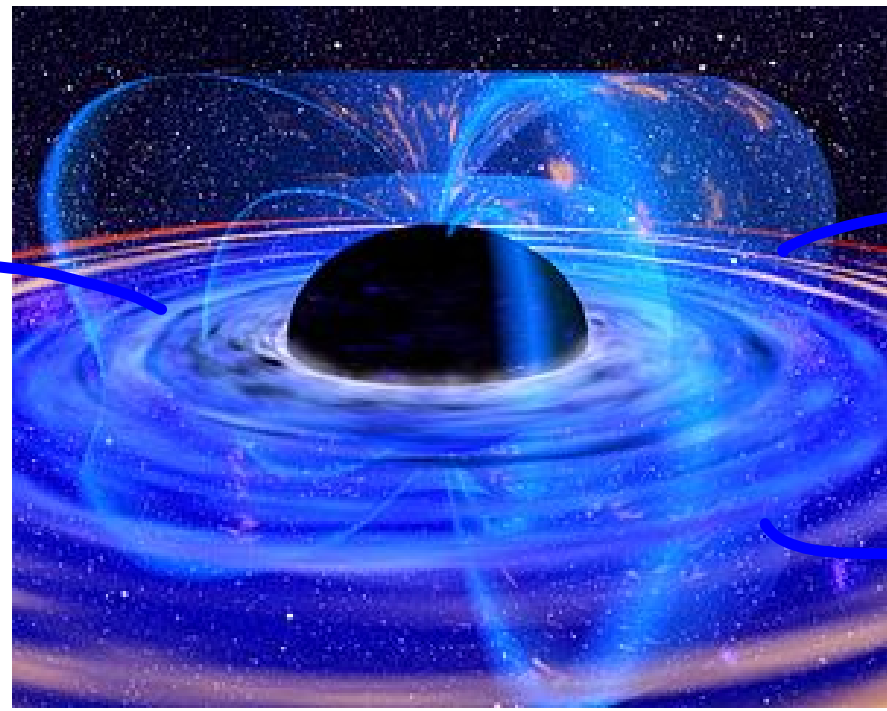
Covariant conserved charges in generally covariant gravitational theories

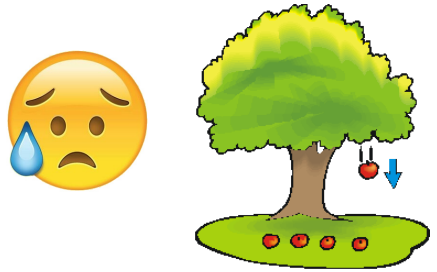
Conference on Gravity and Cosmology 1397



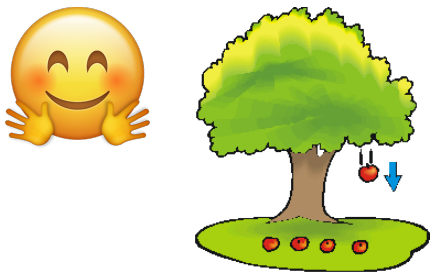
Kamal Hajian
Institute for Research in Fundamental Sciences (IPM)
January 23rd 2019

Outline

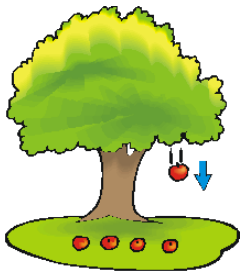
1. **Introduction**
   - Noether formulation



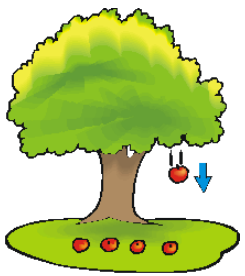
2. **Covariant phase space formulation**



Outline



1. **Introduction**
   - Noether formulation



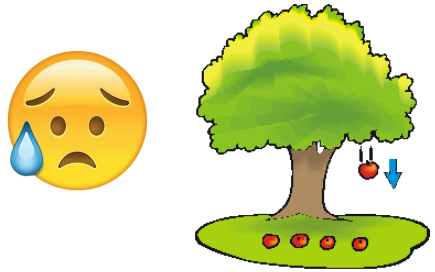
2. **Covariant phase space formulation**

- A. Ashtekar, L. Bombelli, and R. Koul, O. Reula (1987, 1990)
- C. Crnkovic and E. Witten (1987)
- J. Lee and R. M. Wald, *J. Math. Phys.*, **31**, 725–743, (1990)

⋮

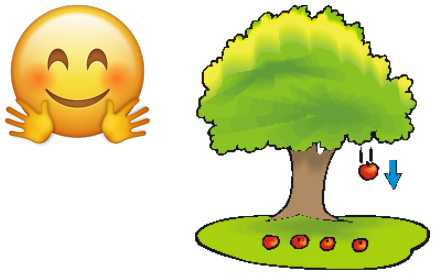
- K. Hajian and M. M. Sheikh-Jabbari, Phys. Rev. D **93**, no. 4, 044074 (2016)
- K. Hajian, Gen. Rel. Grav. **48**, no. 8, 114 (2016)
- M. Ghodrati, K. Hajian and M. R. Setare, Eur. Phys. J. C **76**, no. 12, 701 (2016)
- D. Chernyavsky and K. Hajian, Class. Quant. Grav. **35**, no. 12, 125012 (2018)

Outline



1. **Introduction**
   - Noether formulation

2. **Covariant phase space formulation**



- A. Ashtekar, L. Bombelli, and R. Koul, O. Reula (1987, 1990)
- C. Crnkovic and E. Witten (1987)
- J. Lee and R. M. Wald, *J. Math. Phys.*, **31**, 725–743, (1990)

⋮

- K. Hajian and M. M. Sheikh-Jabbari, Phys. Rev. D **93**, no. 4, 044074 (2016)
- K. Hajian, Gen. Rel. Grav. **48**, no. 8, 114 (2016)
- M. Ghodrati, K. Hajian and M. R. Setare, Eur. Phys. J. C **76**, no. 12, 701 (2016)
- D. Chernyavsky and K. Hajian, Class. Quant. Grav. **35**, no. 12, 125012 (2018)

Introduction

Review: differential forms

D-dimensional spacetime manifold: $(x^0, x^1, \ldots, x^D)$

Differential forms

$a(x)$	0-form	0 index
$a_\mu(x)\ \mathrm{d}x^\mu$	1-form	1 index
$a_{\mu\nu}(x)\ \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$	2-form	2 anti-commuting indices
$\vdots$		
$a_{\mu_1 \ldots \mu_p}(x)\ \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$	p-form	p anti-commuting indices
$\vdots$		
$a_{\mu_1 \ldots \mu_D}(x)\ \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_D}$	D-form (top form)	D anti-commuting indices

Examples

Lagrangian density $\mathcal{L}$	0-form
Electromagnetic gauge field A_μ	1-form
Field strength $F_{\mu\nu}$	2-form

Review: differential forms

Notation: $\boldsymbol{a} \equiv a_{\mu_1 \ldots \mu_p}(x)\ \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$

Exterior derivative: p-form $\boldsymbol{a}$ $\rightarrow$ $(p+1)$-form $\mathrm{d}\boldsymbol{a}$

$$(\mathrm{d}\boldsymbol{a})_{\mu_1 \ldots \mu_{p+1}} = (p+1)\ \partial_{[\mu_1} a_{\mu_1 \ldots \mu_p]}$$

Hodge duality: p-form $\boldsymbol{a}$ $\rightarrow$ $(D-p)$-form $\star\boldsymbol{a}$

$$(\star\boldsymbol{a})_{\mu_{p+1} \ldots \mu_D} = \frac{1}{p!}\sqrt{-g}\ \epsilon_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_D}\ a^{\mu_1 \ldots \mu_p}$$

Wedge: p-form $\boldsymbol{a}_1$ and q-form $\boldsymbol{a}_2$ $\rightarrow$ $(p+q)$-form $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2$

$$(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} a_{1[\mu_1 \ldots \mu_p}\ a_{2\mu_{p+1} \ldots \mu_{p+q}]}$$

Cartan identity

Lie derivative $\mathcal{L}_\xi \mathbf{a} = \xi \cdot \mathrm{d}\mathbf{a} + \mathrm{d}(\xi \cdot \mathbf{a})$ Exterior derivative

Review: differential forms

Cartan identity

Lie derivative $\mathcal{L}_\xi \mathbf{a} = \xi \cdot \mathrm{d}\mathbf{a} + \mathrm{d}(\xi \cdot \mathbf{a})$ Exterior derivative

Simpler way of writing the action:

$$S = \int \mathrm{d}^D x \, \sqrt{g}\mathcal{L} \qquad \xrightarrow{\mathbf{L}=\star\mathcal{L}} \qquad S = \int \mathbf{L}$$

Example: Maxwell theory

Gauge field	$\mathbf{A}$
Field strength	$\mathbf{F} = \mathrm{d}\mathbf{A}$
Action	$S = \int \star\mathbf{F} \wedge \mathbf{F} - \mathbf{A} \wedge \mathbf{J}$
Bianchi identity	$\mathrm{d}\mathbf{F} = \mathrm{d}^2\mathbf{A} = 0$
Equation of motion (e.o.m)	$\mathrm{d} \star \mathbf{F} = \mathbf{J}$

Review: Symmetry

Dynamical fields: Φ $\quad S = \int \mathbf{L}(\Phi; x^\mu)$ $\quad \Phi : \phi,\ A_\mu,\ g_{\mu\nu}, \ldots$

What is a symmetry?

Review: Symmetry

Dynamical fields: Φ $\quad S = \int \mathbf{L}(\Phi; x^\mu)$ $\quad \Phi : \phi,\ A_\mu,\ g_{\mu\nu}, \ldots$

What is a symmetry?

Symmetry

Symmetry is a transformation on Φ such that:

- $S \to S$,

or $\mathbf{L} \to \mathbf{L} + \mathrm{d}\mathbf{K}$,

or e.o.m $\to$ e.o.m,

or set of solutions $\to$ set of solutions,

or its generator G commutes with Hamiltonian $\{G, \mathcal{H}\} = 0$.

Review: Symmetry

Dynamical fields: Φ $\quad S = \int \mathbf{L}(\Phi; x^\mu)$ $\quad \Phi : \phi,\ A_\mu,\ g_{\mu\nu}, \ldots$

What is a symmetry?

Symmetry

Symmetry is a transformation on Φ such that:

- $S \to S$,

or $\mathbf{L} \to \mathbf{L} + \mathrm{d}\mathbf{K}$,

or e.o.m $\to$ e.o.m,

or set of solutions $\to$ set of solutions,

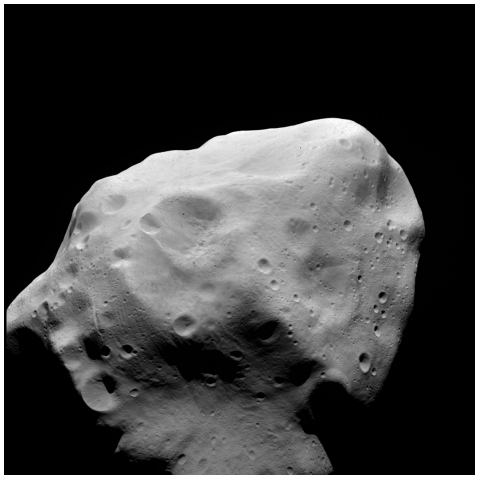
or its generator G commutes with Hamiltonian $\{G, \mathcal{H}\} = 0$.

- Symmetry is an off-shell concept.
- Symmetry is a property of the theory, not a solution.

Symmetry vs. Isometry

- This asteroid is not spherically symmetric.
- Its gravitational dynamics around the sun is spherically symmetric.

$\Rightarrow$ Its angular momenta are conserved.

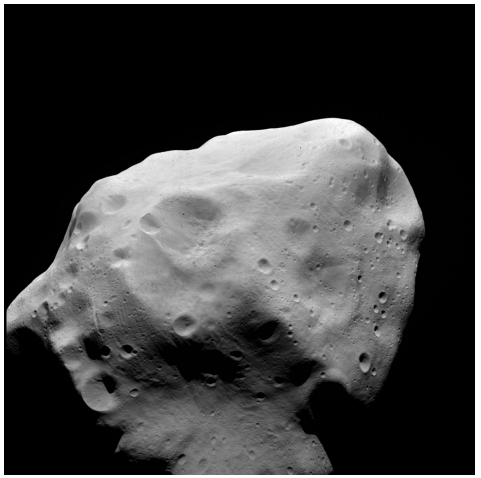


© ESA 2010 MPS/OSIRIS Team

Symmetry vs. Isometry

- This asteroid is not spherically symmetric.
- Its gravitational dynamics around the sun is spherically symmetric.

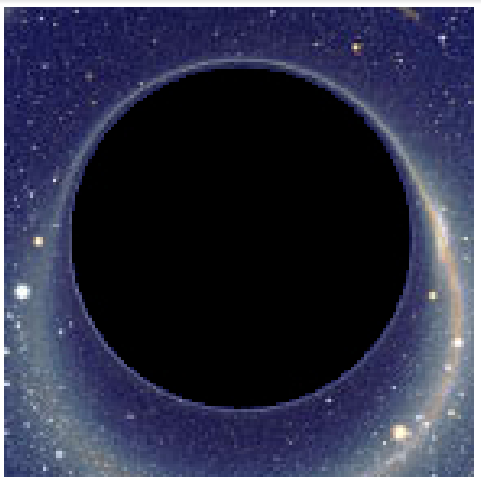
$\Rightarrow$ Its angular momenta are conserved.



© ESA 2010 MPS/OSIRIS Team

Exact symmetry (Isometry):
A transformation which does not change the solution at all.

- Schwarzschild black hole is a solution to Einstein gravity with spherical isometry.



Killing vector

Generally covariant theories

Infinitesimal diffeomorphism: $x^\mu \to x^\mu - \xi^\mu$.

Generally covariant theory: $\mathcal{L}$ is a scalar under diffeomorphisms.

Examples

Maxwell theory	$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$	✓
Einstein-Λ gravity	$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda)$	✓
$f(R)$ gravity	$\mathcal{L} = f(R)$	✓
Gauss-Bonnet gravity	$\mathcal{L} = \frac{1}{16\pi G}(R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta})$	✓
$\vdots$		

- Diffeomorphisms are symmetries of covariant theories

Proof: $\delta_\xi \mathbf{L} = \mathcal{L}_\xi \mathbf{L} = \xi \cdot \cancel{d\mathbf{L}}^{0} + \mathrm{d}(\xi \cdot \mathbf{L}) \quad \Rightarrow \quad \mathbf{L} \to \mathbf{L} + \mathrm{d}(\xi \cdot \mathbf{L})$

Generally covariant theories

Infinitesimal diffeomorphism: $x^\mu \to x^\mu - \xi^\mu$.

Generally covariant theory: $\mathcal{L}$ is a scalar under diffeomorphisms.

Examples

Maxwell theory	$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$	✓
Einstein-Λ gravity	$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda)$	✓
$f(R)$ gravity	$\mathcal{L} = f(R)$	✓
Gauss-Bonnet gravity	$\mathcal{L} = \frac{1}{16\pi G}(R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta})$	✓
$\vdots$		

- Diffeomorphisms are symmetries of covariant theories

Proof: $\delta_\xi \mathbf{L} = \mathcal{L}_\xi \mathbf{L} = \xi \cdot \cancelto{0}{d\mathbf{L}} + \mathrm{d}(\xi \cdot \mathbf{L}) \quad \Rightarrow \quad \mathbf{L} \to \mathbf{L} + \mathrm{d}(\xi \cdot \mathbf{L})$

Symmetry $\rightarrow$ **Conservation**

$$\begin{array}{lcll} \{G,\mathcal{H}\}=0 & \Rightarrow & \delta_{_G}\mathcal{H}=0 & \text{Symmetry} \\ \{\mathcal{H},G\}=0 & \Rightarrow & \delta_{_\mathcal{H}}G=\dot{G}=0 & \text{Conservation} \end{array}$$

Emmy Noether

Noether current

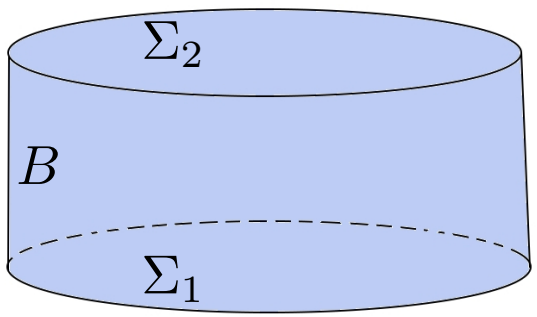
$$\begin{cases} \delta_\xi \mathbf{L} = \mathrm{d}(\xi \cdot \mathbf{L}) \\ \delta_\xi \mathbf{L} = (\mathrm{e.o.m})\delta_\xi \Phi + \mathrm{d}\boldsymbol{\Theta} \end{cases} \xrightarrow{\mathrm{e.o.m}=0} \mathrm{d}\underbrace{(\boldsymbol{\Theta} - \xi \cdot \mathbf{L})}_{\mathbf{J}_\xi} = 0$$

Noether current

$$\begin{cases} \delta_\xi \mathbf{L} = \mathrm{d}(\xi \cdot \mathbf{L}) \\ \delta_\xi \mathbf{L} = (\text{e.o.m})\delta_\xi \Phi + \mathrm{d}\boldsymbol{\Theta} \end{cases} \xrightarrow{\text{e.o.m}=0} \mathrm{d}\underbrace{(\boldsymbol{\Theta} - \xi \cdot \mathbf{L})}_{\mathbf{J}_\xi} = 0$$

Conservation

$$0 = \int_{\text{cylinder}} \mathrm{d}\mathbf{J}_\xi = \cancelto{0}{\oint_B \mathbf{J}_\xi} + \int_{\Sigma_1} \mathbf{J}_\xi - \int_{\Sigma_2} \mathbf{J}_\xi \quad \Rightarrow \quad Q_\xi = \int_\Sigma \mathbf{J}_\xi \quad \text{is conserved.}$$



Noether current

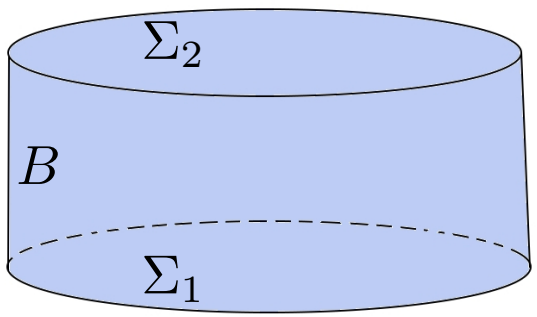
$$\begin{cases} \delta_\xi \mathbf{L} = \mathrm{d}(\xi \cdot \mathbf{L}) \\ \delta_\xi \mathbf{L} = (\text{e.o.m})\delta_\xi \Phi + \mathrm{d}\boldsymbol{\Theta} \end{cases} \xrightarrow{\text{e.o.m}=0} \mathrm{d}\underbrace{(\boldsymbol{\Theta} - \xi \cdot \mathbf{L})}_{\mathbf{J}_\xi} = 0$$

Conservation

$$0 = \int_{\text{cylinder}} \mathrm{d}\mathbf{J}_\xi = \cancelto{0}{\oint_B \mathbf{J}_\xi} + \int_{\Sigma_1} \mathbf{J}_\xi - \int_{\Sigma_2} \mathbf{J}_\xi \quad \Rightarrow \quad Q_\xi = \int_\Sigma \mathbf{J}_\xi \quad \text{is conserved.}$$

Example: free scalar field

Φ	ϕ
$\mathcal{L}$	$-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$
e.o.m	$\nabla_\mu\nabla^\mu\phi = 0$
Θ^μ	$\nabla^\mu\phi\,\delta\phi$
J^μ	$(\nabla^\mu\phi\nabla_\nu\phi - \mathcal{L}\delta^\mu_{\ \nu})\xi^\nu$



Noether current

$$\begin{cases} \delta_\xi \mathbf{L} = \mathrm{d}(\xi \cdot \mathbf{L}) \\ \delta_\xi \mathbf{L} = (\text{e.o.m})\delta_\xi \Phi + \mathrm{d}\boldsymbol{\Theta} \end{cases} \xrightarrow{\text{e.o.m}=0} \mathrm{d}\underbrace{(\boldsymbol{\Theta} - \xi \cdot \mathbf{L})}_{\mathbf{J}_\xi} = 0$$

on-shell

Conservation

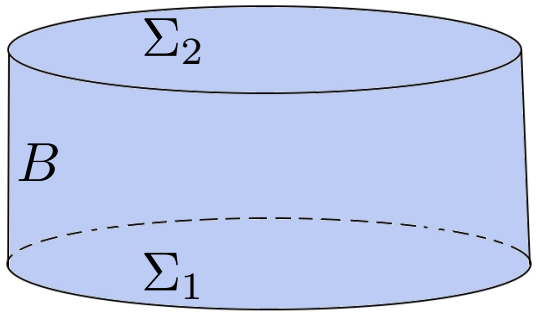
$$0 = \int_{\text{cylinder}} \mathrm{d}\mathbf{J}_\xi = \underbrace{\oint_B \mathbf{J}_\xi}_{0} + \int_{\Sigma_1} \mathbf{J}_\xi - \int_{\Sigma_2} \mathbf{J}_\xi \quad \Rightarrow \quad Q_\xi = \int_\Sigma \mathbf{J}_\xi \quad \text{is conserved.}$$

B.C

Example: free scalar field

Φ	ϕ
$\mathcal{L}$	$-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$
e.o.m	$\nabla_\mu\nabla^\mu\phi = 0$
Θ^μ	$\nabla^\mu\phi\,\delta\phi$
J^μ	$(\nabla^\mu\phi\nabla_\nu\phi - \mathcal{L}\delta^\mu_{\ \nu})\xi^\nu$

$T^\mu{}_\nu\,\xi^\nu$



Einstein-Λ gravity

Φ	$g_{\mu\nu}$
$\mathcal{L}$	$\frac{1}{16\pi G}(R-2\Lambda)$
e.o.m	$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=0$
Θ^μ	$\frac{1}{16\pi G}(\nabla_\nu \delta g^{\mu\nu}-\nabla^\mu \delta g^\nu{}_\nu)$
J^μ	$\frac{1}{16\pi G}\nabla_\nu(\nabla^\nu\xi^\mu-\nabla^\mu\xi^\nu) \xrightarrow{\text{Stokes'}} Q_\xi=\oint_{\partial\Sigma} \mathrm{d}\sigma_{\mu\nu}(\nabla^\nu\xi^\mu-\nabla^\mu\xi^\nu)$

Komar integral

Noether method in gravity produces wrong charges! 😢

- Kerr black hole: $\xi=\partial_t \quad \Rightarrow \quad Q_{\partial_t}=M/2 \quad$ (instead of M)

Covariant phase space formulation

Covariant phase space formulation

Canonical phase space $\qquad\qquad$ Covariant phase space

$$\Phi(\vec{x}), \Pi(\vec{x}) \qquad \leftrightarrow \qquad \Phi(x^\mu)\Big|_{\text{on-shell}}$$

Covariant phase space formulation

Canonical phase space $\qquad\qquad$ Covariant phase space

$$\Phi(\vec{x}), \Pi(\vec{x}) \qquad \leftrightarrow \qquad \Phi(x^\mu)\Big|_{\text{on-shell}}$$

Gravity

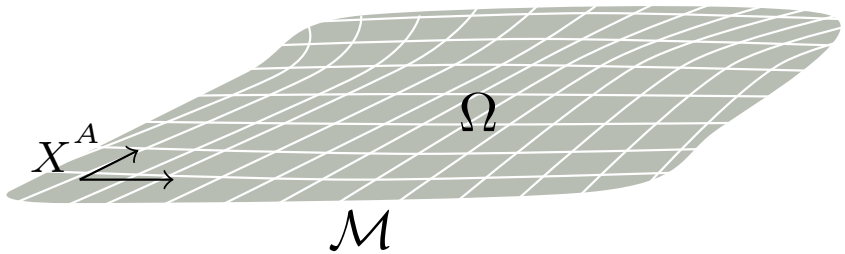
$$g_{\mu\nu}(\vec{x}), \pi^{\mu\nu}(\vec{x}) \qquad \leftrightarrow \qquad g_{\mu\nu}(x^\mu)\Big|_{\text{on-shell}}$$

Phase space is a manifold $\mathcal{M}$ with symplectic 2-form $\Omega = \Omega_{AB}\delta X^A \delta X^B$.

1. Anti-symmetric:
   $\Omega_{AB} = -\Omega_{BA}$
2. Closed:
   $$\delta\Omega = 0$$
3. Non-degenerate:
   $$\Omega_{AB}V^B = 0 \;\Leftrightarrow\; V^B = 0$$

- Ω is invertible:
  $$\Omega^{AB} = (\Omega^{-1})_{AB}$$

Phase Space



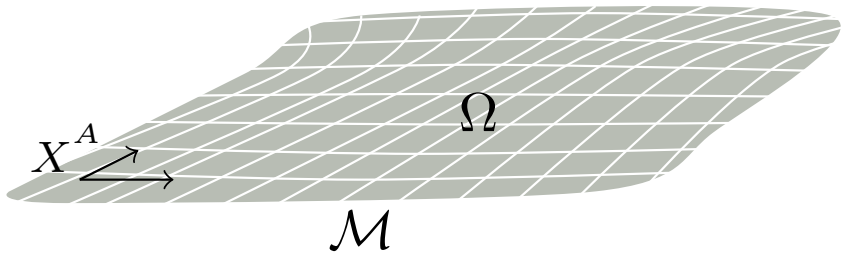
Manifold $\mathcal{M}$ with symplectic 2-form Ω

Phase space is a manifold $\mathcal{M}$ with symplectic 2-form $\Omega = \Omega_{AB}\delta X^A \delta X^B$.

1. Anti-symmetric:
   $\Omega_{AB} = -\Omega_{BA}$
2. Closed:
   $$\delta\Omega = 0$$
3. Non-degenerate:
   $$\Omega_{AB}V^B = 0 \;\Leftrightarrow\; V^B = 0$$

- Ω is invertible:
  $$\Omega^{AB} = (\Omega^{-1})_{AB}$$

Phase Space



Manifold $\mathcal{M}$ with symplectic 2-form Ω

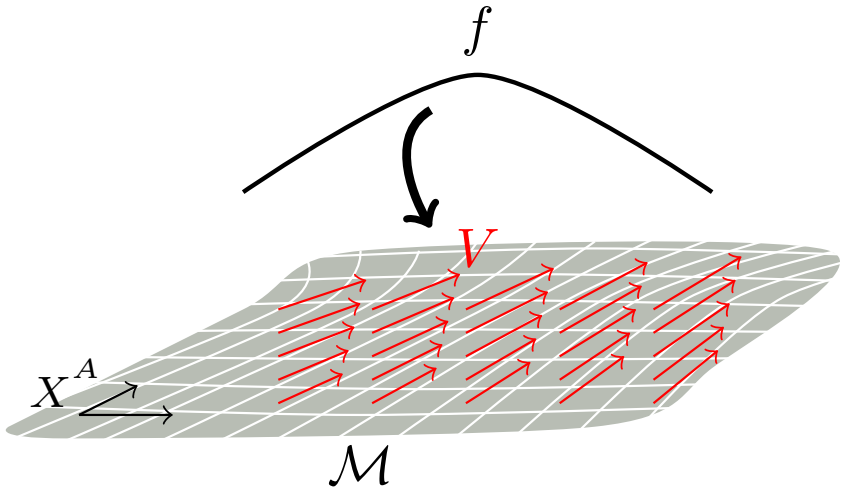
Space time → d Exterior derivative

phase space → δ

Differentiability

For a given function $f(X^A)$, one can find V such that:

$$V \cdot \Omega = \delta f$$



Differentiability

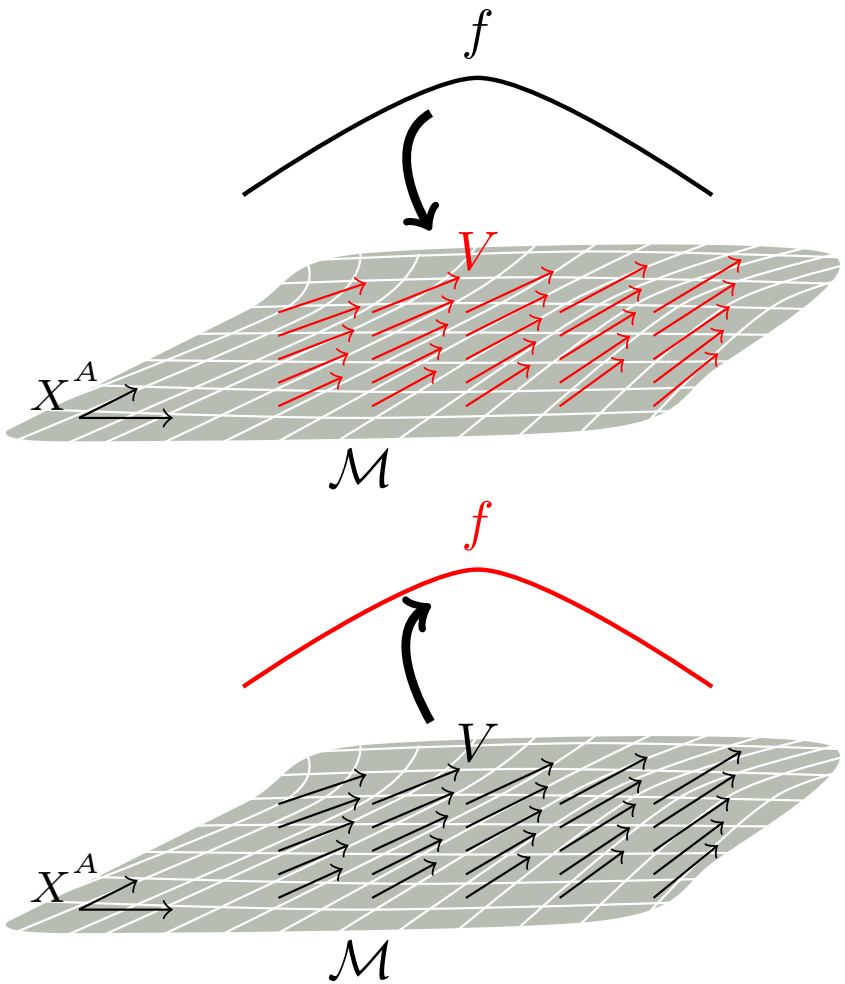
For a given function $f(X^A)$, one can find V such that:

$$V \cdot \Omega = \delta f$$

Integrability

Given a vector V, one can find a function f such that:

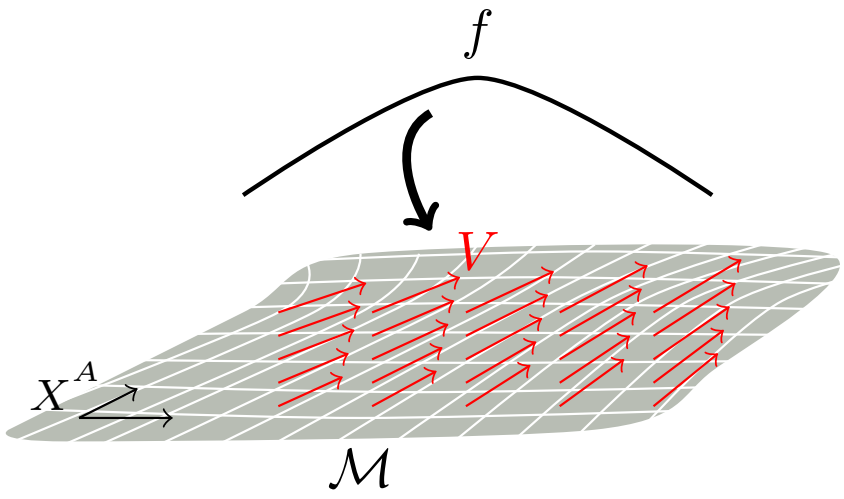
$$\delta f = V \cdot \Omega$$



Differentiability

For a given function $f(X^A)$, one can find V such that:

$$V \cdot \Omega = \delta f$$

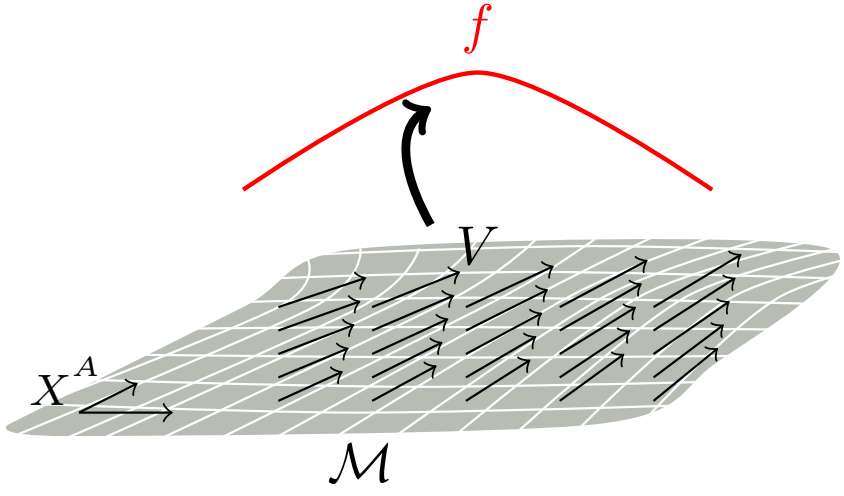


Integrability

Given a vector V, one can find a function f such that:

$$\delta f = V \cdot \Omega$$

charge ↙ ↪ Symmetry



Poisson bracket

$$\{f, g\} \equiv \Omega^{AB} (\delta f)_A (\delta g)_B$$

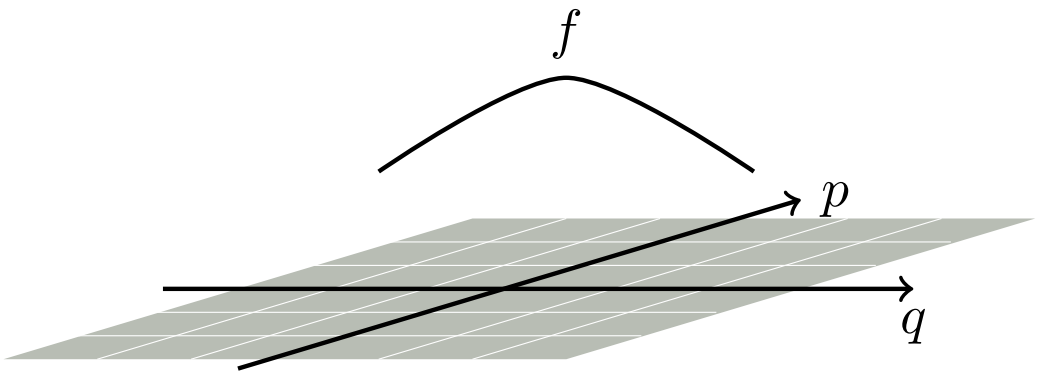
$\Omega \longleftrightarrow \{\cdot,\cdot\}$

Simplest example: one dimensional motion

Particle in one dimension

- $X^1 = q \qquad , \qquad X^2 = p$
- $\Omega_{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\Omega = \delta p \wedge \delta q$

$$\Rightarrow \begin{cases} \{q, p\} = 1 \\ \{q, q\} = \{p, p\} = 0 \end{cases}$$

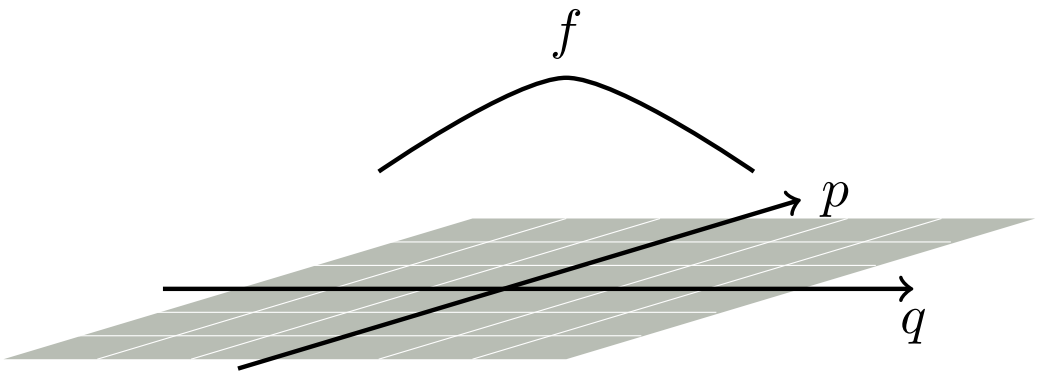


Simplest example: one dimensional motion

Particle in one dimension

- $X^1 = q \qquad , \qquad X^2 = p$
- $\Omega_{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\Omega = \delta p \wedge \delta q$

$$\Rightarrow \begin{cases} \{q, p\} = 1 \\ \{q, q\} = \{p, p\} = 0 \end{cases}$$



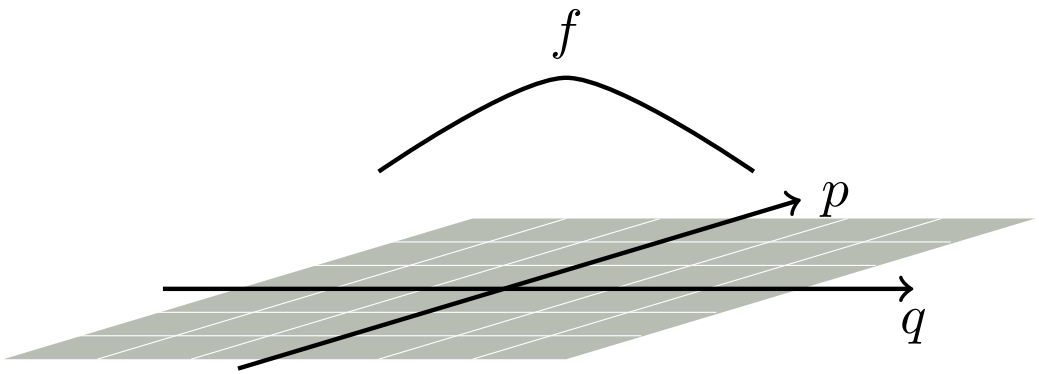
- $f = p \qquad \Rightarrow \qquad \delta p = (\partial_q) \cdot \Omega$
- $f = \mathcal{H} \qquad \Rightarrow \qquad \delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial q} \delta q + \frac{\partial \mathcal{H}}{\partial p} \delta p = (\dot{p} \partial p + \dot{q} \partial_q) \cdot \Omega = (\partial_t) \cdot \Omega$

Simplest example: one dimensional motion

Particle in one dimension

- $X^1 = q \quad , \quad X^2 = p$
- $\Omega_{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\Omega = \delta p \wedge \delta q$

$$\Rightarrow \begin{cases} \{q, p\} = 1 \\ \{q, q\} = \{p, p\} = 0 \end{cases}$$



How to read Ω from Lagrangian

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) \quad \Rightarrow \quad \delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial q} - \partial_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right) \delta q + \frac{\mathrm{d}}{\mathrm{d}t} (p\, \delta q)$$

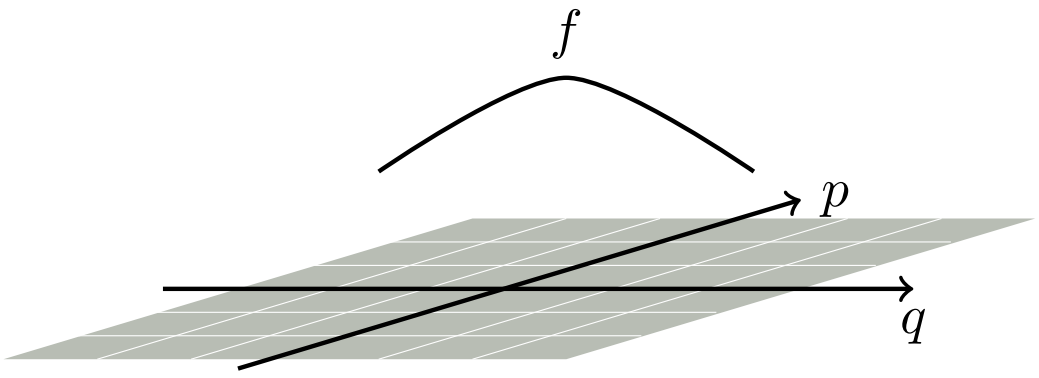
$$\Omega = \delta(p\, \delta q) = \delta p \wedge \delta q$$

Simplest example: one dimensional motion

Particle in one dimension

- $X^1 = q \qquad , \qquad X^2 = p$
- $\Omega_{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\Omega = \delta p \wedge \delta q$

$$\Rightarrow \begin{cases} \{q, p\} = 1 \\ \{q, q\} = \{p, p\} = 0 \end{cases}$$



How to read Ω from Lagrangian

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) \quad \Rightarrow \quad \delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial q} - \partial_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right) \delta q + \frac{\mathrm{d}}{\mathrm{d}t} (p\, \delta q)$$

$$\Omega = \delta(p\, \delta q) = \delta p \wedge \delta q$$

θ

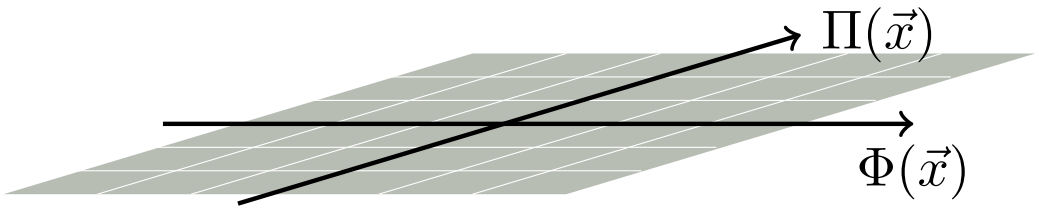
$\Omega = \delta\theta$

Canonical phase space for fields

Canonical phase space

- $\Phi(\vec{x}) \quad , \quad \Pi(\vec{x}) \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{\Phi}}$
- Poisson brackets

$$\{\Phi(\vec{x}_1), \Pi(\vec{x}_2)\} = \delta(\vec{x}_1 - \vec{x}_2)$$
$$\{\Phi(\vec{x}_1), \Phi(\vec{x}_2)\} = 0$$
$$\{\Pi(\vec{x}_1), \Pi(\vec{x}_2)\} = 0$$

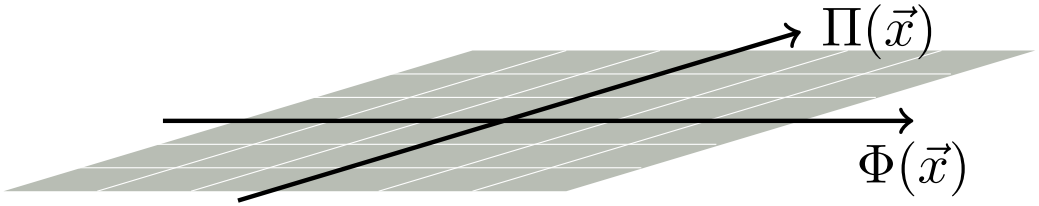


Canonical phase space for fields

Canonical phase space

- $\Phi(\vec{x}) \quad , \quad \Pi(\vec{x}) \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{\Phi}}$
- Poisson brackets

$$\{\Phi(\vec{x}_1), \Pi(\vec{x}_2)\} = \delta(\vec{x}_1 - \vec{x}_2)$$
$$\{\Phi(\vec{x}_1), \Phi(\vec{x}_2)\} = 0$$
$$\{\Pi(\vec{x}_1), \Pi(\vec{x}_2)\} = 0$$

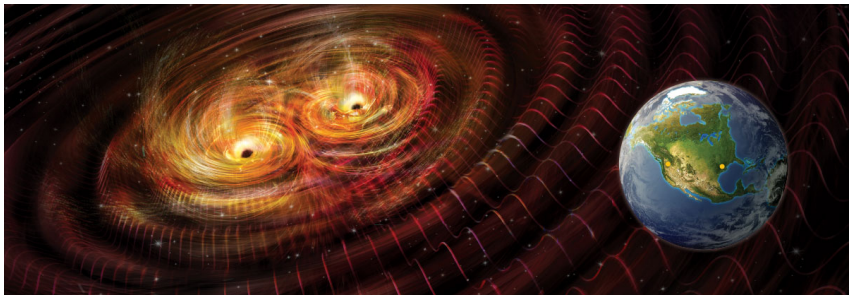


- In canonical method, it is necessary to introduce time foliations

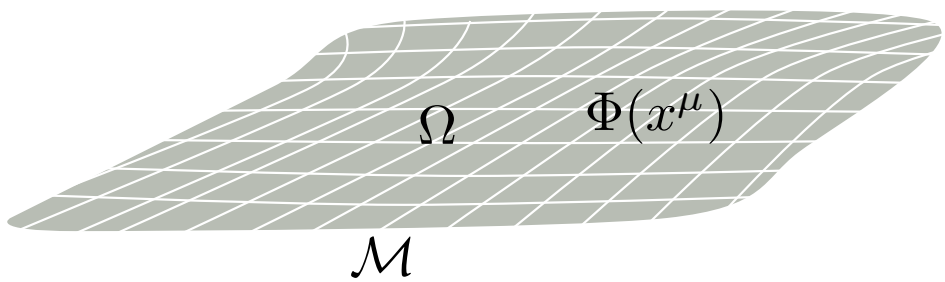
$$x^\mu \to (t, \vec{x})$$

Covariant phase space

- $\Phi(x^\mu)$:
  $g_{\alpha\beta}(x^\mu)$, $A_\nu(x^\mu)$, $\phi(x^\mu)$...
- $\delta \mathbf{L} = (\text{e.o.m})\delta\Phi + \mathrm{d}\boldsymbol{\Theta}$
- $\boldsymbol{\omega} = \delta\boldsymbol{\Theta}$
- $\Omega \equiv \int_\Sigma \boldsymbol{\omega}$

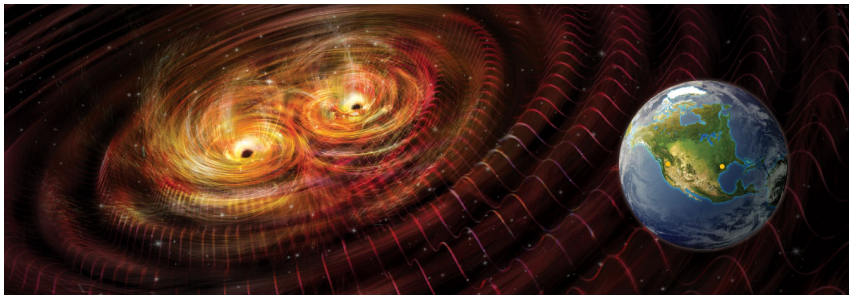


covariant phase space

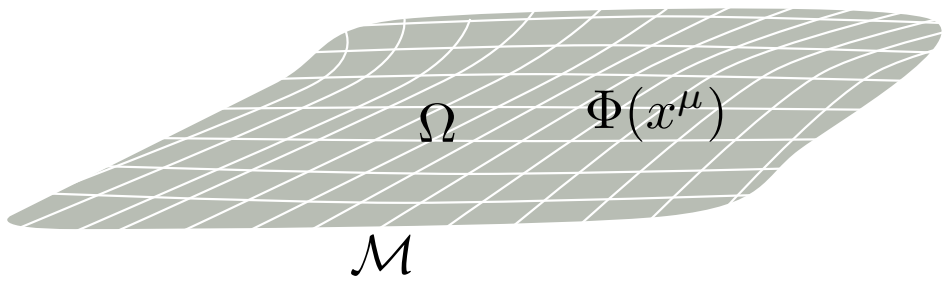


Covariant phase space

- $\Phi(x^\mu)$:
  $g_{\alpha\beta}(x^\mu)$, $A_\nu(x^\mu)$, $\phi(x^\mu)$...
- $\delta \mathbf{L} = (\text{e.o.m})\delta\Phi + \mathrm{d}\boldsymbol{\Theta}$
- $\boldsymbol{\omega} = \delta\boldsymbol{\Theta}$
- $\Omega \equiv \int_\Sigma \boldsymbol{\omega}$



covariant phase space



- Symplectic potential: $\Theta(\delta\Phi, \Phi)$
- Symplectic current: $\boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi) = \delta_1\boldsymbol{\Theta}(\delta_2\Phi, \Phi) - \delta_2\boldsymbol{\Theta}(\delta_1\Phi, \Phi)$
- Conservation: $\mathrm{d}\boldsymbol{\omega} = \mathrm{d}\delta\boldsymbol{\Theta} = \delta\mathrm{d}\boldsymbol{\Theta} = 0 \qquad \oint_B \boldsymbol{\omega} = 0 \qquad \Rightarrow \qquad \Omega = \int \boldsymbol{\omega}.$

Covariant conserved charges

Covariant conserved charge H_ξ Q_ξ Noether conserved charge

Covariant conserved charges

Covariant conserved charge H_ξ $\quad$ Q_ξ Noether conserved charge

Definition of charge variation:	$\delta H_\xi = V_\xi \cdot \Omega$
$V_\xi = \mathcal{L}_\xi \Phi \rightarrow$	$= (\mathcal{L}_\xi \Phi) \cdot \Omega$
$\Omega = \int_\Sigma \boldsymbol{\omega} \rightarrow$	$= (\mathcal{L}_\xi \Phi) \cdot \int_\Sigma \boldsymbol{\omega}$
$\boldsymbol{\omega} = \delta_1 \boldsymbol{\Theta}(\delta_2 \Phi, \Phi) - \delta_2 \boldsymbol{\Theta}(\delta_1 \Phi, \Phi) \rightarrow$	$= \int_\Sigma \delta \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \mathcal{L}_\xi \boldsymbol{\Theta}(\delta \Phi, \Phi)$
$\mathcal{L}_\xi \boldsymbol{\Theta} = \xi \cdot \mathrm{d}\boldsymbol{\Theta} + \mathrm{d}(\xi \cdot \boldsymbol{\Theta}) \rightarrow$	$= \int_\Sigma \delta \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \mathrm{d}\boldsymbol{\Theta} - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\mathrm{d}\boldsymbol{\Theta} = \delta \mathbf{L}$ $\quad$ on-shell $\rightarrow$	$= \int_\Sigma \delta \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \delta \mathbf{L} - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\delta \xi = 0 \rightarrow$	$= \int_\Sigma \delta(\boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \mathbf{L}) - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\mathbf{J}_\xi = \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \mathbf{L} \rightarrow$	$= \int_\Sigma \delta \mathbf{J}_\xi - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\mathbf{J}_\xi = \mathrm{d}\mathbf{Q}_\xi \rightarrow$	$= \int_\Sigma \mathrm{d}(\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\Theta})$
Stokes' theorem $\rightarrow$	$= \int_{\partial\Sigma} \delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\Theta}$
$\int_{\partial\Sigma} \delta \mathbf{Q}_\xi = \delta Q_\xi \rightarrow$	$= \delta Q_\xi - \int_{\partial\Sigma} \xi \cdot \boldsymbol{\Theta}$

Covariant conserved charges

Covariant conserved charge H_ξ $\qquad$ Q_ξ Noether conserved charge

Definition of charge variation:	$\delta H_\xi = V_\xi \cdot \Omega$
$V_\xi = \mathcal{L}_\xi \Phi \rightarrow$	$= (\mathcal{L}_\xi \Phi) \cdot \Omega$
$\Omega = \int_\Sigma \boldsymbol{\omega} \rightarrow$	$= (\mathcal{L}_\xi \Phi) \cdot \int_\Sigma \boldsymbol{\omega}$
$\boldsymbol{\omega} = \delta_1 \boldsymbol{\Theta}(\delta_2 \Phi, \Phi) - \delta_2 \boldsymbol{\Theta}(\delta_1 \Phi, \Phi) \rightarrow$	$= \int_\Sigma \delta \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \mathcal{L}_\xi \boldsymbol{\Theta}(\delta \Phi, \Phi)$
$\mathcal{L}_\xi \boldsymbol{\Theta} = \xi \cdot \mathrm{d}\boldsymbol{\Theta} + \mathrm{d}(\xi \cdot \boldsymbol{\Theta}) \rightarrow$	$= \int_\Sigma \delta \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \mathrm{d}\boldsymbol{\Theta} - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\mathrm{d}\boldsymbol{\Theta} = \delta \mathbf{L}$ $\quad$ on-shell $\rightarrow$	$= \int_\Sigma \delta \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \delta \mathbf{L} - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\delta \xi = 0 \rightarrow$	$= \int_\Sigma \delta(\boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \mathbf{L}) - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\mathbf{J}_\xi = \boldsymbol{\Theta}(\mathcal{L}_\xi \Phi, \Phi) - \xi \cdot \mathbf{L} \rightarrow$	$= \int_\Sigma \delta \mathbf{J}_\xi - \mathrm{d}(\xi \cdot \boldsymbol{\Theta})$
$\mathbf{J}_\xi = \mathrm{d}\mathbf{Q}_\xi \rightarrow$	$= \int_\Sigma \mathrm{d}(\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\Theta})$
Stokes' theorem $\rightarrow$	$= \int_{\partial\Sigma} \delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\Theta}$
$\int_{\partial\Sigma} \delta \mathbf{Q}_\xi = \delta Q_\xi \rightarrow$	$= \delta Q_\xi - \int_{\partial\Sigma} \xi \cdot \boldsymbol{\Theta}$

$\rightarrow K_\xi$

Covariant conserved charges

Covariant conserved charge $$\delta H_\xi = \int_{\partial\Sigma} \boldsymbol{k}_\xi(\delta\Phi, \Phi)$$

Notice that:

- $\mathcal{L} = \alpha\mathcal{L}_1 + \beta\mathcal{L}_2 \quad \Rightarrow \quad \boldsymbol{k} = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$
- $\boldsymbol{k}$ has 3 inputs: $\{\xi^\mu, \Phi, \delta\Phi\}$.
- Φ and $\delta\Phi$ should satisfy e.o.m and linearized e.o.m respectively.
- δH_ξ is a boundary charge.
- Not all ξ^μ have integrable charge.
- Not all ξ^μ have conserved charge.

Covariant conserved charges

Covariant conserved charge $$\delta H_\xi = \int_{\partial\Sigma} \boldsymbol{k}_\xi(\delta\Phi, \Phi)$$

Notice that:

- $\mathcal{L} = \alpha\mathcal{L}_1 + \beta\mathcal{L}_2 \quad \Rightarrow \quad \boldsymbol{k} = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$
- $\boldsymbol{k}$ has 3 inputs: $\{\xi^\mu, \Phi, \delta\Phi\}$.
- Φ and $\delta\Phi$ should satisfy e.o.m and linearized e.o.m respectively.
- δH_ξ is a boundary charge.
- Not all ξ^μ have integrable charge.
- Not all ξ^μ have conserved charge.

Good news

H_ξ is not sensitive to $\mathbf{L} \to \mathbf{L} + \mathrm{d}\boldsymbol{\mu}$

$$\mathbf{L} \to \mathbf{L} + \mathrm{d}\boldsymbol{\mu} \quad \Rightarrow \quad \boldsymbol{\Theta} \to \boldsymbol{\Theta} + \delta\boldsymbol{\mu} \quad \Rightarrow \quad \boldsymbol{\omega} \to \boldsymbol{\omega} + \cancelto{0}{\delta^2\boldsymbol{\mu}}$$

Covariant conserved charges

Covariant conserved charge $$\delta H_\xi = \int_{\partial\Sigma} \boldsymbol{k}_\xi(\delta\Phi, \Phi)$$

Notice that:

- $\mathcal{L} = \alpha\mathcal{L}_1 + \beta\mathcal{L}_2 \quad \Rightarrow \quad \boldsymbol{k} = \alpha\boldsymbol{k}_1 + \beta\boldsymbol{k}_2$
- $\boldsymbol{k}$ has 3 inputs: $\{\xi^\mu, \Phi, \delta\Phi\}$.
- Φ and $\delta\Phi$ should satisfy e.o.m and linearized e.o.m respectively.
- δH_ξ is a boundary charge.
- Not all ξ^μ have integrable charge.
- Not all ξ^μ have conserved charge.

Example: Einstein-Λ theory $\{\xi^\mu, g_{\mu\nu}, h_{\mu\nu}\}$

$$k^{\mu\nu}_\xi = \frac{1}{16\pi G}\Big(\Big[\xi^\nu\nabla^\mu h - \xi^\nu\nabla_\tau h^{\mu\tau} + \xi_\tau\nabla^\nu h^{\mu\tau} + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\tau\nu}\nabla_\tau\xi^\mu\Big] - [\mu \leftrightarrow \nu]\Big)$$

Covariant conserved charges

Covariant conserved charge $\qquad \delta H_\xi = \int_{\partial\Sigma} \boldsymbol{k}_\xi(\delta\Phi, \Phi)$

Let's be specific!

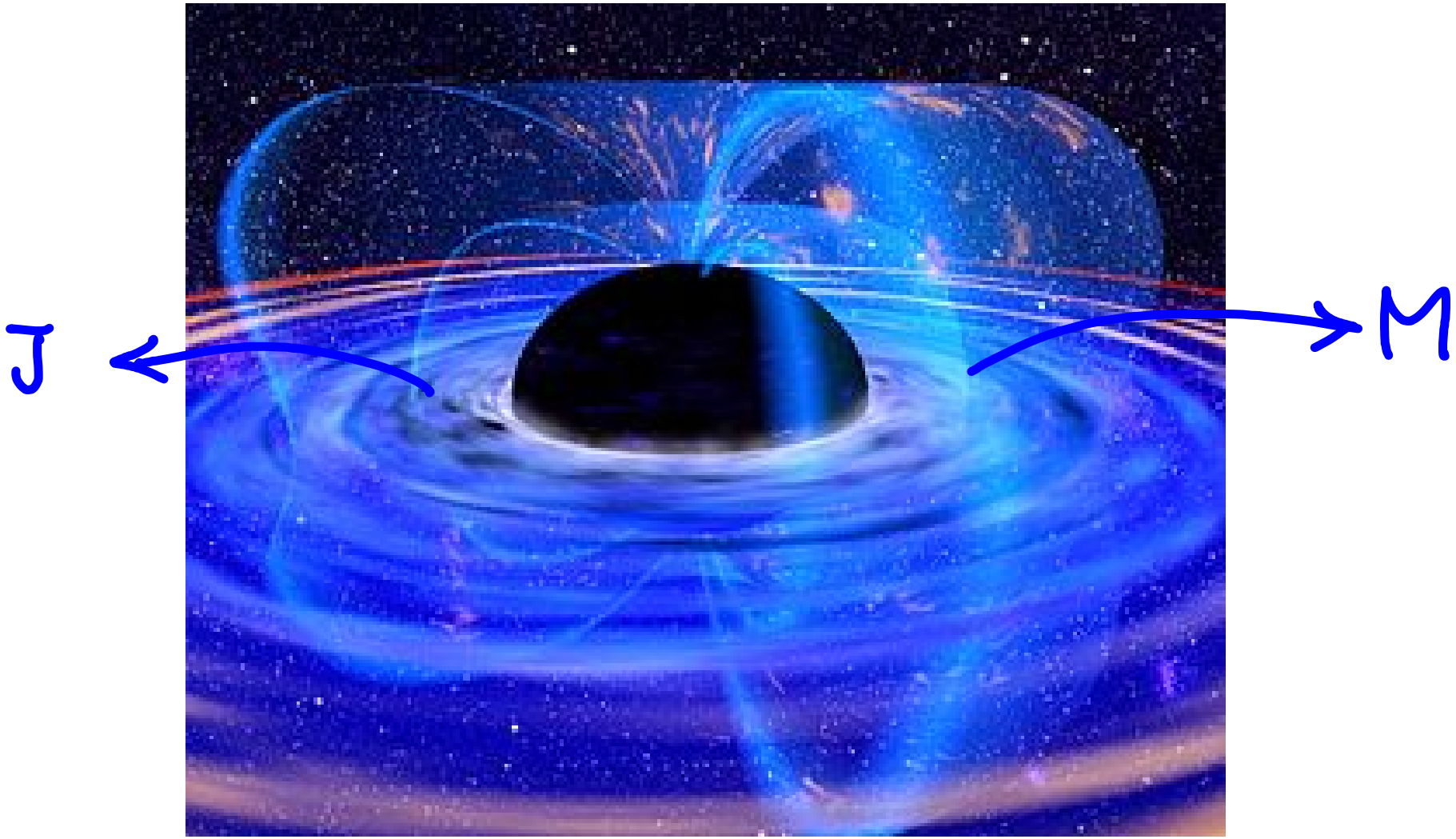
- Exact symmetry $\mathcal{L}_\xi \Phi = 0$ $\qquad$ conservation

$$\boldsymbol{\omega}(\delta\Phi, \mathcal{L}_\xi\Phi, \Phi) = 0 \quad \Rightarrow \quad \oint_B \boldsymbol{\omega} = 0$$

- Parametric solutions: $\Phi(x^\mu; p_i)$ $\qquad$ integration
- Parametric variations: $\delta\Phi = \frac{\partial\Phi}{\partial p_i}\delta p_i$ $\qquad \Rightarrow \delta H_\xi \to H_\xi$

$$H_\xi[\Phi] = \int_{\bar{p}}^{p} \delta H_\xi + H_\xi[\bar{\Phi}]\,,$$

Black holes conserved charges?



$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda)$

Kerr-AdS black hole

$$ds^2 = -\Delta_\theta\Big(\frac{1-\frac{\Lambda r^2}{3}}{\Xi} - \Delta_\theta f\Big)dt^2 - 2\Delta_\theta f a \sin^2\theta\, dt d\varphi + \Big(\frac{r^2+a^2}{\Xi} + f a^2 \sin^2\theta\Big)\sin^2\theta\, d\varphi^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta\,, \quad \Delta_r \equiv (r^2+a^2)(1-\frac{\Lambda r^2}{3}) - 2Gmr,$$

$$\Delta_\theta \equiv 1 + \frac{\Lambda a^2}{3}\cos^2\theta\,, \qquad \Xi \equiv 1 + \frac{\Lambda a^2}{3}\,, \quad f \equiv \frac{2Gmr}{\rho^2\Xi^2} \qquad p_i = \{m, a\}$$

$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda)$ Kerr-AdS black hole

$$ds^2 = -\Delta_\theta(\frac{1-\frac{\Lambda r^2}{3}}{\Xi} - \Delta_\theta f)dt^2 - 2\Delta_\theta f a \sin^2\theta\, dt d\varphi + \left(\frac{r^2+a^2}{\Xi} + f a^2 \sin^2\theta\right)\sin^2\theta\, d\varphi^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta\,, \quad \Delta_r \equiv (r^2+a^2)(1-\frac{\Lambda r^2}{3}) - 2Gmr,$$

$$\Delta_\theta \equiv 1 + \frac{\Lambda a^2}{3}\cos^2\theta\,, \qquad \Xi \equiv 1 + \frac{\Lambda a^2}{3}\,, \quad f \equiv \frac{2Gmr}{\rho^2\Xi^2} \qquad p_i = \{m, a\}$$

$$k^{\mu\nu}_\xi = \frac{1}{16\pi G}\left(\left[\xi^\nu\nabla^\mu h - \xi^\nu\nabla_\tau h^{\mu\tau} + \xi_\tau\nabla^\nu h^{\mu\tau} + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\tau\nu}\nabla_\tau\xi^\mu\right] - [\mu \leftrightarrow \nu]\right)$$

3 inputs:
$$\begin{cases} \Phi(x^\mu; p_i) = g_{\mu\nu}(x^\mu; m, a) \\ \delta\Phi = h_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial m}\delta m + \frac{\partial g_{\mu\nu}}{\partial a}\delta a \\ \text{Mass:} \quad \xi = \partial_t, \qquad\qquad \text{Angular mometnum:} \quad \xi = \partial_\phi \end{cases}$$

Example: Kerr-AdS black hole

Results:

Mass

$$\delta M = \delta H_{\partial_t} = \int_{\partial\Sigma} \boldsymbol{k}_{\partial_t}(h_{\mu\nu}, g_{\mu\nu}) \xrightarrow{\text{computation}} = \frac{\partial\left(\frac{m}{\Xi^2}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{m}{\Xi^2}\right)}{\partial a}\delta a = \delta\left(\frac{m}{\Xi^2}\right)$$

$$\Rightarrow \quad M = \frac{m}{\Xi^2}$$

Angular momentum

$$\delta J = \delta H_{\partial_\varphi} = \int_{\partial\Sigma} \boldsymbol{k}_{\partial_\varphi}(h_{\mu\nu}, g_{\mu\nu}) \xrightarrow{\text{computation}} = \frac{\partial\left(\frac{ma}{\Xi^2}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{ma}{\Xi^2}\right)}{\partial a}\delta a = \delta\left(\frac{ma}{\Xi^2}\right)$$

$$\Rightarrow \quad J = \frac{ma}{\Xi^2}$$

Example: Kerr-AdS black hole

Results:

Mass

$$\delta M = \delta H_{\partial_t} = \int_{\partial\Sigma} \boldsymbol{k}_{\partial_t}(h_{\mu\nu}, g_{\mu\nu}) \xrightarrow{\text{computation}} = \frac{\partial\left(\frac{m}{\Xi^2}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{m}{\Xi^2}\right)}{\partial a}\delta a = \delta\left(\frac{m}{\Xi^2}\right)$$

$$\Rightarrow \quad M = \frac{m}{\Xi^2}$$

Angular momentum

$$\delta J = \delta H_{\partial_\varphi} = \int_{\partial\Sigma} \boldsymbol{k}_{\partial_\varphi}(h_{\mu\nu}, g_{\mu\nu}) \xrightarrow{\text{computation}} = \frac{\partial\left(\frac{ma}{\Xi^2}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{ma}{\Xi^2}\right)}{\partial a}\delta a = \delta\left(\frac{ma}{\Xi^2}\right)$$

$$\Rightarrow \quad J = \frac{ma}{\Xi^2}$$

Soft hairs and soft charges

No-hair Theorem: black holes are identified by M, J, ...

- Soft hairs: other vectors ξ^μ which have non-zero and finite charge.
- Soft charges: H_ξ

Soft hairs and soft charges

No-hair Theorem: black holes are identified by M, J, ...

- Soft hairs: other vectors ξ^μ which have non-zero and finite charge.
- Soft charges: H_ξ

$$\delta H_\xi = \int_{\partial\Sigma} \boldsymbol{k}_\xi(\delta\Phi, \Phi)$$

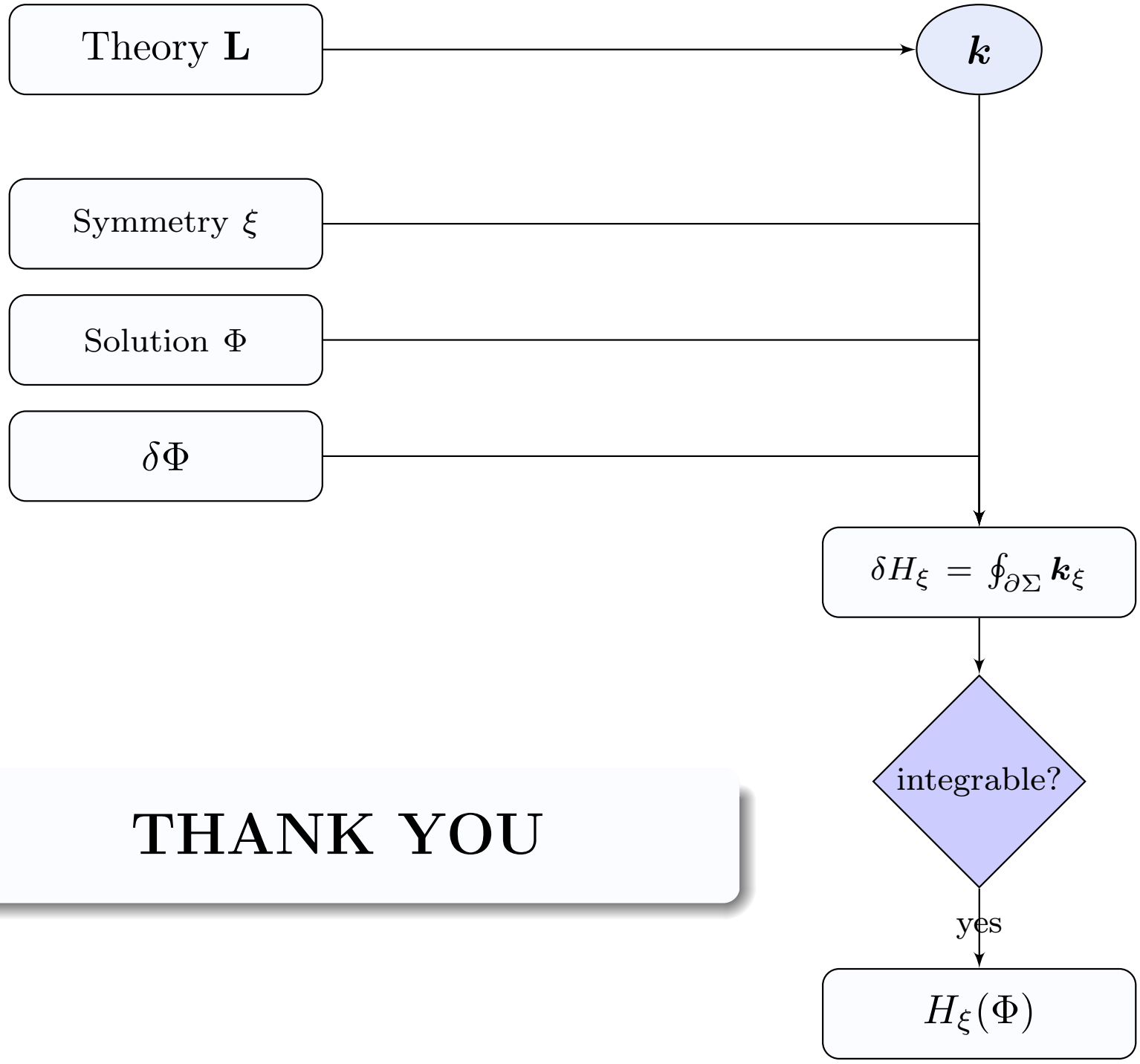
Soft charge algebras: $$\{H_{\xi_1}, H_{\xi_2}\} = \int_{\partial\Sigma} \boldsymbol{k}_{\xi_2}(\mathcal{L}_{\xi_1}\Phi, \Phi)$$

Conclusion

- Covariant conserved charge

$$\delta H_\xi = \oint_{\partial\Sigma} \boldsymbol{k}_\xi, \qquad\qquad \boldsymbol{k}_\xi = \delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\Theta}.$$

- Formulation applicable for any covariant theory.
- Formulation independent of dimensions.
- Formulation independent of asymptotics.
- Formulation independent of $\mathbf{L} \to \mathbf{L} + \mathrm{d}\boldsymbol{\mu}$.


Theory $\mathbf{L}$
$\boldsymbol{k}$
Symmetry ξ
Solution Φ
$\delta\Phi$
$\delta H_\xi = \oint_{\partial\Sigma} \boldsymbol{k}_\xi$
integrable?
yes
$H_\xi(\Phi)$

THANK YOU