A HIERARCHY OF LAX PAIRS FOR THE NEW THIRD ORDER INTEGRABLE SYSTEM

DARYOUSH TALATI

ABSTRACT. We consider the Lax representation of the new two-component coupled integrable system recently discovered by the author. Connection of the hierarchy of infinitely many Lax pairs with each other is presented.

1. INTRODUCTION

Integrable systems are playing an important role in many different aspects of Mathematics and Physics. Nonlinear evolution systems which are solved exactly possesses various surprising features such as infinitely many symmetries and conserved covariants, and they are integrable (see [3] for a recent review). Such models arise in many branches of physics such as classical and quantum field theories, particle physics, relativity, statistical physics and quantum gravity.

The term Lax pair refers to linear systems that are related to nonlinear equations through a compatibility condition. The first part of Lax pair is called the scattering problem, that allows the initial-value problem for the integrable equation to be solved exactly. If a nonlinear equation possesses a Lax pair, then the Lax pair may be used to gather information about the behavior of the solutions to the nonlinear equation. Importantly, the existence of a Lax pair is a signature of integrability of the associated nonlinear equation. In his seminal work [2], Lax suggested a formalism to integrate a class of nonlinear evolution equations. He introduced a pair of linear operators L and M such that

$$L\phi = \lambda\phi,\ \ \phi_t = M\phi, \tag{1.1}$$

where L and M are linear differential operators, λ is an eigenvalue of L, and ϕ is an eigenfunction of L. Assuming $\lambda_t = 0$, differentiating $L\phi$ with respect to t gives

$$L_t\phi + L\phi_t = \lambda\phi_t.$$

Substituting in from (1.1) gives that

$$L_t\phi = ML\phi - LM\phi.$$

And therefore

$$(L_t + [L, M])\phi = 0.$$

Where $[M, L] = ML - LM$ is the operator commutator. Hence,

$$L_t = [M, L], \tag{1.2}$$

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is called Lax equation and contains commutative nonlinear evolution equation for suitable L and M. In [1], it is shown that all scalar evolution equations, which have the form of a conservation law, have a Lax pair with a second order L. For example consider the Lax formalism for the KdV equation

$$L = -\mathcal{D}_x^2 + u + u_1\mathcal{D}_x^{-1},\ M = -4\mathcal{D}_x^3 + 6u\mathcal{D}_x + 9u_1 + 3u_2\mathcal{D}_x^{-1}. \tag{1.3}$$

These operators satisfies the Lax equation (1.2)

$$L_t = u_t + u_{tx}D_x^{-1},\ [M, L] = u_3 - 6uu_1 + (u_4 - 6uu_2 - 6u_1^2)D_x^{-1},$$

if u is solution to the KdV equation:

$$u_t = -u_3 + 6uu_1.$$

Very recently, a integrable system of third order nonlinear equations founded by the author [4]

$$\begin{pmatrix} u \\ \\ v \end{pmatrix}_t = \begin{pmatrix} 48u_{xxx} + 36v_{xxx} - 6uv_{xx} - 12vv_{xx} - 6u_xv_x \\ -2u^2u_x + v^2u_x - 12v_x^2 + 2uvv_x \\ \\ 36u_{xxx} + 12v_{xxx} + 6uu_{xx} + 12vu_{xx} + 6u_x^2 \\ +12u_xv_x + 2uvu_x + v_xu^2 - 2v_xv^2 \end{pmatrix}, \tag{1.4}$$

where x and t are the space and time variables, respectively. Talati [4] showed that this system possesses generalized symmetries of orders 7, 9, 11, 13, 17, ... as well as conserved densities of orders 2, 6, 8, 10, 12, 16, ... and proved that this system is bi-Hamiltonian and therefor possess infinitely many generalized symmetries.

2. Lax pair of system (1.4)

Almost all of the known integrable models possess linear Lax pairs. However in the literature, there is no systematic way of finding whether a given evolution equation possesses a Lax representation and how one can construct the operators $\mathcal{L}$, and $\mathcal{M}$. Our aim of this paper is to construct a Lax pair for the new third order integrable system introduced by the author [4]. For the system (1.4) we shall consider the case where our candidate Lax pair is a differential operator

$$\begin{aligned} \mathcal{M} = &\quad \alpha_0\mathcal{D}_x^i + \alpha_1\mathcal{D}_x^{i-1} + ... + \alpha_{i-1}\mathcal{D}_x + \alpha_i \\ \mathcal{L} = &\quad \beta_0\mathcal{D}_x^j + \beta_1\mathcal{D}_x^{j-1} + ... + \beta_{j-1}\mathcal{D}_x + \beta_j. \end{aligned} \tag{2.1}$$

Here $\alpha = \alpha(u, v, u_x, v_x, ...)$ and $\beta = \beta(u, v, u_x, v_x, ...)$ are the dependent variables and $i, j = 1, 2, 3, ...$. Now the major problem is to find appropriate analytic functions α_i and β_j that satisfy the integrability condition (1.2). From the order of the system (1.4) it is easy to see that we must assign $i = 3$

$$\mathcal{M} = \alpha_0\mathcal{D}_x^3 + \alpha_1\mathcal{D}_x^2 + \alpha_2\mathcal{D}_x + \alpha_3. \tag{2.2}$$

By assuming that α_0 and β_0 are nonzero constants, We carry out the calculations For cases $j = 1, 2, 3, \ldots$ according to (1.2). After lengthy calculations, the integrability condition (1.2) can lead to the following Lax pair:

$$
\begin{aligned}
\mathcal{M} = \quad & -8/3D_x^3 - (4u+4v)D_x^2 - (4u_x + 4v_x + 2u^2 + 4uv \\
& +2v^2)D_x + \tfrac{1}{6}(-260u_{xx} - 152v_{xx} - 30u_x u \\
& -48u_x v + 6v_x u + 24v_x v - 9u^2 v - 9uv^2), \\
\mathcal{L}_{j=1} = \quad & 2D_x + u + v, \\
\mathcal{L}_{j=2} = \quad & D_x^2 + (u+v)D_x + \tfrac{1}{4}(2u_x + 2v_x + u^2 + 2uv + v^2), \\
\mathcal{L}_{j=3} = \quad & 2/3D_x^3 + (u+v)D_x^2 + (u_x + v_x + \tfrac{1}{2}u^2 + uv \\
& +\tfrac{1}{2}v^2)D_x + \tfrac{1}{12}(4u_{xx} + 4v_{xx} + 6u_x u + 6u_x v \\
& +6v_x u + 6v_x v + u^3 + 3u^2 v + 3uv^2 + v^3) \\
& \vdots
\end{aligned} \tag{2.3}
$$

Using the previous result we will state a conjecture about the existence a hierarchy of infinitely many Lax operators of the system (1.4).

Conjecture 2.1. *Consider the differential operators*

$$
\begin{aligned}
\mathcal{M} = \quad & -8/3D_x^3 - (4u+4v)D_x^2 - (4u_x + 4v_x + 2u^2 + 4uv + 2v^2)D_x \\
& +\tfrac{1}{6}(-260u_{xx} - 152v_{xx} - 30u_x u - 48u_x v + 6v_x u + 24v_x v \\
& -9u^2 v - 9uv^2),
\end{aligned}
$$

$$
\mathcal{L}_j = \sum_{n=0}^{j} \beta_n D_x^{j-n}, \; j = 1, 2, 3, \ldots \; . \tag{2.4}
$$

Where

$$
\beta_n = \frac{j-n+1}{2n}(2\frac{d}{dx} + u + v)\beta_{n-1}, \; \beta_0 = \frac{2}{j}.
$$

Then it can be further proved that the operator equation

$$
\mathcal{L}_{j\,t} - [\mathcal{M}, \mathcal{L}_j] = \mathcal{O}, \tag{2.5}
$$

where $\mathcal{O}$ is the zero operator, is equivalent to the system (1.4) in the sense that both sides of the equation turn out to be operators of multiplication by a function. So the system (1.4) admits a hierarchy of infinitely many Lax operators that satisfy the condition $[\mathcal{L}_n, \mathcal{L}_m] = 0$.

Moreover, this equation is claimed to be integrable in [4] and admits the Lax representations but its N -soliton solution is an open question to researchers.

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Department of Engineering Physics, Ankara University 06100 Tandoğan, Ankara
E-mail address: talati@eng.ankara.edu.tr , Daryoush.talati@gmail.com