Holography for 3D Einstein gravity with a conformal scalar field

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Abstract: We review AdS_3/CFT_2 correspondence and discuss its extension to Einstein gravity conformally coupled to a massless scalar field.

Introduction

- AdS_3/CFT_2 correspondence is a concrete example of holography. It follows from Maldacena's conjecture for the D1-D5 brane system.

- The Cardy formula

$$S = 2\pi\sqrt{\frac{c\,\Delta}{6}}$$

correctly reproduces the Bekenstein-Hawking entropy

$$S = \frac{A}{4G}$$

for BTZ black holes.

1. J.David, G. Mandal, S. Wadia, hep-th/0203048.

2. J.D. Brown, M. Henneaux, Commun. Math. Phys. 104, (1986) 207.

3. J.L. Cardy, Nucl. Phys. B. 270, (1986) 186.

4. J. D. Bekenstein, Phys. Rev. D 7, (1973) 2333; S.W. Hawking, Commun. Math. Phys. 43, (1975) 199, [Erratum-ibid. 46, (1976) 206.]

5. M. Banados, C. Teitelboim, J. Zanelli, hep-th/9204099; M. Banados, M. Henneaux, C. Teitelboim, J. Zanelli, gr-qc/9302012.

3D Einstein gravity

Why 3D?

- Theory is general covariant
- There are no local degrees of freedom.

In 3D,

$$G^{\mu}_{\nu} = -\frac{1}{4}\epsilon^{\mu\pi\rho}\epsilon_{\nu\sigma\tau}R_{\pi\rho}{}^{\sigma\tau}$$

Thus the vacuum solution $R_{\mu\nu} = 0$ has a vanishing curvature and can be constructed by gluing together pieces of Minkowski space. The same is true for vacuum field equations with a negative cosmological constant.

[S. Carlip, gr-qc/0503022].

Einstein Gravity with conformal matter

⊛ Einstein gravity $R_{\mu\nu} = 0$ in 3D has no black hole solutions.

⊛ There are black hole solutions at the critical point $M_{\mathsf{Pl}} = 0$, where the field equation is $R = 0$.

With conformal matter ψ, the effective Planck length is given by

$$M_{\mathsf{Pl}}^{\mathsf{eff}} = \left(1 - \frac{\psi^2}{M_{\mathsf{Pl}}}\right) M_{\mathsf{Pl}}$$

Decoupling limit

$$M_{\mathsf{Pl}} \to 0.$$

In string theory this limit corresponds to $\tilde{c} \to \infty$.

[U. Lindström and M. Zabzine, Phys. Lett. B 584 (2004) 178]

[I. Bakas and C. Sourdis, JHEP 06 (2004) 049]

In WZW model,

$$\alpha' = \frac{1}{k - g^{\vee}} \qquad \tilde{c} = \frac{(\dim G)k}{k - g^{\vee}}$$

k is the level of current algebra and $g^{\vee}$ is the dual Coxeter number of G

Outline

- Bekenstein-Hawking entropy and Holography,
- Microscopic description: Near horizon symmetries,
- 3D gravity, AdS_3 and asymptotic symmetries,
- conformal matter.

Holography

Consider the Schwarzschild black hole,

$$ds^2 = -\left(1 - \frac{2G\,M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2G\,M}{r}\right)} + r^2 d\Omega^2$$

Horizon $\quad r_{\mathrm{h}} = 2GM$

⋇ Temperature:

Near horizon geometry $\quad ds^2 = \rho^2 \left(\frac{dt}{4G\,M}\right)^2 + d\rho^2$

Thus, $\beta = (4GM).(2\pi)$.

⋇ Entropy: $dS = \beta dM$ gives,

$$S = 4\pi G M^2 = \frac{A}{4G}, \qquad A = 4\pi\, r_{\mathrm{h}}^2$$

Second law of thermodynamics

$$M_3 = M_1 + M_2$$

$$S_3 = 4\pi G(M_1 + M_2)^2 \geq S_1 + S_2$$

Microscopic description

"$\cdots$ black hole entropy should arguably be a more local property of horizons." [S. Carlip, hep-th/9812013]

$\diamondsuit$B.R. Majhi and T. Padmanabhan, arXiv:1204.1422.

Example: Schwarzschild geometry,

$$ds^2 = -2\kappa x du^2 - 2du dx + dx_\perp^2, \qquad \kappa = 2\pi T$$

Near horizon symmetry

Consider a variation of the metric

$$\delta g_{\mu\nu} = D_\nu \xi_\mu + D_\mu \xi_\nu$$

which leaves the horizon fixed:

$$\delta g_{xx} = 0 \qquad \delta g_{ux} = 0$$

Choose a basis ξ_m^μ such that

$$i[\xi_m, \xi_n]^\mu_{\mathsf{Lie}} = (m-n)\xi^\mu_{m+n}$$

Virasoro algebra of charges

$$i[Q_m, Q_n] = (m-n)Q_{m+n} + \frac{\tilde{c}}{12} m^3 \delta_{m+n,0}$$

▷ central charge:

$$\frac{\tilde{c}}{12} = \frac{A}{16\pi G}\frac{\alpha}{\kappa}$$

▷ energy:

$$\Delta = \frac{A}{8\pi G}\frac{\kappa}{\alpha}$$

▷ entropy: Cardy formula

$$S = 2\pi\sqrt{\frac{\tilde{c}\Delta}{6}} = \frac{A}{4G}$$

3D Einstein gravity

BTZ black holes

By gluing patches of AdS space, one can construct BTZ black holes.

static BTZ: $$ds^2 = -\Big(\frac{r^2}{\ell^2} - 8GM\Big)dt^2 + \frac{dr^2}{\Big(\frac{r^2}{\ell^2} - 8GM\Big)} + r^2 d\phi^2$$

ϕ is periodic modulo 2π.

Asymptotic geometry

$$ds^2 = r^2\Big(- \frac{dt^2}{\ell^2} + d\phi^2\Big) + \frac{\ell^2}{r^2} dr^2$$

Thus the conformal boundary is a cylinder:

$$ds_b^2 = -\frac{dt^2}{\ell^2} + d\phi^2,$$

General asymptotic geometry

Consider a geometry which asymptotes to

$$ds^2 = r^2(-d\mathcal{T}^2 + d\phi^2) + \left(\frac{r}{\ell}\right)^{2z} dr^2 \qquad z \in \mathbb{R}$$

where

$$\mathcal{T} = \frac{t}{\ell},$$

In order to construct the asymptotic symmetries, one may need to define a new radial coordinate x by

$$x^a = \left(\frac{r}{\ell}\right), a \in \mathbb{R}^+$$

- $a(z+1) < \frac{1}{2}$

Asymptotic symmetry

Consider a diffeomorphism given by

$$\xi = \Big(\epsilon + \frac{\bar{\epsilon}}{x^{2a}}\Big)\partial_{\mathcal{T}} + \Big(\lambda + \frac{\bar{\lambda}}{x^{2a}}\Big)\partial_\phi + \alpha x \partial_x$$

where

$$\dot{\epsilon} + a\alpha = 0 \qquad 2\,\bar{\epsilon} + a\ell^2\dot{\alpha} = 0,$$

$$\lambda' + a\alpha = 0, \qquad -2\,\bar{\lambda} + a\ell^2\alpha' = 0$$

$$\dot{\lambda} = \epsilon'$$

This generates the following diffeomorphism

$$\gamma_{\mu\nu} = r^2\gamma^{(0)}_{\mu\nu} \to \gamma_{\mu\nu} = r^2\gamma^{(0)}_{\mu\nu} + \gamma^{(1)}_{\mu\nu} + \cdots$$

$$\gamma^{(0)} = \mathrm{diag}(-1,1) \qquad \gamma^{(1)}_{\pm\pm} = -\frac{\partial^3\epsilon^\pm}{\partial x^{\pm 3}} \qquad \gamma^{(1)}_{+-} = 0$$

Virasoro algebra

It can be seen that

$$\xi = \xi(x^{\pm}) = \sum_{m\in\mathbb{Z}} \xi_m^{\pm}\, \mathsf{exp}[im(x^{\pm})], \qquad x^{\pm} = \mathcal{T} \pm \phi$$

and

$$i[\xi_m^{+}, \xi_n^{+}]_{\mathsf{Lie}} = (m-n)\xi_{m+n}^{+}$$

$$i[\xi_m^{-}, \xi_n^{-}]_{\mathsf{Lie}} = (m-n)\xi_{m+n}^{-}$$

$$[\xi_m^{+}, \xi_n^{-}]_{\mathsf{Lie}} = 0$$

Charges and the central charge

Boundary stress tensor

★ The Brown-York stress tensor is defined by

$$\tau_{\mu\nu} = \frac{1}{8\pi G}(K_{\mu\nu} - K\gamma_{\mu\nu})$$

[J.D. Brown and J.W. York, Phys. Rev. D47 (1993) 1407]

Assume that the metric is given in an ADM-like decomposition

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr)$$

The extrinsic curvature of the boundary is given by

$$K_{\mu\nu} = -\gamma_\mu^\alpha \nabla_\alpha n_\nu$$

n^μ is the outward pointing unit vector to the boundary.

Boundary CFT

Define

$$\tau^{\mathrm{reg}}_{\mu\nu} = \frac{1}{8\pi G}(K_{\mu\nu} - \frac{K}{2}\gamma_{\mu\nu})$$

- $\tau^{\mathrm{reg}}_{\mu\nu}$ is a symmetric tensor with respect to $\gamma_{\mu\nu}$
- $\mathrm{Tr}\tau^{\mathrm{reg}} = 0$
- $\mathcal{D}^{\mu}\tau^{\mathrm{reg}}_{\mu\nu} = 0$

  $\mathcal{D}_{\mu}$ is the covariant derivative compatible with $\gamma_{\mu\nu}$.

Conjecture: $\tau^{\mathrm{reg}}_{\mu\nu}$ corresponds to the CFT stress tensor.

[V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208 (1999) 413]

Mass

$$M = 2\pi R\,\tau_{tt}^{\text{reg}}$$

where R is given by

$$K = \frac{2}{R}$$

★ One can show that a geometry with $M = 0$

$$ds^2 = r^2(-d\mathfrak{T}^2 + d\phi^2) + \left(\frac{r}{\ell}\right)^{2z} dr^2$$

can be deformed to a geometry with $M \neq 0$ by

$$\gamma_{\mu\nu} = r^2\gamma_{\mu\nu}^{(0)} \to \gamma_{\mu\nu} = r^2\gamma_{\mu\nu}^{(0)} + \gamma_{\mu\nu}^{(1)} + \cdots$$

$$\gamma_{\pm\pm}^{(1)} = 4\pi GM \qquad \gamma_{+-}^{(1)} = 0$$

Central charge

$$\tilde{c} = \frac{3R}{2G}$$

$$\delta\tau^{\text{reg}}_{\pm\pm} = -\frac{\tilde{c}}{12\pi}\frac{\partial^3\epsilon^\pm}{\partial z^{\pm 3}}, \qquad z^\pm = \tfrac{R}{\ell}(\mathcal{T} \pm \phi)$$

Trace anomaly $$\text{Tr}\tau^{\text{reg}} = \frac{\tilde{c}}{24\pi}.{}^{(2)}R,$$

Use the identity

$$G_{\mu\nu}n^\mu n^\nu = \frac{1}{2}\big({}^{(2)}R - K_{\mu\nu}K^{\mu\nu} + K^2\big)$$

for $\gamma_{\mu\nu} = x^a\gamma^{(0)}_{\mu\nu} + \gamma^{(1)}_{\mu\nu} + \cdots$

Classification of asymptotic geometries

$$ds^2 = r^2(-d\mathcal{T}^2 + d\phi^2) + \left(\frac{r}{\ell}\right)^{2z} dr^2$$

Asymptotic symmetry is given by two copies of Virasoro algebra with central charge $\tilde{c}$

$$\tilde{c} = \frac{3R}{2G} = \lim_{r\to\infty} \frac{3\ell}{2G} \left(\frac{r}{\ell}\right)^{1+z} = \begin{cases} 0 & z < -1 \\ \frac{3\ell}{2G} & z = -1 \quad AdS_3 \\ \infty & z > -1 \end{cases}$$

Einstein gravity with conformal scalars

The action is given by

$$\int d^3x\sqrt{-g}\left(\frac{(1-\pi G\psi^2)}{16\pi G}R-\frac{1}{2}(\partial\psi)^2\right),$$

For $\psi=(\pi G)^{-1/2}$ the field equation is $R=0$.

Although the Planck mass is effectively zero, but there is a natural mass scale:

$$\langle\psi^2\rangle\sim G^{-1}$$

See [G. Barnich *et al*, arXiv:1204.3288] for asymptotic symmetries in the flat limit of asymptotically AdS_3 spacetimes.

Black hole solutions with a conformal boundary

$$f(r) = \frac{r^2}{\ell^2} - 2q^2, \qquad N^2 = \frac{\left|\frac{r^2}{\ell^2} - q^2\right|^{3/2}}{\frac{r^2}{\ell^2} - 2q^2}$$

Hawking temperature:

$$T = (2\pi\ell)^{-1}\sqrt{\frac{2}{q}}$$

Mass:

$$M = 2\pi R \tau_{tt}^{\mathrm{reg}} = \frac{q^2}{4G}$$

Entropy:

$$S = \left(\frac{2}{5}\right)\frac{2\pi\sqrt{-\det g(r_{\mathrm{h}})}}{4G}$$

★ For static BTZ, $\sqrt{-\det g(r_{\mathrm{h}})} = r_{\mathrm{h}}$

Asymptotic geometry

$$ds^2 = r^2\Big(-\frac{dt^2}{\ell^2} + dx^2\Big) + \frac{r}{\ell}dr^2$$

★ The solution with $q^2 \neq 0$ can be obtained by deforming the above geometry ($q^2 = 0$) by

$$\gamma_{\mu\nu} = r^2\gamma^{(0)}_{\mu\nu} \to \gamma_{\mu\nu} = r^2\gamma^{(0)}_{\mu\nu} + \gamma^{(1)}_{\mu\nu} + \cdots$$

$$\gamma^{(1)}_{\pm\pm} = q^2 \qquad \gamma^{(1)}_{+-} = 0$$

★ $z = \frac{1}{2}$ thus $\tilde{c} \to \infty$

What is the boundary CFT?

- $M_{\mathrm{Pl}} = 0$ ✓

- $\tilde{c} = \infty$ ✓

Recall that in WZW model,

$$\alpha' = \frac{1}{k - g^\vee} \qquad \tilde{c} = \frac{(\mathrm{dim}G)k}{k - g^\vee}$$

Conjecture

The corresponding CFT is a WZW model at critical level

$$k = g^\vee$$

Thank you for your attention.