## Search for Majorana Fermions

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Advanced school on Recent Progress in Condensed Matter Physics IPM, June 2012 A bird's eye view on quasiparticles in CMP Introduction to Majorana fermions (MF) How to make MF

Kitaev Toy model for MF

Topological properties of systems hosting MF

Topological superconductivity (TS)

Possible physical realization of TS

How to use them

Non-Abelian statistics and braiding

**Topological quantum computing** 

How to detect them

Transport properties of MF

#### References

"Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect", **N. Read, D. Green, Phys. Rev. B (2000)** 

*"Unpaired Majorana fermions in quantum wires",* Alexei Kitaev, cond-mat/0010440

"Topological phases and quantum computation", Alexei Kitaev, Chris Laumann, arXiv:0904.2771

"New directions in the pursuit of Majorana fermions in solid state systems", Jason Alicea, arXiv:1202.1293

*"Introduction to topological superconductivity and Majorana fermions",* Martin Leijnse, Karsten Flensberg, arXiv:1206.1736

*"Search for Majorana fermions in Superconductors",* Carlo Beenaker, arXiv:1112.1950

"Non-Abelian anyons and topological quantum computation", Chetan Nayak et al., Rev. Mod. Phys. (2008)

Theory of Everything (TOE)

$$i\hbar \frac{\partial}{\partial t} |\Psi > = \mathcal{H} |\Psi >$$

 $\mathcal{H} = -\sum_{i}^{N_e} \frac{\hbar^2}{2m} \nabla_j^2 - \sum_{\alpha}^{N_i} \frac{\hbar^2}{2M_{\alpha}} \nabla_{\alpha}^2$ 

 $-\sum_{j}^{N_e}\sum_{\alpha}^{N_i}\frac{Z_{\alpha}e^2}{\left|\vec{r}_j-\vec{R}_{\alpha}\right|}+\sum_{j\ll k}^{N_e}\frac{e^2}{\left|\vec{r}_j-\vec{r}_k\right|}+\sum_{\alpha\ll e}^{N_j}\frac{Z_{\alpha}Z_{\beta}e^2}{\left|\vec{R}_{\alpha}-\vec{r}_{\beta}\right|}$ 

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single particle Hilbert space dimension: k

many particle Hilbert space dimension:  $k^N$ 

N=30 k=2 (spin 1/2)  $k^{N}=2^{30}\sim 10^{9}$ 

1 GB Memory to save single state

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microscopic  $\iff$  macroscopic

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#### higher organizing principles in nature

**The** *emergent* **physical phenomena regulated by** *higher organizing principles* **have a property, namely their** *insensitivity to microscopics.* 



#### One main purpose of CMP is to study

#### The elementary excitations



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#### More is Different





## Introduction to Majorana fermions (HEP vs CMP or LEP!)

#### A Majorana fermion is its own antiparticle $\gamma^{\dagger} = \gamma$

- $\diamond$  History: Neutral spin-1/2 particles as real solutions of Dirac eq.
- Particle physics: Neutrinos might be Majorana fermions!
   Neutrinoless double beta decay (Not yet observed)



#### Ettore Majorana

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#### Emergence in condensed matter

- $\diamond$  Prehistory:  $d = (\gamma_1 + i\gamma_2)/\sqrt{2}$
- ◊ Isolated Majoranas are believed to exist in:
   Fractional quantum Hall state at ν = 5/2
   Boundaries of topological superconductors
   (vortex cores and edges)



Ettore Majorana

## Majorana's as protected qubits

#### A big challenge for Quantum Computer: Decoherence

Possible errors destroying the state of qubit (*spin system*)

Classical error:  $\eta_{cl} \sigma_j^x$ flips the *j*th qubit Phase error:  $\eta_{\rm ph} \, \sigma_j^z$ changes the relative sign

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1D spinless Fermionic lattice: No classical error (Fermion parity) Phase error:  $\eta_{ph}a_j^{\dagger}a_j$ Majorana operators:  $c_{2j-1} = a_j + a_j^{\dagger}$ ,  $c_{2j} = \frac{a_j - a_j^{\dagger}}{i}$ Phase error:  $a_j^{\dagger}a_j = \frac{1}{2}(1 + ic_{2j-1}c_{2j})$ Isolated Majorana is immune against any error  $a_1 \quad a_2 \quad \dots \quad a_N$ 

Spinless p-wave superconductor (SPSC)  
$$H_1 = \sum_{j} \left[ -w(a_j^{\dagger} a_{j+1} + a_{j+1}^{\dagger} a_j) - \mu \left( a_j^{\dagger} a_j - \frac{1}{2} \right) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^{\dagger} a_j^{\dagger} \right]$$



Alexi Kitaev

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Majorana operators:  

$$c_{2j-1} = \exp\left(i\frac{\theta}{2}a_{j} + \exp\left(-i\frac{\theta}{2}a_{j}\right)\right)$$

$$c_{2j} = -i\exp\left(i\frac{\theta}{2}a_{j} + i\exp\left(-i\frac{\theta}{2}a_{j}\right)\right)$$

$$H_{1} = \frac{i}{2}\sum_{j} \left[ -\mu c_{2j-1}c_{2j} + (w + |\Delta|)c_{2j}c_{2j+1} + (-w + |\Delta|)c_{2j-1}c_{2j+2} \right]$$



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(a)  $|\Delta| = w = 0, \ \mu < 0$ 

$$H_{1} = -\mu \sum_{j} \left(a_{j}^{\dagger}a_{j} - \frac{1}{2}\right) = \frac{i}{2}(-\mu) \sum_{j} c_{2j-1}c_{2j}$$
(b)  $|\Delta| = w > 0, \ \mu = 0$ 

$$H_{1} = iw \sum_{j} c_{2j}c_{2j+1}$$

$$a \qquad c_{1} \quad c_{2} \quad c_{3} \quad c_{4} \quad c_{2L-1} \quad c_{2L}}{e \quad \Phi} \qquad b$$

Alexi Kitaev

- (a) The Majorana operators from the same site paired, to form a ground state with occupation number 0;
- (b) The Majorana operators at the ends remain unpaired, leading to a two-fold degenerate ground state,

$$-ic_1c_{2N}|\psi_0\rangle = |\psi_0\rangle \qquad -ic_1c_{2N}|\psi_1\rangle = -|\psi_1\rangle$$
$$c_1 = d + d^{\dagger}, \quad c_{2N} = (d - d^{\dagger})/i \implies d^{\dagger}d|\psi_0\rangle = 0, \quad d^{\dagger}d|\psi_1\rangle = |\psi_1\rangle$$

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$$|\Delta| = w = 0, \ \mu < 0$$
  $H_1 = -\mu \sum_j \left( a_j^{\dagger} a_j - \frac{1}{2} \right) = \frac{i}{2} (-\mu) \sum_j c_{2j-1} c_{2j}$ 

(b) 
$$|\Delta| = w > 0, \mu = 0$$
  $H_1 = iw \sum_j c_{2j} c_{2j+1}$ 





Alexi Kitaev







 $\mathbb{Z}_2$  symmetry Majorana:  $P_{\text{maj}} = \prod_{k=1}^{N} (-ic_{2k-1}c_{2k}) \implies -ic_1c_{2L}$ 



$$\mathbb{Z}_2$$
 symmetry Majorana:  $P_{\text{maj}} = \prod_{k=1}^{N} (-ic_{2k-1}c_{2k}) \implies -ic_1c_{2L}$ 

fermionic parity cannot be disturbed by local perturbation, but the spin  $\mathbb{Z}_2$  symmetry can be removed by  $h_x$ 

Kitaev Hamiltonian in k-space:

$$C_{k}^{\dagger} = [c_{k}^{\dagger}, c_{-k}] \qquad H = \frac{1}{2} \sum_{k \in BZ} C_{k}^{\dagger} \mathcal{H}_{k} C_{k}, \quad \mathcal{H}_{k} = \begin{pmatrix} \epsilon_{k} & \tilde{\Delta}_{k}^{*} \\ \tilde{\Delta}_{k} & -\epsilon_{k} \end{pmatrix} \text{ (exercise 2)}$$

$$\epsilon_{k} = -t \cos k - \mu \qquad \Longrightarrow \qquad E(k) = \sqrt{\epsilon_{k}^{2} + |\tilde{\Delta}_{k}|^{2}}$$

$$\tilde{\Delta}_{k} = -i\Delta e^{i\phi} \sin k$$

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$$|g.s.\rangle \propto \prod_{0 < k < \pi} [1 + \varphi_{C.p.}(k)c_{-k}^{\dagger}c_{k}^{\dagger}]|0\rangle \qquad \varphi_{C.p.}(k) = (E - \varepsilon)/\tilde{\Delta}_{k}$$

$$|\varphi_{C.p.}(x)| \sim \begin{cases} e^{-|x|/\zeta}, & \mu < -t \\ \text{const.} & |\mu| < t \end{cases}$$

$$\begin{cases} \text{const,} & |\mu| < t \\ \text{const,} & |\mu| < t \end{cases}$$

$$\begin{cases} \text{topological} \\ (\text{weak pairing}) \\ \text{non-topological} \end{cases}$$

$$\mu = -t$$

(strong pairing)

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Strong  $(|\mu| > t)$  vs. weak pairing  $(|\mu| < t)$ BCS-BEC crossover (still no guarantee for topologically distinct phases)



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$$k$$

$$(c)$$

 $\hat{\mathbf{h}}(k) = \mathbf{h}(k)/|\mathbf{h}(k)|$  $\hat{\mathbf{h}}(0) = s_0 \hat{\mathbf{z}}, \quad \hat{\mathbf{h}}(\pi) = s_\pi \hat{\mathbf{z}},$ 

 $Z_2$  topological invariant $u = s_0 s_\pi$ 





Transition takes place when the gap closes; i.e. in the boundaries where h(k) is ill-defined.

#### **Topological superconductor**

#### Spinless $p_x + ip_y$ superconductor

$$\mathcal{H}_{BdG}(\mathbf{k}) = [\mathcal{H}_0(\mathbf{k}) - \mu]\tau_z + \Delta_0(k_x\tau_x + ik_y\tau_y)$$

particle-hole symmetry:

 $\Xi \mathcal{H}_{BdG}(\mathbf{k})\Xi^{-1} = -\mathcal{H}_{BdG}(-\mathbf{k}) \quad (\Xi = \tau_x \mathcal{C})$ 

 $\Gamma_{-E} = \Gamma_{E}^{\dagger} \ (E = 0 \text{ if exists is protected})$ 





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(a) 1D T-SC Protected states can exist in the boundaries: Majorana edge states (2D) Majorana bound states at the ends (1D) Majorana bound to vortex core (2D)



(a) (a)

(b)

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2D T - SC

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# Bulk-edge correspondence in topologically nontrivial state of matter

(a)



1D T-SC

## Role of topology in 2D spinless p-wave superconductor

#### Hamiltonian in k-space:

$$H = \frac{1}{2} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \Psi^{\dagger}(\mathbf{k}) \mathcal{H}(\mathbf{k}) \Psi(\mathbf{k}), \qquad \epsilon(k) = \frac{k^2}{2m} - \mu$$
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1

(b)(a)  $\frac{k^2}{2m}$  $\mathcal{H}_k = \mathbf{h}(k) \cdot \boldsymbol{\sigma}$ C = 0(trivial) Chern number k $C = \int \frac{d^2 \mathbf{k}}{4\pi} [\mathbf{\hat{h}} \cdot (\partial_{k_x} \mathbf{\hat{h}} \times \partial_{k_y} \mathbf{\hat{h}})]$ topological (C) (weak pairing)  $\mu = 0$ non-topological  $C \neq 0$  nontrivial topology (topological) (strong pairing)

## Vortex core states in 2D spinless p-wave superconductor

$$H_{\text{edge}} = \frac{1}{2} \int d^2 \mathbf{r} \Psi'^{\dagger}(\mathbf{r}) \mathcal{H}(\mathbf{r}) \Psi'(\mathbf{r}), \qquad \qquad \psi = e^{-i\theta/2} \psi'$$
$$\mathcal{H}(\mathbf{r}) = \begin{pmatrix} -\mu(r) & \Delta e^{-i\phi}(-\partial_r + \frac{i\partial_\theta}{r}) \\ \Delta e^{i\phi}(\partial_r + \frac{i\partial_\theta}{r}) & \mu(r) \end{pmatrix} \qquad \Psi'^{\dagger}(\mathbf{r}) = [\psi'^{\dagger}(\mathbf{r}), \ \psi'(\mathbf{r})]$$

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$$\chi_n^{\text{out}}(\mathbf{r}) = e^{in\theta} e^{\frac{1}{\Delta} \int_{R_{\text{out}}}^r dr' \mu(r')} \begin{pmatrix} ie^{-i\phi/2} \\ -ie^{i\phi/2} \end{pmatrix} \quad \chi_n^{\text{in}}(\mathbf{r}) = e^{in\theta} e^{-\frac{1}{\Delta} \int_{R_{\text{in}}}^r dr' \mu(r')} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}$$

 $E_{\text{out/in}} = \pm n\Delta/R_{\text{out/in}}$  for half-integer *n* (anti-periodic boundary condition)



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$$H_{\text{edge}} = \frac{1}{2} \int d^2 \mathbf{r} \Psi'^{\dagger}(\mathbf{r}) \mathcal{H}(\mathbf{r}) \Psi'(\mathbf{r}), \qquad \psi = e^{-i\theta/2} \psi'$$
$$\mathcal{H}(\mathbf{r}) = \begin{pmatrix} -\mu(r) & \Delta e^{-i\phi}(-\partial_r + \frac{i\partial_\theta}{r}) \\ \Delta e^{i\phi}(\partial_r + \frac{i\partial_\theta}{r}) & \mu(r) \end{pmatrix} \quad \Psi'^{\dagger}(\mathbf{r}) = [\psi'^{\dagger}(\mathbf{r}), \ \psi'(\mathbf{r})]$$

$$\chi_n^{\text{out}}(\mathbf{r}) = e^{in\theta} e^{\frac{1}{\Delta} \int_{R_{\text{out}}}^r dr' \mu(r')} \begin{pmatrix} ie^{-i\phi/2} \\ -ie^{i\phi/2} \end{pmatrix} \quad \chi_n^{\text{in}}(\mathbf{r}) = e^{in\theta} e^{-\frac{1}{\Delta} \int_{R_{\text{in}}}^r dr' \mu(r')} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}$$

 $E_{\text{out/in}} = \pm n\Delta/R_{\text{out/in}}$  for half-integer *n* (*anti-periodic boundary condition*)

A vortex carrying a quantum flux



periodic boundary condition energies with integer *n n=0 Majorana bound state* 



#### Physical realization: Artificial topological superconductor



FIG. 4. (Color online) Boundary status dancy topolog conductor (TSC). (a) A Horse perconductor du deit hybo at its ends. The end statignsplotnengenferal (b) Hangoha topological 2D superconductory with ketal chiral Majo mode. (c) A vortex with flux  $\Phi = h/2e$  is associated where the particle-hole operator mode. =+1. Equation (15) follows

the odd parity of the real  $\Delta$ eigenstate of  $\mathcal{U}_{-}$ , with energy

#### Physical realization: Artificial topological superconductor



## Physical realization: Nanowire with RashbahSpin-onbitual

Spatially varying A For the case where A depends which B = B + b n where A depends which a constant

#### Nanowire on top of a superconductor

He y dependent Bases for if exchange a page F and gas in Eags E, is State to be 4) and (5). Therefore the the departments states 'in erge sitive & positive his case in exactly the hansa way way as above to with of his entry he diagonalizing matrices being 24 - 4 (1) 2 (7, 7, 4) zero-energy houndrenewed theorefica

with b and  $\Delta$  exchanged with d and  $B^d$  respectively in the point a zero-energy bound the b and  $\Delta$  exchanged with d and  $B^d$  respectively. Both halves have contract the resulting wave function. This yield strict and are joined by short junctions with a  $e_{ij} = e_{ij} = m_{ij} = m_{ij} + m$  $q_d = \gamma_d^{\dagger} = (\eta_1 - \eta_2) / \sqrt{2}$ . cles) are formed at the junctions. (b) Majorana state in the sector p = 0 when B varies. The gap in the finite-p sector remains finite in the entire wire. (c) Majorana state in the sector p = 0 when  $\Delta$  varies. (d) Majorana state in the sector p = 0 when  $\mu$  varies... (e) "p-wave" Majorana state when  $\Delta$ changes sign. The sector p = 0 remains gapped in the entire d)

wife Each crossing with  $\Delta = 0$  hosts two Majorana states. B Spatiality varying  $\mu$ . If  $B_{\text{ext}}$  in the entire wire,  $\overline{d}$ then at the interface between spin-gap regions with  $\mu^2 < 1$ FIG. 2. (a) Wire in a ring geometry  $\Delta Boundhave gap$  regions with  $|\mu|^2 > B^2 - \Delta^2$ , a tant parameters and are-joined abor shoret state with also form (Fig.2d). In this case, we inearly varying parameter. Majoranaestlates  $\mu$  in an type of the states  $\mu_{\ell}$  in an type of the states  $\mu_{\ell}$  is the state of the 14) are formed into the junition section and in the condition for the Majorana and parameters and are Tome by this finit functions

it is preferable to have a stricting splithed remains finite in the entire wire (c) Majorana state in the sector  $\mu_r^2$  the direction of propagation, requiring that bector p = 0 when  $\Delta$  varies. (d) Majorana state in the sector  $\mu_r^2$  the direction of propagation, requiring that es) are tormed at the principations (b) Majorana stal at pent, along its axisatz Alternatively, She mag atares sten 0 Twhener Br war in sem line reader in the family and the analysis of a lost right for the wire-Majora



states.

both with and with Several candidat spin-orbit interaction orbit coupling arise it is preferable to ha the direction of pro bent along its axis. a strong electric fie haps a more promis wurtzite structure orbit coupling [17]. (1) is related to the  $\lambda_{\rm S0} = 100nm = m$  $u \sim \hbar 2 \Delta_{\rm SO} \lambda_{\rm SO} \approx 7$  $0.015m_e$ , with  $m_e$ bers (with  $\Delta = 280$ 

#### Bhysical realization: Nanowire with Rashba Spin orbit $\Delta Patially varying (A) = R + min u = 0 and a constant$ states. E(pNanowire on top of a superconductor B<sub>so</sub>i Ne y dependent Bases for the wenexchange F an Eq. in Eags E, is State to 1pt Nanowire 4) and (5). Therefore the the dylangoranes statice emerge sitive plansisis. S-wave superconductor his case in exactly the hansame was a searce to extent of riscention he diagonalizing matrices being 24 - 4 (1) Parts - the sta b and $\Delta$ exchanged with d and $B^{\prime}$ respectively, port a zero-energy is and $\Delta$ exchanged with d and $B^{\prime}$ respectively. Both halves have consulting wave function. This welds after not $-\infty$ in the second secon mundrenewed theore exercise 4 lat The state of the s spin-orbit interaction cles) are formed at the junctions. (b) Majorana state in the $\sqrt{2}$ . orbit coupling arise sector p = 0 when B varies. The gap in the finite-p sector it is preferable $t_{i}^{p}$ matrix ranains/finite in the entire wire. (c), Majorana state in the the direction of pro s = 0 when $\Delta$ varies. (d Najorana state in the sector f) $\Delta = 0.3 B = 1/4 \mu = 0$ bent along its axis. = 0 when u varies. (e) "p-wase" Majoran = 0 when $\mu$ varies...(e) "p-wave" Majorana state when $\Delta$ nanges sign. The sector p = 0 remains gapped in the entire a strong electric fi wire Each crossing with $\Delta = 0$ hosts two Majorana states s-wave superconductor states haps a more promis $d)^{-}$ wurtzite structure Spatiality varying p. If Best $\Delta_1$ in the entire p ine, orbit coupling $[1l'_{l}]$ . Set the interface betweer spin-gap regions with $\mu^2 S_{(h)}$ $(\underline{S}d)_{C}$ is related to the (a) Wire in a role deonetry ABOM (Ration ) gap regions with $|\mu|^2 > B^2 -$ FIG. = 100nm =/n $\lambda_{S0}$ tant parameters and are foind abor theretotactionals form (Fig.2d). In this case, we $u \sim \hbar 2 \Delta_{\rm SO} \not\approx 7$ inearly varying parameter. Major maestates $\mu_{\rm mankps}$ abruptly $A = \frac{1}{2} \frac{12}{4} = \frac{1}{2} \frac{12}{4} = \frac{1}{2} \frac{12}{4} = \frac{1}{2} \frac{1}{4} = \frac{1}{2} \frac{1}{4} = \frac{1}{2} \frac{1}{4} = \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} = \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} = \frac{1}{4} \frac{1}{4}$ $0.015m_e$ , with $m_e$ IES) Fre (armediate the junition of the distributed by Barty 0. The condition for the Majorana = 280bers (with $\Delta$ and parameters and are proved by the finite one on the parameters and are proved by the short $\frac{1}{2}$ funce on the parameters and the parameters are proved by the parameters of the paramete it is preferable to have a swittig spirit of remains finite in the entire wire (c) Majorana state in the sector $\mu_r^2 > tRe^2$ direction of propagation, requiring that p = 0 when $\Delta$ varies. (d) algorang state in the sector $\mu_r^2 > tRe^2$ direction of propagation, requiring that es) arentarmed at the principant on the prana stal at pent, along the axisatz Alternatively, one cou Atome sign O Twhen the war in sem line ware the family and the family and the second of the wire-Majora



G. 21 (a) Wire in a ring geometry in Both halves h at parameters and are joined by short junction: early varying parameter. Majorana states in the states tor n = 0 when B varies. The provided at the junctions of the direction is the direction of propagation for the direction of propagation

![](_page_44_Figure_0.jpeg)

#### **Physical realization:** Experiment

#### **Signatures of Majorana Fermions in Hybrid Superconductor-Semiconductor Nanowire Devices**

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wide gates

![](_page_45_Figure_3.jpeg)

![](_page_45_Figure_4.jpeg)

fine gates

silicon oxide

### Physical realization: Experiment

![](_page_46_Figure_1.jpeg)

#### non-degenerate braiding: $\Psi \mapsto e^{i\alpha} \Psi$ degenerate braiding: $\Psi \mapsto U \Psi$

#### non-degenerate braiding: $\Psi \mapsto e^{i\alpha} \Psi$ degenerate braiding: $\Psi \mapsto U \Psi$

Cooper pairs acquires  $2\pi$  phase upon crossing the brach cut, Majorana catch half a such phase

#### non-degenerate braiding: $\Psi \mapsto e^{i\alpha} \Psi$ degenerate braiding: $\Psi \mapsto U \Psi$

Clockwise exchange of two MF Cooper pairs acquires  $2\pi$  phase upon crossing the brach cut, Majorana catch half a such phase

$$\gamma_1 \rightarrow -\gamma_2$$

 $\gamma_2 \rightarrow + \gamma_1$ 

$$\gamma_i \to B_{12} \gamma_i B_{12}^{\dagger}$$
$$B_{12} = \frac{1}{\sqrt{2}} \left( 1 + \gamma_1 \gamma_2 \right)$$

![](_page_49_Picture_7.jpeg)

#### non-degenerate braiding: $\Psi \mapsto e^{i\alpha} \Psi$ degenerate braiding: $\Psi \mapsto U \Psi$

Clockwise exchange of two MF

$$\gamma_1 \rightarrow -\gamma_2$$
  
 $\gamma_2 \rightarrow +\gamma_1$ 

$$\gamma_i \to B_{12} \gamma_i B_{12}^{\dagger}$$
$$B_{12} = \frac{1}{\sqrt{2}} \left( 1 + \gamma_1 \gamma_2 \right)$$

Cooper pairs acquires  $2\pi$  phase upon crossing the brach cut, Majorana catch half a such phase

Rotating one MF around the other

 $B_{12}^2 = \gamma_1 \gamma_2$ 

$$\gamma_1 \to (\gamma_1 \gamma_2) \gamma_1 (\gamma_1 \gamma_2)^{\dagger} = -\gamma_1$$
  
$$\gamma_2 \to (\gamma_1 \gamma_2) \gamma_2 (\gamma_1 \gamma_2)^{\dagger} = -\gamma_2$$

![](_page_50_Picture_9.jpeg)

$$B_{12}|0\rangle = \frac{1}{\sqrt{2}} (1+i)|0\rangle$$
$$B_{12}|1\rangle = \frac{1}{\sqrt{2}} (1-i)|1\rangle$$

$$|1\rangle = f_1^{\dagger}|0\rangle$$
$$f_1 = (\gamma_1 + i\gamma_2)/2$$

$$B_{12}|00\rangle = \frac{1}{\sqrt{2}} (1+i)|00\rangle,$$
  

$$B_{23}|00\rangle = \frac{1}{\sqrt{2}} (|00\rangle + i|11\rangle),$$
  

$$B_{34}|00\rangle = \frac{1}{\sqrt{2}} (1+i)|00\rangle,$$

#### non-Abelian statistics MF exchange

$$[B_{i-1,i}, B_{i,i+1}] = \gamma_{i-1}\gamma_{i+1}$$

![](_page_51_Figure_6.jpeg)

#### Properties: topological quantum computing

$$\begin{split} |\bar{0}\rangle &\equiv |00\rangle, \ |\bar{1}\rangle \equiv |11\rangle \\ &-i\gamma_1\gamma_2 = -i\gamma_3\gamma_4 = \sigma_z, \\ &-i\gamma_2\gamma_3 = \sigma_x, \\ &-i\gamma_1\gamma_3 = -i\gamma_2\gamma_4 = \sigma_y, \end{split}$$

$$B_{12} = B_{34} = e^{-\frac{i\pi}{4}\sigma_z},$$
  
$$B_{23} = e^{-\frac{i\pi}{4}\sigma_x},$$

![](_page_52_Figure_3.jpeg)

Measure 
$$(|0_{12}0_{34} > + |1_{12}1_{34} >)/\sqrt{2}$$
  
1 2 3 4  
1 2 3 4  
1 t  
Braid  
3 3 0 2 3 0 4  
0 12 0 23  
Create 0 12 0 34

![](_page_52_Figure_5.jpeg)

#### Peculiar characteristics (Tools for detection)

Nonlocal transport (through coupled Majoranas)

1. Direct coupling:  $H_M = i\epsilon_M \gamma_1 \gamma_2$ ,  $\epsilon_M \propto \exp(l/\xi)$ 

2. Coulomb interaction in mesoscopic TS islands

Fractional or  $4\pi$ -periodic Josephson effect Half-integer conductance quantization

![](_page_53_Figure_6.jpeg)

Other detection mechanisms? (theory & experiment) nonlocal transport and entanglement

Universal quantum computing (theory) Non-topological operations Coupling topological qubit with conventional qubits

Majorana in other systems (theory & experiment)

Topological superconductivity in  $Sr_2RuO_4$ 

FQHE at  $\nu~=~5/2$