Jacobi-Lie symmetry in H_4-WZW model

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Outline

- Poisson-Lie symmetry
- Jacobi structure and Jacobi-Lie bialgebra
- Jacobi-Lie symmetry
- Jacobi-Lie symmetry in H_4-WZW model

Poisson bracket

A Poisson bracket is real bilinear map that is satisfying in the following conditions:

$$\{,\} : C^{\infty}(M,R) \times C^{\infty}(M,R) \mapsto C^{\infty}(M,R)$$

$$\{c_1 f + c_2 g, h\} = c_1\{f,h\} + c_2\{g,h\} \tag{1.1}$$

$$\{f,g\} = -\{g,f\} \tag{1.2}$$

$$\{fg,h\} = f\{g,h\} + g\{h,f\} \tag{1.3}$$

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0 \tag{1.4}$$

Poisson structure

$$\{f, g\} = \Pi(df, dg) \tag{1.5}$$

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$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

$$\downarrow$$

$$[\Pi, \Pi] = 0 \tag{1.6}$$

where 2-vector Π is called Poisson structure. The manifold M endowed with a Poisson structure is called a Poisson manifold.

A. Lichnerowicz, J. Differential Geometry. **12** (1977) 253-300.

Lie bialgebra

A Lie bialgebra structure on Lie algebra is a skew-symmetric linear map δ

1) $\delta : \mathbf{g} \to \mathbf{g} \otimes \mathbf{g}$

$$\forall X, Y \in \mathbf{g} \quad \delta([X, Y]) = [\delta(X), 1\otimes Y + Y\otimes 1] + [1\otimes X + X\otimes 1, \delta(Y)], \tag{1.7}$$

2) the dual map $\delta^t : \mathbf{g}^* \otimes \mathbf{g}^* \to \mathbf{g}^*$ is a Lie bracket on $\mathbf{g}^*$

$$\forall X \in \mathbf{g}, \xi, \eta \in \mathbf{g}^* \quad (\xi \otimes \eta, \delta(X)) = (\delta^t(\xi \otimes \eta), X) = ([\xi, \eta]_*, X). \tag{1.8}$$

The Lie bialgebra defined in this way will be denoted by $(\mathbf{g}, \mathbf{g}^*)$ or $(\mathbf{g}, \delta)$.

V. G. Drinfel'd, Sov. Math. Dokl. **27** (1983) 68-71.

Also, condition (1.7) can be rewritten in the following form

$$d_*[X, Y]^{\mathbf{g}} = [X, d_* Y]^{\mathbf{g}} - [Y, d_* X]^{\mathbf{g}}. \qquad (1.9)$$

where d_* being the Chevalley-Eilenberg differential of $\mathbf{g}^*$ acting on $\mathbf{g}$.

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where d_* being the Chevalley-Eilenberg differential of $\mathbf{g}^*$ acting on $\mathbf{g}$. By choosing

$$d_* X_i = -\frac{1}{2}\tilde{f}^{jk}{}_i X_j \wedge X_k, \tag{1.10}$$

we have a well-known Bianchi identity

$$f_{ij}{}^k \tilde{f}^{mn}{}_k = f_{ik}{}^m \tilde{f}^{kn}{}_j + f_{ik}{}^n \tilde{f}^{mk}{}_j + f_{kj}{}^m \tilde{f}^{kn}{}_i + f_{kj}{}^n \tilde{f}^{mk}{}_i \tag{1.11}$$

where

$$[X_i, X_j] = f_{ij}{}^k X_k \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}{}_k \tilde{X}^k.$$

Poisson-Lie symmetry

This symmetry does not require the isometry in the original and dual target manifolds

$$d \star J_i = -\frac{1}{2}\tilde{f}^{jk}{}_i \star J_j \wedge \star J_k, \qquad (1.12)$$

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$$\downarrow$$

$$\mathcal{L}_{v_i}\mathcal{E}_{\mu\nu} = \tilde{f}^{jk}{}_i v_j{}^{\lambda} v_k{}^{\eta}\mathcal{E}_{\mu\eta}\mathcal{E}_{\lambda\nu}. \tag{1.13}$$

Then, the integrability condition for $\mathcal{L}_{v_i}\mathcal{E}_{\mu\nu}$ gives the Bianchi identity

$$f_{ij}{}^k\tilde{f}^{mn}{}_k = f_{ik}{}^m\tilde{f}^{kn}{}_j + f_{ik}{}^n\tilde{f}^{mk}{}_j + f_{kj}{}^m\tilde{f}^{kn}{}_i + f_{kj}{}^n\tilde{f}^{mk}{}_i \tag{1.14}$$

C. Klimčik and P. Ševera, Phys. Lett. B. **351** (1995) 455-462.

Jacobi bracket

A Jacobi bracket is real bilinear map that is satisfying in the following conditions:

$$\{,\} : C^\infty(M,R) \times C^\infty(M,R) \mapsto C^\infty(M,R)$$

$$\{c_1 f + c_2 g, h\} = c_1\{f,g\} + c_2\{g,h\} \tag{2.1}$$

$$\{f,g\} = -\{g,f\} \tag{2.2}$$

$$\{fg,h\} = f\{g,h\} + g\{h,f\} - fg\{1,h\} \tag{2.3}$$

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0 \tag{2.4}$$

A. Kirillov, Russ. Math. Surv. **31** (1976) 55-75.

Jacobi structure

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$$\downarrow$$
$$[\Lambda,\Lambda] = 2E\wedge\Lambda\ ,\ [E,\Lambda] = 0. \qquad (2.6)$$

A Jacobi structure on M is a pair (Λ, E), where Λ is a 2-vector and E is a vector field on M.

A. Lichnerowicz, J. Math. Pures Appl. **57** (1978) 453-488.

Jacobi-Lie bialgebra

A Jacobi-Lie bialgebra is a pair $((\mathbf{g}, \phi_0), (\mathbf{g}^*, X_0))$, where $(\mathbf{g}, [,]^{\mathbf{g}})$ is a real Lie algebra of finite dimension and $\mathbf{g}^*$ is dual space of $\mathbf{g}$ with Lie bracket $[,]^{\mathbf{g}^*}$, such that we have

$$d_{*X_0}[X, Y]^{\mathbf{g}} = [X, d_{*X_0} Y]^{\mathbf{g}}_{\phi_0} - [Y, d_{*X_0} X]^{\mathbf{g}}_{\phi_0}, \tag{2.7}$$

$$\phi_0(X_0) = 0, \tag{2.8}$$

$$i_{\phi_0}(d_* X_0) + [X_0, X] = 0. \tag{2.9}$$

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The $X_0 \in \mathbf{g}$ and $\phi_0 \in \mathbf{g}^*$ are 1-cocycles on $\mathbf{g}^*$ and $\mathbf{g}$, respectively, i.e. we must have

$$d_* X_0 = 0, \tag{2.10}$$

$$d\phi_0 = 0. \tag{2.11}$$

D. Iglesias and J. C. Marrero, J. Geom. Phys. **40** (2001) 176-199.

We expand $X_0 \in \mathbf{g}$ and $\phi_0 \in \mathbf{g}^*$ in terms of the basis of the Lie algebras $\mathbf{g}$ and $\mathbf{g}^*$

$$X_0 = \alpha^i X_i \qquad , \qquad \phi_0 = \beta_j \tilde{X}^j, \tag{2.12}$$

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Now, according to the generalized Chevalley-Eilenberg differential d_{*X_0} as follows

$$\forall Y \in \mathbf{g} \qquad\qquad d_{*X_0} Y = d_* Y + X_0 \wedge Y, \tag{2.13}$$

and using

$$d_* X_i = -\frac{1}{2} \tilde{f}^{jk}{}_i X_j \wedge X_k, \tag{2.14}$$

we have the new definition of Jacobi-Lie bialgebras for physical applications.

A Jacobi-Lie bialgebra is a pair $((\mathbf{g}, \phi_0), (\mathbf{g}^*, X_0))$ where $(\mathbf{g}, [,]^{\mathbf{g}})$ is a real Lie algebra of finite dimension with the basis $\{X_i\}$, and $\mathbf{g}^*$ is dual space of $\mathbf{g}$ with Lie bracket $[,]^{\mathbf{g}^*}$ and basis $\{\tilde{X}^i\}$, such that $X_0 = \alpha^i X_i \in \mathbf{g}$ and $\phi_0 = \beta_j \tilde{X}^j \in \mathbf{g}^*$ are 1-cocycles on $\mathbf{g}^*$ and $\mathbf{g}$, respectively, i.e

$$\alpha^i \tilde{f}^{mn}{}_i = 0, \tag{2.15}$$

$$\beta_i f_{mn}{}^i = 0, \tag{2.16}$$

and we have

$$\begin{aligned} f_{ij}{}^k \tilde{f}^{mn}{}_k - f_{ik}{}^m \tilde{f}^{kn}{}_j - f_{ik}{}^n \tilde{f}^{mk}{}_j - f_{kj}{}^m \tilde{f}^{kn}{}_i - f_{kj}{}^n \tilde{f}^{mk}{}_i \\ + \beta_i \tilde{f}^{mn}{}_j - \beta_j \tilde{f}^{mn}{}_i + \alpha^m f_{ij}{}^n - \alpha^n f_{ij}{}^m + (\alpha^k f_{ik}{}^m - \alpha^m \beta_i)\delta_j{}^n \\ -(\alpha^k f_{jk}{}^m - \alpha^m \beta_j)\delta_i{}^n - (\alpha^k f_{ik}{}^n - \alpha^n \beta_i)\delta_j{}^m + (\alpha^k f_{jk}{}^n - \alpha^n \beta_j)\delta_i{}^m = 0, \end{aligned} \tag{2.17}$$

$$\alpha^i \beta_i = 0, \tag{2.18}$$

$$\alpha^n f_{ni}{}^m - \beta_n \tilde{f}^{nm}{}_i = 0. \tag{2.19}$$

A. Rezaei-Aghdam and M. Sephid, Int. J. Geom. Methods Mod. Phys. **13** (2016) 1650087.

(12) A. Rezaei-Aghdam and M. Sephid, Int. J. Geom. Methods Mod. Phys. **14** (2017) 1750007.

Jacobi-Lie symmetry

We consider two-dimensional sigma model on the target space M with background matrix $\mathcal{E}_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ in the presence of a σ-function on M ($\sigma \in C^\infty(M)$)

$$S = \frac{1}{2}\int_\Sigma d\xi^+ \wedge d\xi^- \; e^{\sigma(x)}\mathcal{E}_{\mu\nu}(x)\; \partial_+ x^\mu \partial_- x^\nu \tag{3.1}$$

where $\xi^\pm$ and $\{x^\mu\}$ are coordinates of the world sheet Σ and manifold M, respectively.

One can consider the free action of Lie group G on manifold M by the following transformation

$$x^{\mu} \rightarrow x^{\mu} + \epsilon^{i}(\xi^{+}, \xi^{-}) v_{i}{}^{\mu}. \tag{3.2}$$

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$$x^\mu \to x^\mu + \epsilon^i(\xi^+, \xi^-) {v_i}^\mu. \tag{3.2}$$

The variation of action (3.1) under transformation (3.2) is calculated to be

$$\delta S = \frac{1}{2}\int d\xi^+ \wedge d\xi^- \; e^{\sigma(x)} \epsilon^i (\mathcal{L}_{v_i}\mathcal{E}_{\mu\nu} + {v_i}^\lambda \partial_\lambda \sigma \; \mathcal{E}_{\mu\nu}) \, \partial_+ x^\mu \partial_- x^\nu \\ - \frac{1}{2}\int d\epsilon^i \wedge \star J_i, \tag{3.3}$$

where Hodge star of the Noether's currents have the following form

$$\star J_i = e^{\sigma(x)} (\mathcal{E}_{\mu\gamma} \; {v_i}^\gamma \; \partial_+ x^\mu d\xi^+ - \mathcal{E}_{\gamma\nu} \; {v_i}^\gamma \; \partial_- x^\nu d\xi^-). \tag{3.4}$$

If we consider $\phi_0 = d\sigma \in \Omega^1(M)$ as a 1-cocycle on $\mathbf{g}$ (Lie algebra of the Lie group G) with values in $\mathbf{g}^*$, then we have ϕ_0-Lie derivative in the sense of D. Iglesias and J. C. Marrero as

$$(\mathcal{L}_{\phi_0})_{v_i}\mathcal{E}_{\mu\nu} = \mathcal{L}_{v_i}\mathcal{E}_{\mu\nu} + < \phi_0, v_i > \mathcal{E}_{\mu\nu}. \tag{3.5}$$

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Therefore, one can rewrite the variation of S using the above definition for ϕ_0-Lie derivative as follows

$$\delta S = \frac{1}{2}\int d\xi^+ \wedge d\xi^- \; e^{\sigma(x)}\epsilon^i(\mathcal{L}_{\phi_0})_{v_i}\mathcal{E}_{\mu\nu}\; \partial_+ x^\mu \partial_- x^\nu - \frac{1}{2}\int d\epsilon^i \wedge \star J_i. \qquad (3.6)$$

Now, with $\delta S = 0$ we have

$$d \star J_i = e^{\sigma(x)}[(\mathcal{L}_{\phi_0})_{v_i}\mathcal{E}_{\mu\nu}\, \partial_+ x^\mu \partial_- x^\nu] d\xi^- \wedge d\xi^+. \tag{3.7}$$

Now, with $\delta S = 0$ we have

$$d \star J_i = e^{\sigma(x)}[(\mathcal{L}_{\phi_0})_{v_i}\mathcal{E}_{\mu\nu}\ \partial_+ x^\mu \partial_- x^\nu] d\xi^- \wedge d\xi^+. \qquad (3.7)$$

We assume that the 1-forms $\star J_i$ are not closed and they obey the following generalized Maurer-Cartan equation

$$d \star J_i = -\frac{1}{2} e^{-\sigma}(\tilde{f}^{jk}{}_i - \alpha^j \delta^k{}_i + \alpha^k \delta^j{}_i) \star J_j \wedge \star J_k, \qquad (3.8)$$

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Using (3.7) and (3.8) the condition of *Jacobi-Lie symmetry* can be formulated in the following form

$$(\mathcal{L}_{\phi_0})_{v_i}\mathcal{E}_{\mu\nu} = (\tilde{f}^{jk}{}_i - \alpha^j \delta^k{}_i + \alpha^k \delta^j{}_i) v_j{}^\lambda v_k{}^\eta \mathcal{E}_{\mu\eta}\mathcal{E}_{\lambda\nu}. \qquad (3.9)$$

Note that, this symmetry is a generalization of the Poisson-Lie symmetry and subsequently isometry symmetry.

A. Rezaei-Aghdam and M. Sephid, Nucl. Phys. B. **926** (2018) 602-613.

Now, we will consider the integrability condition for the ϕ_0-Lie derivative

$$(\mathcal{L}_{\phi_0})_{[v_i,v_j]}\mathcal{E}_{\mu\nu} = (\mathcal{L}_{\phi_0})_{v_i}(\mathcal{L}_{\phi_0})_{v_j}\mathcal{E}_{\mu\nu} - (\mathcal{L}_{\phi_0})_{v_j}(\mathcal{L}_{\phi_0})_{v_i}\mathcal{E}_{\mu\nu}. \quad (3.10)$$

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According to the definition of the ϕ_0-Lie derivative presented in (3.5), and 1-cocycle condition on ϕ_0 i.e.,

$$\beta_k {f_{ij}}^k = 0, \quad (3.11)$$

where $\beta_k \equiv {v_k}^\lambda \partial_\lambda \sigma$, the integrability condition for ϕ_0-Lie derivative is equivalent to the integrability condition for usual Lie derivative

$$\mathcal{L}_{[v_i,v_j]}\mathcal{E}_{\mu\nu} = \mathcal{L}_{v_i}\mathcal{L}_{v_j}\mathcal{E}_{\mu\nu} - \mathcal{L}_{v_j}\mathcal{L}_{v_i}\mathcal{E}_{\mu\nu}. \quad (3.12)$$

Now, using (3.5) and (3.9), after some computations, the integrability condition (3.12) gives the first condition of Jacobi-Lie bialgebra $((\mathbf{g}, \phi_0), (\mathbf{g}^*, X_0))$, i.e.,

$$f_{ij}{}^{k}\tilde{f}^{mn}{}_{k} - f_{ik}{}^{m}\tilde{f}^{kn}{}_{j} - f_{ik}{}^{n}\tilde{f}^{mk}{}_{j} - f_{kj}{}^{m}\tilde{f}^{kn}{}_{i} - f_{kj}{}^{n}\tilde{f}^{mk}{}_{i}$$
$$+\beta_i\tilde{f}^{mn}{}_{j} - \beta_j\tilde{f}^{mn}{}_{i} + \alpha^m f_{ij}{}^{n} - \alpha^n f_{ij}{}^{m} + (\alpha^k f_{ik}{}^{m} - \alpha^m\beta_i)\delta_j{}^{n}$$
$$-(\alpha^k f_{jk}{}^{m} - \alpha^m\beta_j)\delta_i{}^{n} - (\alpha^k f_{ik}{}^{n} - \alpha^n\beta_i)\delta_j{}^{m} + (\alpha^k f_{jk}{}^{n} - \alpha^n\beta_j)\delta_i{}^{m} = 0. \tag{3.13}$$

In other words, the integrability condition of the ϕ_0-Lie derivative together with the Jacobi-Lie symmetry (3.9) result that the ϕ_0 is 1-cocycle and the first condition of the Jacobi-Lie bialgebra $((\mathbf{g}, \phi_0), (\mathbf{g}^*, X_0))$ is satisfied.

In the same way, one can consider the dual sigma model on the manifold $\tilde{M}$ in the presence of a $\tilde{\sigma}$-function

$$\tilde{S} = \frac{1}{2}\int d\xi^{+} \wedge d\xi^{-}\ e^{\tilde{\sigma}(\tilde{x})}\tilde{\mathcal{E}}_{\mu\nu}(\tilde{x})\ \partial_{+}\tilde{x}^{\mu}\partial_{-}\tilde{x}^{\nu}. \tag{3.14}$$

In the same way, one can consider the dual sigma model on the manifold $\tilde{M}$ in the presence of a $\tilde{\sigma}$-function

$$\tilde{S} = \frac{1}{2}\int d\xi^+ \wedge d\xi^- \; e^{\tilde{\sigma}(\tilde{x})}\tilde{\mathcal{E}}_{\mu\nu}(\tilde{x})\;\partial_+\tilde{x}^\mu\partial_-\tilde{x}^\nu. \qquad (3.14)$$

If we consider $X_0 = d\tilde{\sigma} \in \Omega^1(\tilde{M})$ as a 1-cocycle on $\mathbf{g}^*$ with values in $\mathbf{g}$, then, we have X_0-Lie derivative as

$$(\mathcal{L}_{*X_0})_{\tilde{v}^i}\tilde{\mathcal{E}}_{\mu\nu} = \mathcal{L}_{*\tilde{v}^i}\tilde{\mathcal{E}}_{\mu\nu} + < X_0, \tilde{v}^i > \tilde{\mathcal{E}}_{\mu\nu}. \qquad (3.15)$$

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If we consider $X_0 = d\tilde{\sigma} \in \Omega^1(\tilde{M})$ as a 1-cocycle on $\mathbf{g}^*$ with values in $\mathbf{g}$, then, we have X_0-Lie derivative as

$$(\mathcal{L}_{*X_0})_{\tilde{v}^i}\tilde{\mathcal{E}}_{\mu\nu} = \mathcal{L}_{*\tilde{v}^i}\tilde{\mathcal{E}}_{\mu\nu} + < X_0, \tilde{v}^i > \tilde{\mathcal{E}}_{\mu\nu}. \tag{3.15}$$

Using the equations of motion related to $\tilde{S}$ and the following generalized Maurer-cartan equation

$$d \star \tilde{J}^i = -\frac{1}{2}e^{-\tilde{\sigma}}({f_{jk}}^{i} - \beta_j{\delta_k}^{i} + \beta_k{\delta_j}^{i}) \star \tilde{J}^j \wedge \star\tilde{J}^k, \tag{3.16}$$

$$\downarrow$$

$$(\mathcal{L}_{*X_0})_{\tilde{v}^i}\tilde{\mathcal{E}}_{\mu\nu} = ({f_{jk}}^{i} - \beta_j{\delta_k}^{i} + \beta_k{\delta_j}^{i})\tilde{v}^{j\lambda}\tilde{v}^{k\eta}\tilde{\mathcal{E}}_{\mu\eta}\tilde{\mathcal{E}}_{\lambda\nu}. \tag{3.17}$$

The integrability condition for $(\mathcal{L}_{*X_0})_{\tilde{v}^i}$

$$(\mathcal{L}_{*X_0})_{[\tilde{v}^m,\tilde{v}^n]}\tilde{\mathcal{E}}_{\mu\nu} = (\mathcal{L}_{*X_0})_{\tilde{v}^m}(\mathcal{L}_{*X_0})_{\tilde{v}^n}\tilde{\mathcal{E}}_{\mu\nu} - (\mathcal{L}_{*X_0})_{\tilde{v}^n}(\mathcal{L}_{*X_0})_{\tilde{v}^m}\tilde{\mathcal{E}}_{\mu\nu}, \tag{3.18}$$

with the 1-cocycle condition for X_0 i.e.,

$$\alpha^k \tilde{f}^{mn}{}_k = 0, \tag{3.19}$$

is equivalent to the integrability condition for $(\mathcal{L}_*)_{\tilde{v}^i}$ as follows

$$\mathcal{L}_{*[\tilde{v}^m,\tilde{v}^n]}\tilde{\mathcal{E}}_{\mu\nu} = \mathcal{L}_{*\tilde{v}^m}\mathcal{L}_{*\tilde{v}^n}\tilde{\mathcal{E}}_{\mu\nu} - \mathcal{L}_{*\tilde{v}^n}\mathcal{L}_{*\tilde{v}^m}\tilde{\mathcal{E}}_{\mu\nu}. \tag{3.20}$$

The integrability condition for $(\mathcal{L}_{*X_0})_{\tilde{V}^i}$

$$(\mathcal{L}_{*X_0})_{[\tilde{V}^m,\tilde{V}^n]}\tilde{\mathcal{E}}_{\mu\nu} = (\mathcal{L}_{*X_0})_{\tilde{V}^m}(\mathcal{L}_{*X_0})_{\tilde{V}^n}\tilde{\mathcal{E}}_{\mu\nu} - (\mathcal{L}_{*X_0})_{\tilde{V}^n}(\mathcal{L}_{*X_0})_{\tilde{V}^m}\tilde{\mathcal{E}}_{\mu\nu}, \tag{3.18}$$

with the 1-cocycle condition for X_0 i.e.,

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is equivalent to the integrability condition for $(\mathcal{L}_*)_{\tilde{V}^i}$ as follows

$$\mathcal{L}_{*[\tilde{V}^m,\tilde{V}^n]}\tilde{\mathcal{E}}_{\mu\nu} = \mathcal{L}_{*\tilde{V}^m}\mathcal{L}_{*\tilde{V}^n}\tilde{\mathcal{E}}_{\mu\nu} - \mathcal{L}_{*\tilde{V}^n}\mathcal{L}_{*\tilde{V}^m}\tilde{\mathcal{E}}_{\mu\nu}. \tag{3.20}$$

Using (3.15) and (3.17), relation (3.20) is converted to the first condition of the Jacobi-Lie bialgebras $((\mathbf{g}^*, X_0), (\mathbf{g}, \phi_0))$

$$\begin{gathered} f_{ij}{}^k \tilde{f}^{mn}{}_k - f_{ik}{}^m \tilde{f}^{kn}{}_j - f_{ik}{}^n \tilde{f}^{mk}{}_j - f_{kj}{}^m \tilde{f}^{kn}{}_i - f_{kj}{}^n \tilde{f}^{mk}{}_i \\ +\beta_i \tilde{f}^{mn}{}_j - \beta_j \tilde{f}^{mn}{}_i + \alpha^m f_{ij}{}^n - \alpha^n f_{ij}{}^m + (\beta_k \tilde{f}^{mk}{}_i - \alpha^m \beta_i)\delta_j{}^n \\ -(\beta_k \tilde{f}^{mk}{}_j - \alpha^m \beta_j)\delta_i{}^n - (\beta_k \tilde{f}^{nk}{}_i - \alpha^n \beta_i)\delta_j{}^m + (\beta_k \tilde{f}^{nk}{}_j - \alpha^n \beta_j)\delta_i{}^m = 0. \end{gathered} \tag{3.21}$$

Finally, with subtraction of the relations (3.13) and (3.21), we arrive to the third condition of Jacobi-Lie bialgebras $((\mathbf{g},\phi_0),(\mathbf{g}^*,X_0))$ and $((\mathbf{g}^*,X_0),(\mathbf{g},\phi_0))$, i.e.,

$$\alpha^k f_{ik}{}^m - \beta_k \tilde{f}^{mk}{}_i = 0. \tag{3.22}$$

Finally, with subtraction of the relations (3.13) and (3.21), we arrive to the third condition of Jacobi-Lie bialgebras $((\mathbf{g},\phi_0),(\mathbf{g}^*,X_0))$ and $((\mathbf{g}^*,X_0),(\mathbf{g},\phi_0))$, i.e.,

$$\alpha^k f_{ik}{}^m - \beta_k \tilde{f}^{mk}{}_i = 0. \tag{3.22}$$

If α^i and β_i in relations (3.11), (3.13), (3.19), (3.21) and (3.22) are satisfying in the following relation

$$\alpha^i \beta_i = 0, \tag{3.23}$$

then, we have the second condition of the Jacobi-Lie bialgebras $((\mathbf{g},\phi_0),(\mathbf{g}^*,X_0))$ and $((\mathbf{g}^*,X_0),(\mathbf{g},\phi_0))$. The equation (3.23) can be obtained from the isometric symmetry in direction of 1-cocycles X_0 and ϕ_0, i.e.,

$$(\mathcal{L})_{X_0}\mathcal{E}_{\mu\nu} = 0 \qquad , \qquad (\mathcal{L}_*)_{\phi_0}\tilde{\mathcal{E}}_{\mu\nu} = 0. \tag{3.24}$$

WZW model on the Heisenberg Lie group H_4

The WZW action on Lie group G is written as

$$S_{WZW}(g) = \frac{K}{4\pi}\int_{\Sigma} d\xi^{+}\wedge d\xi^{-}\; L^{i}_{+}\;\Omega_{ij}\;L^{j}_{-} + \frac{K}{24\pi}\int_{B} d^{3}\xi\;\varepsilon^{\gamma\alpha\beta}L^{i}_{\gamma}\;\Omega_{il}\;L^{j}_{\alpha}\;f_{jk}{}^{l}\;L^{k}_{\beta}, \tag{4.1}$$

where $\Omega_{ij} =< X_i, X_j >$ is non-degenerate ad-invariant bilinear form in the following relation

$$< X_i, [X_j, X_k] >=< [X_i, X_j], X_k >, \tag{4.2}$$

C. R. Nappi and E. Witten, Phys. Rev. Lett. **71** (1993) 3751.

The oscillator Lie algebra h_4 (related to Heisenberg Lie group H_4) with four generators $\{a, a^\dagger, N = aa^\dagger, M\}$

$$[N, a^\dagger] \;=\; a^\dagger, \qquad [N, a] = -a, \qquad [a, a^\dagger] \;=\; M, \tag{4.3}$$

has ad-invariant symmetric bilinear form Ω_{ij} as

$$\Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 & -\kappa \\ 0 & 0 & \kappa & 0 \\ 0 & \kappa & 0 & 0 \\ -\kappa & 0 & 0 & \kappa' \end{pmatrix}, \qquad \kappa \in \Re - \{0\}, \quad \kappa' \in \Re. \tag{4.4}$$

Using the parametrization $g = e^{vX_4}\, e^{uX_3}\, e^{xX_1}\, e^{yX_2}$ and $\kappa' = 0$ H_4-WZW action is written as

$$S_{wzw} = \frac{\kappa K}{4\pi} \int d\xi^+ \wedge d\xi^- \Big\{ -\partial_+ x \partial_- v - \partial_+ v \partial_- x + e^x \Big(\partial_+ y \partial_- u + \partial_+ u \partial_- y + y \partial_+ u \partial_- x - y \partial_+ x \partial_- u \Big) \Big\}. \tag{4.5}$$

A. Eghbali and A. Rezaei-Aghdam, Nucl. Phys. B. **899** (2015) 165.

Jacobi-Lie symmetry in H_4-WZW model

From the Jacobi-Lie symmetry on (4.5) the non-zero commutation relations of the dual pair to the Heisenberg Lie algebra h_4 and 1-cocycles X_0 and ϕ_0 are found to be

i) $((\mathbf{h_4}, \mathbf{0}), (\mathcal{A}_\mathbf{2} \oplus \mathbf{2}\mathcal{A}_\mathbf{1}, \mathbf{0}))$

$$[\tilde{X}^2, \tilde{X}^4] = \tilde{X}^2 \tag{4.6}$$

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ii) $((\mathbf{h_4}, \mathbf{0}), (\mathcal{V} \oplus \Re.\mathbf{i}, \mathbf{X_4}))$

$$[\tilde{X}^1, \tilde{X}^4] = -\tilde{X}^1 \quad [\tilde{X}^3, \tilde{X}^4] = -\tilde{X}^3 \tag{4.7}$$

iii) $((\mathbf{h_4}, \mathbf{0}), (\mathcal{A}^{\mathbf{a},\mathbf{a}}_{\mathbf{4},\mathbf{5}}.\mathbf{i}, \frac{\mathbf{a}}{\mathbf{a}-\mathbf{1}}\mathbf{X_4}))$

$$[\tilde{X}^1, \tilde{X}^4] = -\frac{a}{a-1}\tilde{X}^1 \quad [\tilde{X}^2, \tilde{X}^4] = -\frac{1}{a-1}\tilde{X}^2 \quad [\tilde{X}^3, \tilde{X}^4] = -\frac{a}{a-1}\tilde{X}^3 \tag{4.8}$$

iv) $((\mathbf{h_4}, \mathbf{0}), (\mathcal{A}^{\mathbf{a},\mathbf{1}}_{\mathbf{4},\mathbf{5}}.\mathbf{i}, \frac{\mathbf{1}}{\mathbf{1}-\mathbf{a}}\mathbf{X_4}))$

$$[\tilde{X}^1, \tilde{X}^4] = \frac{1}{a-1}\tilde{X}^1 \quad [\tilde{X}^2, \tilde{X}^4] = \frac{a}{a-1}\tilde{X}^2 \quad [\tilde{X}^3, \tilde{X}^4] = \frac{1}{a-1}\tilde{X}^3 \tag{4.9}$$

$((\mathbf{h_4}, \mathbf{0}), (\mathcal{A}_{4,5}^{-1,1}.\mathbf{i}, \frac{1}{2}\mathbf{X_4}))$

$$S = \frac{1}{2}\int d\xi^+ \wedge d\xi^- \Big\{ -\partial_+ x \partial_- v - \partial_+ v \partial_- x + e^x \Big(\partial_+ y \partial_- u + \partial_+ u \partial_- y + y \partial_+ u \partial_- x - y \partial_+ x \partial_- u\Big)\Big\}. \quad (4.10)$$

$$\Updownarrow$$

$$\tilde{S} = \frac{1}{2}\int d\xi^+ \wedge \, d\xi^- \Big\{ - e^{\frac{\tilde{v}}{2}}(\partial_+ \tilde{x} \partial_- \tilde{v} + \partial_+ \tilde{v} \partial_- \tilde{x}) + \frac{1}{3 - 2e^{\frac{-\tilde{v}}{2}}}(\partial_+ \tilde{y} \partial_- \tilde{u} + \tilde{u} \partial_+ \tilde{y} \partial_- \tilde{v} + \tilde{y} \partial_+ \tilde{v} \partial_- \tilde{u}) + \frac{1}{1 - 2e^{\frac{-\tilde{v}}{2}}}(-\partial_+ \tilde{u} \partial_- \tilde{y} + \tilde{y} \partial_+ \tilde{u} \partial_- \tilde{v} + \tilde{u} \partial_+ \tilde{v} \partial_- \tilde{y}) - \frac{2\tilde{y}\tilde{u}}{(1 - 2e^{\frac{-\tilde{v}}{2}})(3 - 2e^{\frac{-\tilde{v}}{2}})}\partial_+ \tilde{v} \partial_- \tilde{v}\Big\}. \quad (4.11)$$

A. Rezaei-Aghdam and M.Sephid, arXiv:1804.07948 [hep-th].

Thank you

Let G be a connected Lie group with Lie algebra $\mathbf{g}$. Let $\Phi : G \times V \to V$ be a representation of G on a vector space V and $T_{\mathbf{e}}\Phi : \mathbf{g} \times V \to V$ be the induced representation of $\mathbf{g}$ on V.

If the map $\phi : G \to V$ is a 1-cocycle on G relative to Φ, i.e., if for $h, g \in G$

$$\phi(hg) = \phi(h) + \Phi(h, \phi(g)), \tag{1}$$

then $\epsilon =: (\delta\phi)(\mathbf{e}) : \mathbf{g} \to V$ is a 1-cocycle on $\mathbf{g}$ relative to $T_{\mathbf{e}}\Phi$, i.e., for $X, Y \in \mathbf{g}$

$$T_{\mathbf{e}}\Phi(X, \epsilon(Y)) - T_{\mathbf{e}}\Phi(Y, \epsilon(X)) = \epsilon([X, Y]^{\mathbf{g}}). \tag{2}$$