(2\cos^2\varphi_0 - \sin^2\varphi_0)\phi]\phi + \sigma_x(\vec{r},t)$$

$$u_y = -\alpha c_0^2 \cos\varphi_0 [2\sqrt{\frac{t_0}{\mu}} \sin\varphi_0 \cos\varphi_0 + c_0 \frac{t_0}{\mu}(2\sin^2\varphi_0 - \cos^2\varphi_0)\phi]\phi + \sigma_y(\vec{r},t)$$

$$\rho = \rho_0[1 - 2\alpha c_0 \sqrt{\frac{t_0}{\mu}} \tan\varphi_0 \phi + 2\alpha c_0^2 \frac{t_0}{\mu}[(2\alpha - 1)\tan^2\varphi_0 - 1]\phi^2] + \xi_x(\vec{r},t)$$

$$P = P_0[1 - 2(\alpha+1)c_0 \sqrt{\frac{t_0}{\mu}} \tan\varphi_0 \phi + 2(\alpha+1)c_0^2 \frac{t_0}{\mu}[(2\alpha - 1)\tan^2\varphi_0 - 1]\phi^2] + \psi_x(\vec{r},t)$$

2-Diffusing Gaussian Solutions

$$\zeta = \frac{m\overline{x} + n\overline{y}}{\sqrt{m^2 + n^2}}, \qquad \chi = \frac{-n\overline{x} + m\overline{y}}{\sqrt{m^2 + n^2}}$$

$$\frac{\partial \phi}{\partial t} + \sqrt{m^2 + n^2}\,\phi \frac{\partial \phi}{\partial \zeta} - \left(\frac{\partial^2 \phi}{\partial \zeta^2} + \frac{\partial^2 \phi}{\partial \chi^2}\right) = 0$$

$$\phi = \frac{C}{2\sqrt{\mu}(\overline{t}+\tau_0)^{3/2}} \exp\left(-\frac{\chi^2}{4(\overline{t}+\tau_0)}\right) + \frac{1}{(\overline{t}+\tau_0)} \frac{\zeta}{\sqrt{m^2+n^2}}$$

$$\overline{t} \geq -\tau_0$$

$$\phi \to \frac{c_0}{\sqrt{\mu t_0}} \frac{m\sin\varphi_0 - n\cos\varphi_0}{m^2+n^2}, \qquad t \to +\infty$$