(Super) Poisson Lie symmetry in WZW models based on Drinfel'd (Super)doubles

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Abstract

It is shown that the Poisson-Lie T-duality relates the Heisenberg WZW model to a sigma model defined on the Heisenberg Lie group when the dual Lie group is $A_2 \oplus 2A_1$. In this respect, the mutually T-dual sigma models on Drinfel'd double generated by the Heisenberg Lie group and its dual pair as target space are constructed in such a way that the original model is the same as the Heisenberg WZW model. Finally, we discuss a hierarchy of WZW models based on Lie supergroups related by the super Poisson-Lie T-duality.

1 Poisson-Lie symmetry in the WZW model based on the H_4 Lie group

The WZW models on non-semi-simple Lie groups [1]-[4] play an important role in string theory, since some of them provide exact string backgrounds having a target space dimension equal to the integer Virasoro central charge of the affine non-semi-simple algebra [3]. The first of these models was based on the group E_2^c, a central extension of the two-dimensional Euclidean group, and the corresponding sigma model describes string propagation on a four-dimensional space-time in the background of a gravitational plane wave [1]. This construction was subsequently extended to other non-semi-simple Lie groups [3], [4].

On the other hand, the T-duality is a very important symmetry of string theories, or more generally, two-dimensional sigma models [5], and the Poisson-Lie T-duality [6], [7] is a generalization of Abelian and non-Abelian target space duality (T-duality). Until now, there is one example for conformal sigma models related by Poisson-Lie T-duality [8] in such a way that the duality relates the standard $SL(2,R)$ WZW model to a constrained sigma model defined on the $SL(2,R)$ group space. Moreover, we have recently shown that the WZW models on the Lie supergroups $GL(1|1)$ [9] and (C^3+A) [10] contain super Poisson-Lie symmetry such that in this process the dual Lie supergroups are the respective $B \oplus A \oplus A_{1,1|}.i$ and $C^3 \oplus A_{1,1|}.i$.

In the present contribution, based on our previous works [9], [10], we will describe a new example of a WZW model that contains Poisson-Lie symmetry. The model is constructed on the Heisenberg Lie group, a non-semi-simple Lie group of dimension four (here and in the following we indicate it by H_4). The WZW model based on the Heisenberg group was, for the first time, introduced by Kehagias and Meessen [4]. Here, we first obtain the WZW model on the Heisenberg group with a new background. Then, we will show that the model has Poisson-Lie symmetry. Before proceeding to construct the model, let us first introduce the h_4 Lie algebra of the H_4 Lie group (the oscillator Lie algebra). The h_4 Lie algebra is generated by the generators $\{N, A_+, A_-, M\}$ with Lie brackets

$$[N, A_+] \;=\; A_+, \qquad [N, A_-] \;=\; -A_-, \qquad [A_-, A_+] \;=\; M, \qquad [M, .] \;=\; 0. \tag{1}$$

We should consider that the h_4 Lie algebra is a Drinfel'd double because one can show that the h_4 Lie algebra is isomorphic to a two-dimensional Lie bialgebra, i.e., four-dimensional Lie algebra of Drinfel'd double $(\mathcal{A}_2, \mathcal{I}_2)$ [11], in which, $\mathcal{A}_2$ is two-dimensional Lie algebra and $\mathcal{I}_2$ is two-dimensional Abelian Lie algebra. Let us now turn into our model. In general, given a Lie algebra with generators X_i and structure constants ${f_{ij}}^k$, to define a WZW model, one needs a non-degenerate symmetric bilinear form $\Omega_{ij} =< X_i, X_j >$, which is ad-invariant metric on Lie algebra $\mathcal{G}$ [1]. Using the commutation relations (1), one can obtain a non-degenerate invariant bilinear form on the h_4 Lie algebra as

$$< N \;,\; A_{\pm} >=< M \;,\; A_{\pm} >= 0, \qquad\qquad < A_+ \;,\; A_- >= - < N \;,\; M >= a, \tag{2}$$

where a is a non-zero real constant. The WZW model based on a Lie group G is defined on a Riemann surface Σ by the action [9]

$$S_{WZW}(g) = \frac{k}{4\pi}\int_{\Sigma} d\sigma^+ d\sigma^- \;< g^{-1}\partial_+ g \;,\; g^{-1}\partial_- g > + \frac{k}{24\pi}\int_B d^3\sigma \;< g^{-1}\, dg \;,\; [g^{-1}dg \;,\; g^{-1}dg] >, \tag{3}$$

where $g : \Sigma \to G$ is an element of Lie group G and $\sigma^{\pm} \equiv \frac{1}{2}(\tau \pm \sigma)$ are the standard light-cone variables on the worldsheet. Here B is a three-manifold bounded by worldsheet Σ. By introducing the new generators $\{X_1, X_2, X_3, X_4\}$ instead of $\{N, A_+, A_-, M\}$, respectively, we parametrize the corresponding Lie group H_4 with coordinates $x^{\mu} = \{x, y, u, v\}$ so that its elements can be written as

$$g \;=\; e^{vX_4}\; e^{uX_3}\; e^{xX_1}\; e^{yX_2}. \tag{4}$$

Hence, using relations (1) and (2) and some algebraic calculations, the WZW action on the H_4 Lie group is worked out to be of the form

$$S_{WZW}(g) = \frac{ak}{4\pi}\int d\sigma^+\, d\sigma^- \Big[-\partial_+ x\partial_- v - \partial_+ v\partial_- x + e^x\Big(\partial_+ y\partial_- u + \partial_+ u\partial_- y + y\partial_+ u\partial_- x - y\partial_+ x\partial_- u\Big)\Big]. \tag{5}$$

By identifying the above action with the sigma model of the form [6]

$$S = \frac{1}{2}\int d\sigma^+\, d\sigma^-\; \partial_+ x^{\mu}\; \mathcal{E}_{\mu\nu}\; \partial_- x^{\nu}, \tag{6}$$

we can read off the background matrix $\mathcal{E}_{\mu\nu}$. As we know the sigma model (6) has Poisson-Lie symmetry with respect to the Lie group $\tilde{G}$ (the dual Lie group to G with the same dimension G) when background matrix satisfies in the following relation[1] [7]

$$\mathcal{L}_{V_i}(\mathcal{E}_{\mu\nu}) \;=\; \mathcal{E}_{\mu\rho}\; (V^t)^{\rho}{}_{j}\; \tilde{f}^{kj}{}_{i}\; V_k{}^{\lambda}\; \mathcal{E}_{\lambda\nu}, \tag{7}$$

where $\mathcal{L}_{V_i}$ stands for the Lie derivative corresponding to the left invariant vector field V_i, and $\tilde{f}^{jk}{}_{i}$ are the structure constants of the dual Lie algebra $\tilde{\mathcal{G}}$. In the following we will show that the WZW model on the H_4 Lie group has Poisson-Lie symmetry. To this end, we need the left invariant vector fields on the H_4. Now, by substituting the background matrix of the action (5) and the left invariant vector fields calculated by use of the parametrization (4) on the right hand side of (7) and, then, by direct calculation of Lie derivative of $\mathcal{E}_{\mu\nu}$ with respect to V_i, one can obtain the dual pair to the h_4 Lie algebra in such a way that only non-zero commutation relation of the dual pair is

$$[\tilde{X}^2, \tilde{X}^4] \;=\; \tilde{X}^2. \tag{8}$$

The Lie algebra deduced in this process is a four-dimensional decomposable Lie algebra. We indicate it by $\mathcal{A}_2 \oplus 2\mathcal{A}_1$, in which $\mathcal{A}_1$ is one-dimensional Lie algebra. Having a Drinfel'd double which is simply a Lie group D, we can construct the Poisson-Lie symmetric sigma models on it. So we will, first, form the Drinfel'd double generated by the h_4 Lie algebra and its dual pair $\mathcal{A}_2 \oplus 2\mathcal{A}_1$. In the next section, we will construct a pair of Poisson-Lie T-dual sigma models which is associated with the $(H_4\;,\; A_2 \oplus 2A_1)$ Drinfel'd double and will show that the original sigma model on the H_4 is the same as the WZW model obtained in (5).

2 Poisson-Lie symmetric sigma models on the $(H_4, A_2 \oplus 2A_1)$ Drinfel'd double

As mentioned in the above, having Drinfel'd doubles, we can construct the Poisson-Lie T-dual sigma models on them. The construction of the models has been described in [6] and [7]. The models have target spaces in the Lie groups G and $\tilde{G}$, and are defined by the actions

$$S = \frac{1}{2}\int d^2\sigma (\partial_+ g g^{-1})^i\; E^+_{ij}(g)(\partial_- g g^{-1})^j, \quad \tilde{S} = \frac{1}{2}\int d^2\sigma (\partial_+ \tilde{g}\tilde{g}^{-1})_i\; \tilde{E}^{+ij}(\tilde{g})(\partial_- \tilde{g}\tilde{g}^{-1})_j, \tag{9}$$

where $E^+(g) = \big(\Pi(g) + (E^+_0)^{-1}(e)\big)^{-1}$ and analogously $\tilde{E}^+(\tilde{g}) = \Big(\tilde{\Pi}(\tilde{g}) + (\tilde{E}^+_0)^{-1}(\tilde{e})\Big)^{-1}$ [6], [7]. To construct the mutually T-dual sigma models with the group manifold of the $(H_4\;,\; A_2 \oplus 2A_1)$ Drinfel'd double, we use the same parametrization (4) for both models. Then, with a convenient choice of the constant matrix $E^+_0(e)$ at the unit element of H_4, T-dual sigma models are found to be given by

$$S \;=\; \frac{1}{2}\int d\sigma^+\, d\sigma^- \Big[\partial_+ x\partial_- v + \partial_+ v\partial_- x - e^x\Big(\partial_+ y\partial_- u + \partial_+ u\partial_- y + y\partial_+ u\partial_- x - y\partial_+ x\partial_- u\Big)\Big], \tag{10}$$

$$\begin{aligned}\tilde{S} \;=\;& \frac{1}{2}\int d\sigma^+\, d\sigma^- \Big[\partial_+\tilde{x}\partial_-\tilde{v} + \partial_+\tilde{v}\partial_-\tilde{x} - \partial_+\tilde{u}\partial_-\tilde{y} + \tilde{u}\partial_+\tilde{v}\partial_-\tilde{y} + \tilde{y}\partial_+\tilde{u}\partial_-\tilde{v} \\ &+ \frac{e^{-\tilde{v}}}{e^{-\tilde{v}} - 2}\Big(\partial_+\tilde{y}\partial_-\tilde{u} + \tilde{u}\partial_+\tilde{y}\partial_-\tilde{v} + \tilde{y}\partial_+\tilde{v}\partial_-\tilde{u} + 2\tilde{y}\tilde{u}e^{\tilde{v}}\partial_+\tilde{v}\partial_-\tilde{v}\Big)\Big].\end{aligned} \tag{11}$$

[1] The superscript "t" means transposition of the matrix.

The dual model is obtained by the exchange $G \leftrightarrow \tilde{G}$, $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$, and $E_0^+(e) \leftrightarrow \tilde{E}_0^+(\tilde{e}) = (E_0^+)^{-1}(e)$. By rescaling $a = -\frac{2\pi}{k}$ in action (5), one can conclude that action (10) is nothing but the WZW action based on the H_4 Lie group. Thus, we showed that the Poisson-Lie T-duality relates the H_4 WZW model to a sigma model defined on the Lie group H_4 when the dual Lie group is $A_2 \oplus 2A_1$. On the other hand, it is seen that in this case the Poisson-Lie T-duality transforms the rather extensive and complicated action (11) to much simpler form such as (10).

3 A hierarchy of WZW models related by the super Poisson-Lie T-duality

As mentioned in the beginning of the paper, we have recently found a pair of conformal super Poisson-Lie T-dual sigma models which is associated with the $\left((C^3 + A)\ ,\ C^3 \oplus A_{1,1|}.i\right)$ Drinfel'd superdouble in such a way that the original sigma model defined on the $(C^3 + A)$ Lie supergroup becomes equal to the WZW model based on the $(C^3 + A)$ Lie supergroup [10]. The original and dual sigma models have been presented in [10]. Also, to prove the conformal invariance of the dual model we have looked at the one-loop β-function equations [10]. Then, we showed that the dual model on the $C^3 \oplus A_{1,1|}.i$ Lie supergroup is identical to the WZW model on isomorphic Lie supergroup $(C^3 + A).i$ [10]. In the same way, one can show that the WZW model based on the isomorphic $(C^3 + A).i$ Lie supergroup has also super Poisson-Lie symmetry. In this process, the dual pair to the $(C^3 + A).i$ is found to be the $(2A_{1,1} + 2A)^0.i$ Lie supergroup [12]. Then, the corresponding dual sigma model with the $(2A_{1,1} + 2A)^0.i$ Lie supergroup is

$$\tilde{S} = \frac{1}{2}\int d\sigma^+ d\sigma^- \ (\partial_+\tilde{y}\partial_-\tilde{x} + \partial_+\tilde{x}\partial_-\tilde{y} - \partial_+\tilde{\psi}\ \partial_-\tilde{\chi} + \partial_+\tilde{\chi}\ \partial_-\tilde{\psi} - \frac{1}{2}\partial_+\tilde{y}\tilde{\chi}\partial_-\tilde{\chi} - \frac{1}{6}\partial_+\tilde{\chi}\tilde{\chi}\partial_-\tilde{y} + \frac{1}{3}\tilde{y}\partial_+\tilde{\chi}\partial_-\tilde{\chi}). \quad (12)$$

Similar to the above discussion, it is shown that the dual sigma model (12) is equal to the WZW model on Lie supergroup $(C^3 + A).ii$ (note that the Lie supergroups $(C^3 + A)$, $(C^3 + A).i$ and $(C^3 + A).ii$ are isomorphic to each other). Correspondingly, one can show that the $(C^3 + A).ii$ WZW model contains super Poisson-Lie symmetry. It seems this process can continue incessantly, that is, we can obtain a hierarchy of WZW models related by the super Poisson-Lie T-duality. To obtain an exact hierarchy of these models, one, firstly, needs to classify all Lie superbialgebras of the $(\mathcal{C}^3 + \mathcal{A})$ Lie superalgebra.

4 Concluding remarks

As the gist of the argument, first we have constructed a WZW model based on the Heisenberg Lie group (the Drinfel'd double H_4) by choosing a convenient parametrization of the group. In this way, it is found to correspond to a four-dimensional new string background of plane wave type. Then we have shown that the Poisson-Lie T-duality relates the H_4 WZW model to a sigma model defined on the Lie group H_4 when the dual Lie group is $A_2 \oplus 2A_1$. We showed that there is a hierarchy of WZW models on the Drinfel'd superdouble $(C^3 + A)$ and its isomorphisms related by the super Poisson-Lie T-duality. Therefore, the study carried out in this paper raises one question: Do all WZW models constructed on Drinfel'd (super)doubles have the (super) Poisson-Lie symmetry? Answer to this question is currently under investigation. We hope to answer this question and complete the hierarchy discussed in the last section of this paper in a later publication.

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