Anderson localization

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Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

MARCH 1, 1958
Short History of Anderson Localization

1958 → Anderson localization of electron wavefunctions in disordered solids

1977 : Nobel Prize in Physics goes to Philip Anderson

1985 → Growing interest in Anderson localization of classical waves (light, sound, …)

1995 → Claims of experimental observations of Anderson localization of light, microwaves, ultrasound, …

2000 → Search for Anderson localization of Bose-Einstein condensates in disordered potentials
Localized and extended states in 1D chain

![Graphs showing extended and localized states](graph.png)
A non-interacting electron moving in random potential

Quantum interference of scattering waves

Anderson localization of electrons

extended

localized

$E_{\text{loc}}$

$E$

critical

extended
Scaling theory (1979)

Conductance $g$ changes when system size $L$ is

Metal: $g \propto \frac{\text{area}}{\text{length}} = L^{d-2}$

Insulator: $g \propto e^{-L/\xi}$

$$\beta(g) = \frac{d \ln g}{d \ln L} = d - 2 - \mathcal{O}(g^{-1})$$

$g \gg 1$

All wave functions are localized below two dimensions!

A metal-insulator transition at $g = g_c$ is continuous ($d > 2$).

But the localization phenomena is Multifractal!
Multifractality: scaling behavior of moments of (critical) wave functions

Critical wave function at a metal-insulator transition point

Multifractal exponents $\tau_q$

$$L^d |\psi(r)|^{2q} \sim L^{-\tau_q}$$

$$\tau_q = d(q - 1) + \Delta_q$$

$D_q = \tau_q / (q - 1)$

fractal dimension

Continuous set of independent and universal critical exponents

$\Delta_q$: anomalous scaling dimensions

Singularity spectrum

$$f(\alpha) = q\alpha - \tau_q$$

$$\alpha = \frac{d\tau_q}{dq}$$

$\alpha > 0$
Anderson Localization (1957)

**What?** Quantum diffusion in a random potential stops due to interference effects.

Goal of localization theory

\[ D_{\text{quant}} = f(d, \text{dis})? \]

\[ a = ? \]

Delocalization

\[ |\Psi_\alpha(r)| \propto 1/\sqrt{V} \]

Localization

\[ |\Psi_\alpha(r)| \propto e^{-r/\xi_{\text{loc}}} \]
Self-consistent theory of Anderson localization
Self-consistent theory of Anderson localization

The presence of loops increases return probability as compared to ‘normal’ diffusion

Diffusion slows down

Diffusion constant should be renormalized \( D_B \rightarrow D < D_B \)
Generalization to open media

Loops are less probable near the boundaries

Slowing down of diffusion is spatially heterogeneous

Diffusion constant becomes position-dependent

\[ D_B \to D(r) < D_B \]
Classical localization

\[
\frac{\psi_N}{\psi_1} = \exp(-\frac{L}{\zeta})
\]

\[
\zeta(E) = \lim_{N \to \infty} \frac{1}{N} \ln \left| \frac{\psi_N}{\psi_1} \right|
\]
Multiple Scattering of Waves

Incident wave

Random medium

Detector
Multiple Scattering of Waves

Incident wave

Detector

Random medium

\[ L \]

\[ \lambda \]
From Single Scattering to Anderson Localization

Wavelength $\lambda$, Mean free paths $l, l^*$, Localization length $\xi$

- Ballistic propagation (no scattering)
- Single scattering
- Multiple scattering (diffusion)
- Strong (Anderson) localization

Size $L$ of the medium

0

‘Strength’ of disorder
Experimental Evidence

Exponential scaling of average transmission with $L$

Diffusive regime:
$$\langle T \rangle \propto \frac{\ell}{L}$$

Localized regime:
$$\langle T \rangle \propto \exp \left( -\frac{L}{\xi} \right)$$

$T$
What if We Look at the Dynamics?

$I_0(t)$

$\langle T(t) \rangle = ?$
Outline

1) Non-perturbative method in 1D
   a) random mass oscillators
   b) Disordered Schrödinger equation
   c) Magnon modes of spin glass

2) Perturbative method
   a) Mapping the quenched random differential equation to a Martin-Siggia-Rose action (elastic wave)

3) Numerical method
   a) Red-Shift of spectral density (light localization)

4) Acoustic wave in random Dimer media
Part – I

**a) Random Mass (1D)**

\[
\frac{\partial^2}{\partial x^2} \psi + m(x) \omega^2 \psi = 0.
\]

Random density \( m(x) = \bar{m} + c(x) \) (\( \bar{m} \) is mean density)

\( c(x) \) is Gaussian uncorrelated random function

\(< c(x) >= 0, \)

\(< c(x)c(x') >= \sigma^2 \delta(x - x'). \)
Random Mass

The log-derivative of $\psi$, \( f(x) = \frac{\psi'(x)}{\psi(x)} \)

\[
f' + f^2 + \omega^2 m(x) = 0. \quad f' = \frac{df}{dx}.
\]

\[
D^{(1)} = -f^2 - \omega^2 \bar{m}
\]

\[
D^{(2)} = \omega^2 \sigma.
\]
The probability distribution function (PDF) of $f(x)$, i.e. $P(\xi, x)$,

Fokker-Planck equation

$$\frac{\partial}{\partial x} P(\xi, x) = \frac{\partial}{\partial \xi} \left( \xi^2 + \omega^2 \bar{m} + \frac{\sigma^2 \omega^4}{2} \frac{\partial}{\partial \xi} \right) P(\xi, x).$$

$P(\xi, x)d\xi$ is the probability of finding $f(x)$ between $\xi$ and $\xi + d\xi$

$$P(\xi, x) = \langle \delta(f(x) - \xi) \rangle$$
Random Mass

The homogeneous solution (or at $x \to \infty$) of Fokker-Planck equation

$$
(\xi^2 + \omega^2 \bar{m} + \frac{\sigma^2 \omega^4}{2} \frac{\partial}{\partial \xi}) p(\xi) = p_0.
$$

$$
p(\xi) = \lim_{x \to \infty} P(\xi, x).
$$

$$
p(\xi) = \frac{2p_0}{\sigma^2 \omega^4} \exp \left\{ -\frac{2}{\sigma^2 \omega^4} \frac{\xi^3}{3} + \omega^2 \bar{m} \xi \right\}
$$

$$
\times \int_{-\infty}^{\xi} \, d\eta \, \exp \left\{ \frac{2}{\sigma^2 \omega^4} \frac{\eta^3}{3} + \omega^2 \bar{m} \eta \right\}.
$$
Random Mass (generalized PDF)

\[ \mathcal{P}(\xi, x), \text{ which is defined as } \mathcal{P}(\xi, x) = \langle \delta(f(x) - \xi) \left| \frac{\partial f}{\partial \omega^2} \right| \rangle \]

Dos:

\[ \rho(\omega^2) = \lim_{L \to \infty} \frac{1}{L} \mathcal{P}(\xi_L, L) . \]

sign of \( \partial f / \partial \omega^2 \) \( \frac{\partial}{\partial x} \frac{\partial f}{\partial \omega^2} + 2f \frac{\partial f}{\partial \omega^2} + m(x) = 0. \)

\[ \frac{\partial f}{\partial \omega^2} = -\int_0^x dy \frac{\psi^2(y)}{\psi^2(x)} m(y). \]

has negative sign if \( \sigma / \tilde{m} \ll 1 \)

positivity condition of mass
\[ \mathcal{P}(\xi, x) = - \left< \frac{\partial f}{\partial \omega^2} \delta(f(x) - \xi) \right>. \]

\[ \frac{\partial \mathcal{P}(\xi, x)}{\partial x} = \bar{m}P(\xi, x) + \omega^2 \sigma^2 \frac{\partial P(\xi, x)}{\partial \xi} - \]

\[ \left[ (\xi^2 + \omega^2 \bar{m}) \frac{\partial}{\partial \xi} + \frac{\omega^4 \sigma^2}{2} \frac{\partial^2}{\partial \xi^2} \right] \mathcal{P}(\xi, x). \]

\[ \rho(\omega^2) = \bar{m} \int p(-\xi) p(\xi) d\xi + \omega^2 \sigma^2 \int p(-\xi) \frac{\partial p}{\partial \xi} d\xi. \]
Random Mass

\[ p(\xi) = \frac{2p_0}{\sigma^2\omega^4} \exp\left\{ -\frac{2}{\sigma^2\omega^4}\left(\frac{\xi^3}{3} + \omega^2\bar{m}\xi\right) \right\} \]
\[ \times \int_{-\infty}^{\xi} d\eta \exp\left\{ \frac{2}{\sigma^2\omega^4}\left(\frac{\eta^3}{3} + \omega^2\bar{m}\eta\right) \right\} . \]

\[ \frac{\partial p_0}{\partial \omega^2} = \rho(\omega^2), \quad p_0 = \mathcal{N}(\omega^2) \]

\[ \frac{1}{\mathcal{N}(\omega^2)} = \frac{\sqrt{2\pi}}{(\omega^2\sigma)^{\frac{3}{2}}} \int_{0}^{\infty} \frac{dx}{\sqrt{x}} \exp\left[ -\frac{1}{6} x^3 - \frac{2\bar{m}}{\omega^2 \sigma^{\frac{4}{3}}} x \right] . \]

In the limit \( \sigma \to 0 \), \( \mathcal{N}(\omega^2) = 1/\pi\sqrt{\bar{m}\omega^2} \)
Lyapunov Exponents

\[ f = \frac{\partial \ln(\psi)}{\partial x}, \]

\[ \gamma = \lim_{x \to \infty} \langle \frac{\partial}{\partial x} \ln(\psi) \rangle = \langle f \rangle_\infty. \]

\[ \gamma = \int \xi p(\xi) d\xi. \]

\[ \gamma = \frac{\omega^2}{2} \left( \frac{I_+}{I_-} \right), \]

\[ I_\pm = \int_0^\infty du u^{(\pm)\frac{1}{2}} \exp \left\{ -\frac{\omega^2}{6\sigma^2} u^3 - \frac{2\bar{m}}{\sigma^2} u \right\}. \]
Lyapunov Exponents

in limit of $\omega \to 0$,

$$\gamma = \frac{\sigma^2 \omega^2}{8 \bar{m}},$$

$\omega \to \infty$

$$\gamma = \frac{6^{\frac{1}{3}} \sqrt{\pi}}{2 \times \Gamma(\frac{1}{6})} \sigma^{2/3} \omega^{4/3}.$$  

localization length scales as $l \sim \omega^{-4/3}$
b) Disordered Schrodinger Equation (1D)

\[ H = -\frac{1}{2} \Delta + V(x) \]

potential \( V(x) \) is a zero mean, Gaussian random white noise

\[ \langle V(x)V(x') \rangle = \delta(x - x'). \]

\[ \gamma = \frac{1}{2E} \left( \frac{I_+}{I_-} \right), \quad I_{\pm} = \int_0^\infty \! du u^{(\pm)\frac{1}{2}} \exp \left\{ - \frac{1}{24E^3} u^3 - u \right\}. \]

In the limit \( (E \to \infty) \) \[ \gamma \sim 1/4E \]
c) HARMONIC MAGNON MODES
HEISENBERG--MATTIS SPIN GLASSES

\[ H = \sum_{i,j} J_{ij} S_i \cdot S_j. \]

\[ P(J_{ij}) = \frac{1}{2} \delta(J_{ij} - J) + \frac{1}{2} \delta(J_{ij} + J) \]

coupling \( J \) is a constant \( S^\pm = S_x \pm iS_y \)

\[ H = \sum_{i,j} J_{ij} \left[ \frac{1}{2} (S^+_i S^-_j + S^-_i S^+_j) + S^z_i S^z_j \right]. \]
HEISENBERG--MATTIS SPIN GLASSES

\[ i\hbar \frac{\partial S^+_i}{\partial t} = \sum_{j \neq i} J_{ij} \left\{ \frac{1}{2} [S^+_i, S^+_j S^-_j + S^-_j S^+_j] + [S^+_i, S^z_i S^z_j] \right\}. \]

\[ [S^+_i, S^-_j] = 2\delta_{ij} S^z_i \text{ and } [S^z_i, S^\pm_j] = \pm\delta_{ij} S^\pm_i \]

\[ i\hbar \frac{\partial S^+_i}{\partial t} = \sum_{j \neq i} J_{ij} (S^z_i S^+_j - S^z_j S^+_i). \]

\[ i\hbar \frac{\partial S^+_i}{\partial t} = -(J_{i,i+1} S^z_{i+1} + J_{i,i-1} S^z_{i-1}) S^+_i + J_{i,i+1} S^z_i S^+_i S^+_i + J_{i,i-1} S^z_i S^+_i S^+_i - 1. \]
HEISENBERG--MATTIS SPIN GLASSES

Heisenberg–Mattis spin glass.

Now assuming to have small spin wave amplitude

\[ S^x, S^y \ll S, \quad S_i^z \approx \zeta_i S, \]

where \( S \) is the length of spin vectors and \( \zeta_i \) equates \( \pm 1 \).

\[ (2 - \zeta_i \Omega) \mu_i = \mu_{i+1} + \mu_{i-1}. \]

\[ \mu_i = \zeta_i S_i^+ \]

zero average mass

\[ \omega^2 \equiv \Omega \]

\[ \gamma = \frac{6^{1/3} \sqrt{\pi}}{2 \times \Gamma(\frac{1}{6})} (\sigma \Omega)^{2/3}. \]
Part – ( II-a)

Elastic wave localization

Martin-Siggia-Rose action
II-a) The Model (Scalar Field)

The scalar wave equation:

$$\frac{\partial^2}{\partial t^2} \psi(x, t) - \nabla \cdot [\lambda(x) \nabla \psi(x, t)] = 0,$$

where $\psi(x, t)$ is the wave amplitude, and $\lambda(x) = e(x)/m$ the ratio of the elastic stiffness $e(x)$ and the medium’s mean density $m$. We then write $\lambda$ as,

$$\lambda(x) = \lambda_0 + \eta(x),$$

where $\lambda_0 = \langle \lambda(x) \rangle$. In the present paper $\eta(x)$ is assumed to be a Gaussian random process with a zero mean and the covariance,

$$\langle \eta(x) \eta(x') \rangle = 2C(|x - x'|) = 2D_0 \delta^d(x - x') + 2D\rho |x - x'|^{2\rho-d}.$$
Propagation of Wave Component with Frequency $\omega$

\[ \nabla^2 \psi(x, \omega) + \frac{\omega^2}{\lambda_0} \psi(x, \omega) + \nabla \cdot \left( \frac{\eta(x)}{\lambda_0} \nabla \psi(x, \omega) \right) = 0 \]
The Martin-Siggia-Rose Action

\[ S_e[\psi_I, \psi_R, \tilde{\psi}, \chi, \chi^*] = \]
\[ \int dx dx' \left[ (i\tilde{\psi}_I(x')(\nabla^2 + \frac{\omega^2}{\lambda_0})\psi_I(x) + i\tilde{\psi}_R(x')(\nabla^2 + \frac{\omega^2}{\lambda_0})\psi_R(x) \right. \]
\[ + \chi^*(x')(\nabla^2 + \frac{\omega^2}{\lambda_0})\chi(x))\delta(x - x') \]
\[ + (i\nabla\tilde{\psi}_I \nabla\psi_I + i\nabla\tilde{\psi}_R \nabla\psi_R + \nabla\chi \nabla\chi) \frac{K(x - x')}{\lambda_0^2} \left( i\nabla\tilde{\psi}_I \nabla\psi_I \nabla\psi_R + \nabla\chi \nabla\chi \right) \]

- Two coupling constants:

\[ g_0 = D_0/\lambda_0^2, \quad g_\rho = D_\rho/\lambda_\rho^2 \]

- RG analysis to one-loop order in the limit, \( \omega^2/\lambda_0 \to 0 \), to determine the two beta functions.
**Diagrammatic Representation and One-Loop Corrections**

\[
\frac{1}{k^2 - i\omega/\lambda_e} = \frac{k, \omega}{k, \omega} = \frac{1}{k^2 - i\omega/\lambda_e} = \frac{k, \omega}{k, \omega}
\]

\[-4g_0(k_1, k_2)(k_3, k_4)\delta(\sum_{i=1}^4 k_i) = \quad = \quad = \quad = \frac{-1}{2}\]

\[-4g_\rho(k_1, k_2)(k_3, k_4)\kappa^{-2\rho}\delta(\sum_{i=1}^4 k_i) = \quad = \quad = \quad = -\frac{1}{2}\]

**FIG. 2.** Diagrammatic representations of the propagators and vortices in the effective action \( S_e \).

\[
\bullet = \quad + \quad + \quad + \frac{1}{2} \quad I_1 \quad + \quad 2 \quad I_2 \quad + \quad 2 \quad I_3
\]

\[
+ \quad 2 \quad I_4 \quad + \quad 2 \quad I_5 \quad + \quad I_6 \quad + \quad I_7
\]

\[
+ \quad 2 \quad I_8 \quad + \quad 2 \quad I_9 \quad + \quad 2 \quad I_{10}
\]

**FIG. 3.** One-loop corrections to the four-point correlation function.
The Beta Functions

The functions $\beta(\tilde{g}_0)$ and $\beta(\tilde{g}_\rho)$ are then given by,

$$
\beta(\tilde{g}_0) = \frac{\partial \tilde{g}_0}{\partial \ln l} = -d\tilde{g}_0 + 8\tilde{g}_0^2 + 10\tilde{g}_\rho^2 + 20\tilde{g}_0\tilde{g}_\rho ,
$$

$$
\beta(\tilde{g}_\rho) = \frac{\partial \tilde{g}_\rho}{\partial \ln l} = (2\rho - d)\tilde{g}_\rho + 12\tilde{g}_0\tilde{g}_\rho + 16\tilde{g}_\rho^2 ,
$$

where $l > 1$ is the re-scaling parameter, and

$$
\tilde{g}_0 = k_d \left[ \frac{d + 5}{2d(d + 2)} \right] g_0 ,
$$

$$
\tilde{g}_\rho = k_d \left[ \frac{d + 5}{2d(d + 2)} \right] g_\rho ,
$$
Phase Space and Fixed Points

Three sets of fixed points for \(0 < \rho < d/2\):

Trivial FP (Gaussian) at \(g_0^* = g_\rho^* = 0\) (stable)

Non-trivial FPs, one at \(g_0^* = d/8, g_\rho^* = 0\), and the other at

\[
\begin{align*}
g_0^* &= -\frac{4}{41} \left[ d + \frac{5}{16}(2\rho - d) \right] \\
&- \frac{4}{41} \sqrt{\left[ d + \frac{5}{16}(2\rho - d) \right]^2 + \frac{205}{256}(2\rho - d)^2},
\end{align*}
\]

\[
g_\rho^* = \frac{3}{4}g_0^* + \frac{1}{16}(d - 2\rho),
\]

Stable in one eigendirection, but unstable in the other eigendirection.
Phase Space (Continued)

Thus, a system with uncorrelated disorder is unstable against disorder with long-range correlations towards a new FP.

Thus, with increasing disorder, extended $\rightarrow$ localized
Phase Space (Continued)

FIG. 5. Flows in the coupling constants space for $0 < \rho < \frac{1}{2} d$. 
Two sets of fixed points for $\rho > d/2$:

- Gaussian FP, stable on the $g_0$ axis, but not on the $g_\rho$ axis
- Non-trivial FP at $g_0^* = d/8$, $g_\rho^* = 0$, unstable in all directions.
- Thus, power-law disorder relevant, but no new FP.
FIG. 6. Flows in the coupling constants space for $\rho > \frac{1}{2}d$. 

Phase Space (Continued)
Frequency-Dependence of Localization Length

Localization length $\xi$ as a function of the frequency $\omega$ for $\sigma < \sigma_c \simeq 2.4$ and $\sigma > \sigma_c$. The system size is $N = 6 \times 10^6$. The results represent averages over 6000 realizations.
Wave Front

FIG. 7. The wave front in a 2D anisotropic system at (dimensionless) times, $t_1 = 328$, $t_2 = 384$, and $t_3 = 440$, with $\rho = 1.5$. 
Roughness of Wave Front: Self-Affine Fronts

Computing the correlation function

\[ C(r) = \langle [d(x) - d(x + r)]^2 \rangle \]

\( d(x) = \) distance from the source along the propagation direction

\[ C(r) \sim r^{2\alpha} \]

\( \alpha = H = \rho - 1 \)

The Shape of Wave Front and its Evolution

$H=0.3$

$H=0.75$
Part – III

Numerical Method
Red-Shift of Spectral Density
(Light Localization)
Transfer Matrix
(random mass)

\[ \psi_{i+1} + \psi_{i-1} - (2 - m_i \omega^2) \psi_i = 0 , \]

\[
\begin{pmatrix}
\psi_{i+1} \\
\psi_i
\end{pmatrix} =
\begin{pmatrix}
2 - m_i \omega^2 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_i \\
\psi_{i-1}
\end{pmatrix}
\]

with initial normalized vector, \( \mathbf{v} = (1/\sqrt{2}, 1/\sqrt{2})^T \).

Oseledec theorem:

To ensure the stability of the numerical method, after multiplying the transfer matrices \( M \) times, we checked the length of the resulting vector, normalized it again, and then continued with the new vector.
Transfer Matrix
(random mass)

The Lyapunov exponent is then expressed in terms of vector lengths $d_{\alpha}$ obtained after $N$ normalization of $v$.

$$
\gamma = \frac{1}{MN} \sum_{\alpha=1}^{N} \ln(d_{\alpha}).
$$

The error in estimating $\gamma$ is given by,

$$
\frac{\Delta \gamma}{\gamma} = \frac{1}{\sqrt{N}} \sqrt{\langle (\ln d_{\alpha})^2 \rangle - \langle \ln d_{\alpha} \rangle^2} / \langle \ln d_{\alpha} \rangle.
$$

$\gamma$ is a self-averaged quantity, and the error of its estimates approaches zero as $1/\sqrt{N}$, as $N$ increases.
Dynamics of Pulse in Random Media

“3D Light localization”

FIG. 1. The observed pulse in $Z = 500d$, where $d$ is the lattice spacing and the initial pulse at $Z = 0$ (inset).
Red-Shift of Spectral Density

\[ \psi(\omega, Z) \sim \exp\{-Z/\xi(\omega)\} \]

\[ \tilde{SD}(Z) \sim \exp\{-2Z/\xi(\omega)\}. \]

\[ \xi^{-1}(\omega) = - \lim_{Z \to \infty} (2Z)^{-1} \ln \left( \frac{SD(\omega, Z)}{SD(\omega, 0)} \right). \]

FIG. 2. The initial spectral density of pulse at \( Z = 0 \) and the observed ones (others) in different location from the source. Here \( \omega_0 \) is the maximum frequency of spectrum. There is a red-shift in the spectral density due to the localization of different frequencies in different length scales.
Localized Behavior

FIG. 3. Semi-Log plot of amplitude difference of spectral density verses distance $Z$, for different disordered strengths, (for frequency $\omega = 2\omega_0$, where $\omega_0$ is the frequency that the spectral density of the incident pulse has maximum).
Localization length

FIG. 4. The frequency dependence of the localization length for different disordered strengths. The modes with $\xi(\omega)/L > 1$ and $\xi(\omega)/L < 1$ are delocalized and localized, respectively.
Black-Body radiation

FIG. 5. Spectral density of black body radiation and its shape deformation by propagating in for instance, $Z$ direction.
Red-shift vs Distance from the source.

FIG. 6. Peak frequency of the spectral density versus the location of observation for different noise strengths.

For small disorder strength \[ \Delta \omega / \omega_0 \propto Z / d, \]
\[ \Delta \omega / \omega_0 \propto W/W_0. \]
\begin{equation}
\frac{(T_0 - T(Z))}{T_0} \propto Z
\end{equation}

As an example, suppose that the source at \( Z = 0 \) is at temperature 1000 K. At a distance 92 \( \mu \text{m} \) (which is about 500 characteristic length of the disorder), the apparent measured temperature of the source would be about \( 900 \pm 50 \) K. This difference in the actual and apparent temperatures is obtained for the disorder strength \( \sigma = 0.3 \).
Acoustic waves in the random dimer media
Random Dimer

\( k_A, k_A, k_A, k_A, k_B, k_B, k_A, k_A, k_A, k_B, k_B, \cdots \)
Critical exponents

\[ \xi \propto |\omega - \omega_c|^{-\nu} \]

- \( \omega > \omega_c \quad \nu \approx 2.20 \)
- \( \omega < \omega_c \quad \nu \approx 1.85 \)
- \( \omega \rightarrow 0^+ \quad \nu \approx 1.94 \)

\[ \omega_c^2 = 2k_B \]
Thanks