Holographic entanglement entropy

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Plan of the talk

- Entanglement entropy
- Holography and Holographic entanglement entropy
- Thermalization and entanglement entropy
- The first law of entanglement thermodynamics
- Summary

Entanglement entropy

Consider a state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, which evolves in time by its Hamiltonian H

Physical quantities are computed as expectation values of operators as follows

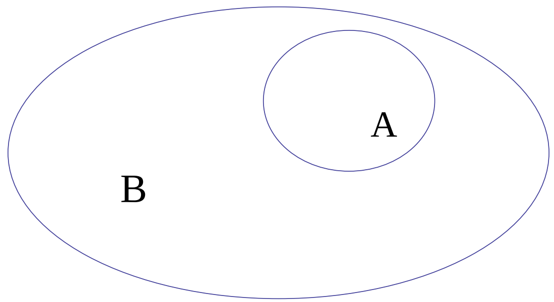
$$\langle O \rangle = \langle \psi | O | \psi \rangle = \mathrm{Tr}(\rho O)$$

where we defined the density matrix $\rho = |\psi\rangle\langle\psi|$. This system is called a pure state as it is described by a unique wave function $|\psi\rangle$.

In mixed states, the system is described by a density matrix ρ. An example of a mixed state is the canonical distribution

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

Assume that the quantum system has multiple degrees of freedom and so one can decompose the total system into two subsystems A and B



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

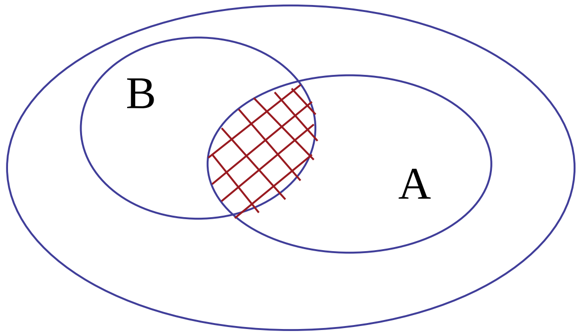
The reduced density matrix of the subsystem A

$$\rho_A = \mathrm{Tr}_B(\rho)$$

Then the entanglement entropy is defined as the von-Neumann entropy for A

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

Properties of Entanglement entropy



1. For pure state

$$S_A = A_B$$

2. For two subspace A and B, the strong subadditivity is

$$S_A + S_B \leq S_{A\cup B} + S_{A\cap B}$$

3. Leading divergence term is proportional to the area of the boundary ∂A

$$S_A = c_0 \frac{\text{Area}}{\epsilon^{d-1}} + O(\epsilon^{-(d-2)}),$$

where c_0 is a numerical constant; ϵ is the ultra-violet(UV) cut off in quantum field theories.

Example

Two spins $\frac{1}{2}$

$$|\psi\rangle = |\uparrow\rangle_A \otimes |\uparrow\rangle_B, \qquad |\tilde{\psi}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$$

The corresponding density matrices are

$$\rho = |\psi\rangle\langle\psi|, \qquad \tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$$

$$\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{\rho}_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

So that $S = 0$ while $\tilde{S} = \ln 2$.

Holography and holographic entanglement entropy

AdS/CFT correspondence

Basically AdS/CFT correspondence is a duality or a relation between two theories one with a gravity and the other without gravity.

The gravitational theory is usually defined in higher dimension.

Well developed case is the one where the gravity is defined on an AdS geometry where the dual theory is a CFT living in the conformal boundary of AdS space.

Classical gravity on an asympotically locally AdS$_{d+1}$ background is dual to a d-dimensional Large N strongly coupled field theory with a UV fixed point on its boundary.

AdS$_{d+1}$ metric in Poincare coordinates

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2.$$

AdS$_{d+1}$ metric in global coordinates

$$ds^2 = -(1 + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_{d-1}^2.$$

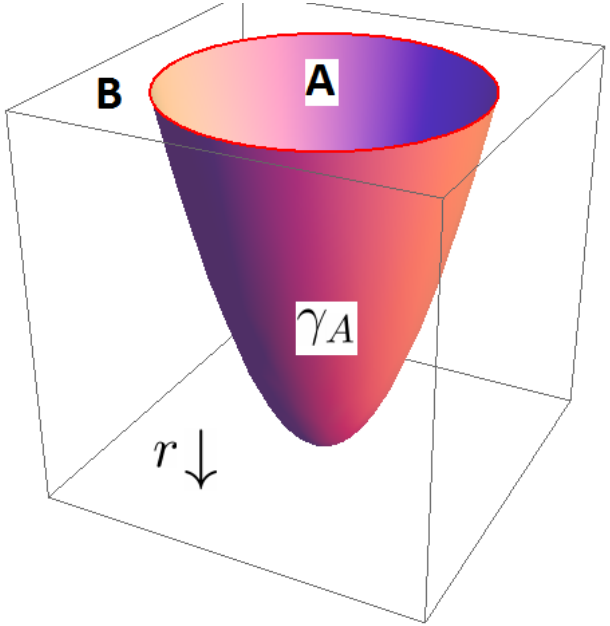
Here boundary is at $r \to \infty$

There is one to one correspondence between objects in CFT and those in the gravitational theory on AdS space.

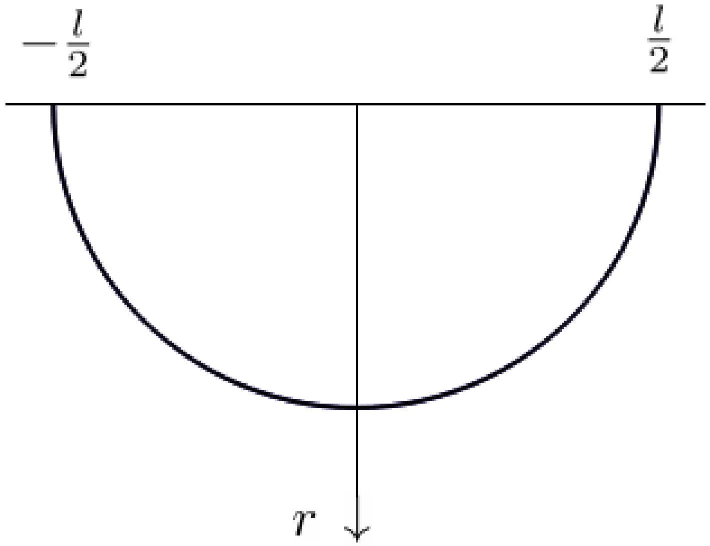
Gravity	$\Longleftrightarrow$	Field theory
$\{r, t, \vec{x}\}$	$\Longleftrightarrow$	$\{$scale of energy$, t, \vec{x}\}$
Near boundary Near horizon	$\Longleftrightarrow$	UV, IR regions
Symmetries	$\Longleftrightarrow$	Symmetries
Fields $\Phi(r, t, \vec{x})$	$\Longleftrightarrow$	Operators $\mathcal{O}(t, \vec{x})$
On shell action	$\Longleftrightarrow$	Generating function

Holographic Formula for Entanglement Entropy

For static background and fixed time divide the boundary into A and B. Extend this division $A \cup B$ to of the bulk spacetime. Extend ∂A to a surface γ_A in the entire spacetime such that $\partial \gamma_A = \partial A$.



$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}$$



Consider a strip in a d dimensional CFT at fixed time

$$t = \text{fixed}, \qquad -\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}, \qquad 0 \leq x_i \leq L, \quad i = 2, \cdots, d-1.$$

$$dS^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx_1^2 + dx_i^2), \qquad z = \frac{1}{r}$$

Consider a profile in the bulk $x_1 = x(z)$, so that the induced metric reads

$$dS_{\text{ind}}^2 = \frac{R^2}{z^2}[(1 + x'^2)dz^2 + dx_i^2].$$

The area of the induced metric is

$$A = L^{d-2} R^{d-1} \int dz \, \frac{\sqrt{1+x'^2}}{z^{d-1}}$$

One needs to minimize the area

$$\frac{x'}{z^{d-1}\sqrt{1+x'^2}} = \text{constant} = \frac{1}{z_t^{d-1}}$$

The width and the entanglement entropy are

$$\frac{\ell}{2} = \int_0^{z_t} dz \, \frac{(z/z_t)^{d-1}}{\sqrt{1-(\frac{z}{z_t})^{2(d-1)}}}, \qquad S = \frac{L^{d-2} R^{d-1}}{2G} \int_\epsilon^{z_t} \frac{dz}{z^{d-1}\sqrt{1-(\frac{z}{z_t})^{2(d-1)}}},$$

where z_t is a turning point and ϵ is a UV cut-off.

$$S = \begin{cases} \frac{L^{d-2}R^{d-1}}{2G}\left(-\frac{1}{(d-1)\epsilon^{d-2}} + \frac{c_0}{\ell^{d-2}}\right) & \text{for } d \neq 2, \\ \\ \frac{R}{2G}\ln\frac{\ell}{\epsilon}, & \text{for } d = 2, \end{cases}$$

with c_0 being a numerical factor

$$c_0 = \frac{2^{d-2}\pi^{\frac{d-1}{2}}}{d-2}\left(\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right)^{d-1}$$

For holographic entanglement entropy

1. The formula leads to the area law (for Einstein gravity).

2. The strong subadditivity can also be holographically proven (for static background)

3. For 2D CFT using AdS_3 one finds

$$S_A = \frac{c}{3} \ln \frac{\ell}{\epsilon}$$

where ℓ width of strip, $c = \frac{3R}{2G}$.

Time-dependent backgrounds

So far we have considered static case where we have a time slice on which we can define minimal surfaces. In the time-dependent case there is no a natural choice of the time-slices.

In Lorentzian geometry there is no minimal area surface. In order to resolve this issue we use the covariant holographic entanglement entropy which is

$$S_A(t) = \frac{\mathrm{Area}(\gamma_A(t))}{4G_N^{(d+2)}}$$

where $\gamma_A(t)$ is the extremal surface in the bulk Lorentzian spacetime with the boundary condition $\partial\gamma_A(t) = \partial A(t)$.

Example of time-dependent case: Black hole formation or Thermalization

$$\text{Geometry} \iff \text{State}$$

$$AdS \text{ solution} \iff \text{Vaccum state}$$
$$\text{Black hole} \iff \text{Excited state; thermal}$$

Let us perturbe a system so that the end point of the time evolution would be a thermal state. This might be done by a global quantum quench. Typically during evolution the system is out of equilibrium.

The thermalization process after a global quantum quench may be map to a black hole formation due to a gravitational collapse.

A quantum quench in the field theory may occurs due to a sudden change in the system which might be caused by turning on the source of an operator in an interval $\delta t \to 0$.

This change can excite the system to an excited state with non-zero energy density that could eventually thermalize to an equilibrium state.

From gravity point of view this might be described by a gravitational collapse of a thin shell of matter which can be modelled by an AdS-Vaidya metric.

$$dS^2 = \frac{R^2}{r^2}[f(r,v)dv^2 - 2drdv + d\vec{x}^2], \quad f(\rho,v) = 1 - m\theta(v)r^d$$

where r ρ is the radial coordinate, x_is are spatial boundary coordinates and v is the null coordinate. Here $\theta(v)$ is the step function and therefore for $v < 0$ the geometry is an AdS metric while for $v > 0$ it is an AdS-Schwarzschild black hole.

To compute the entanglement entropy for a strip with width ℓ, let us consider the following strip

$$-\frac{\ell}{2} \leq x_1 = x \leq \frac{\ell}{2}, \qquad 0 \leq x_a \leq L, \quad \text{for } \; a = 2, \cdots, d.$$

Since the metric is not static one needs to use the covariant proposal for the holographic entanglement entropy. Therefore the corresponding co-dimension two hypersurface in the bulk may be parametrized by $v(x)$ and $\rho(x)$. Then the induced metric on the hypersurface is

$$ds^2_{\text{ind}} = \frac{1}{\rho^2}\left[\left(1 - f(\rho, v)v'^2 - 2v'\rho'\right)dx^2 + dx_a^2\right),$$

The area of the hypersurface reads

$$\mathcal{A} = \frac{L^{d-2}}{2}\int_{-\ell/2}^{\ell/2} dx\, \frac{\sqrt{1-2v'\rho'-v'^2 f}}{\rho^{d-1}}$$

We note, however, that since the action is independent of x the corresponding Hamiltonian is a constant of motion

$$\rho^{d-1}\mathcal{L} = H = \text{constant}.$$

Moreover we have two equations of motion for v and ρ. Indeed, by making use of the above conservation law the corresponding equations of motion read

$$\partial_x P_v = \frac{P_\rho^2}{2}\frac{\partial f}{\partial v}, \qquad \partial_x P_\rho = \frac{P_\rho^2}{2}\frac{\partial f}{\partial \rho} + \frac{d-1}{\rho^{2d-1}}H^2,$$

where

$$P_v = \rho' + \rho^{1-z}v' f, \qquad P_\rho = v',$$

are the momenta conjugate to v and ρ up to a factor of H^{-1}, respectively.

These equations have to be supplemented by the following boundary conditions

$$\rho(\frac{\ell}{2}) = 0, \qquad v(\frac{\ell}{2}) = t, \qquad \rho'(0) = 0, \qquad v'(0) = 0,$$

and

$$\rho(0) = \rho_t, \qquad v(0) = v_t,$$

where (ρ_t, v_t) is the coordinate of the extremal hypersurface turning point in the bulk.

In what follows we will consider the case of $\ell \gg \rho_H$

The process we will be considering for the thermalization after a global quantum quench consists of three phases: initial phase, intermediate phase and final phase.

- Early times growth where $t \ll \rho_H$

$$\Delta S \approx \frac{L^{d-2} m}{8G}\, t^2,$$

- The intermediate region where $\frac{\ell}{2} \gg t \gg \rho_H$

$$\Delta S = L^{d-2} S_{\mathrm{th}}\, v_E\, t,$$

where

$$v_E = \left(\frac{d-2}{2(d-1)}\right)^{\frac{d-1}{d}} \sqrt{\frac{d}{d-2}}, \qquad S_{\mathrm{th}} = \frac{1}{4G\rho_H^{d-1}}$$

- Late time saturation $t \sim \frac{\ell}{2}$

$$\Delta S = \frac{L^{d-2}\ell}{4G\rho_H^{d-1}} + \cdots \ .$$

The first law of entanglement thermodynamics

Thermodynamics provides a useful tool to study a system when it is in the thermal equilibrium. In this limit the physics may be described in terms of few macroscopic quantities such as energy, temperaure, pressure, entropy.

There are also laws of thermodynamics which describe how these quantities behave under various conditions. In particular the first law of thermodynamics which is energy conservation, tells us how the entropy change as one changes the energy of the system.

There are several interesting phenomena which occur when the system is far from thermal equilibrium.

The entanglement entropy may provide a useful quantity to study excited quantum systems which are far from thermal equilibrium. For a generic quantum system it is difficult to compute the entanglement entropy. Nevertheless, at least, for those quantum systems which have holographic descriptions, one may use the holographic entanglement entropy to explore the behavior of the system.

Another quantity which can be always defined is the energy (or energy density) of the system. It is then natural to pose the question whether there is a relation between the entanglement entropy of an excited state and its energy.

For sufficiently small subsystem, the entanglement entropy is proportional to the energy of the subsystem. The proportionality constant is indeed given by the size of the entangling region.

Recall

Gravity on an asymptotically locally AdS provides a holographic description for a strongly coupled quantum field with a UV fixed point.

The information of quantum state in the dual field theory is encoded in the bulk geometry. In particular the AdS geometry is dual to the ground state of the dual conformal field theory.

Exciting the dual conformal field theory from the ground state to an excited state holographically corresponds to modifying the bulk geometry from AdS solution to a general asymptotically locally AdS solution.

The aim is to compute the entanglement entropy of an excited state for the case where the entangling region is sufficiently small.

Let's now compute the holographic entanglement entropy for a strip in an AdS black hole geometry.

$$dS^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + g(r)dr^2 + dx_1^2 + dx_{d-2}^2\right), \qquad f(r) = g(r)^{-1} = 1 - mr^d$$

For the strip, the induced metric on this hypersurface

$$dS^2_{\mathrm{ind}} = \frac{R^2}{r^2}\left[\left(g(r) + x'^2\right)dr^2 + d\vec{x}^2\right].$$

Therefore the area A reads

$$A = L^{d-2}R^{d-1}\int dr \frac{\sqrt{g + x'^2}}{r^{d-1}},$$

$$\frac{\ell}{2} = \int_0^{r_t} dr \frac{\sqrt{g(r)}\left(\frac{r}{r_t}\right)^{d-1}}{\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}, \qquad S = \frac{L^{d-2}R^{d-1}}{2G_N}\int_\epsilon^{r_t} \frac{\sqrt{g(r)}dr}{r^{d-1}\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}$$

In the limit of $ml^d \ll 1$ the change of the entanglement entropy is

$$\Delta S = S - S_0 = \frac{L^{d-2}R^{d-1}}{32(d+1)G_N}\frac{ml^2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2(d-1)})^2\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d}{2(d-1)})^2\Gamma(\frac{1}{2}+\frac{1}{d-1})}$$

On the other hand since the change of the energy is

$$\Delta E = \frac{(d-1)L^{d-2}R^{d-1}ml}{16\pi G_N}$$

Therefore one finds

$$\Delta S = c_0 \ell \Delta E$$

which can be recast to the following first law of the entanglement entropy

$$\Delta E = T_E \Delta S$$

The entanglement temperature is

$$T_E \sim \frac{1}{\ell}$$

Summary

1. Entanglement entropy is a good order parameter

2. There is very nice simple holographic description of entanglement entropy

4. Entanglement entropy might be a useful quantity to probe time dependent system

3. One may define a framework for entanglement entropy such as thermo-dynamics