Search for Majorana Fermions

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Advanced school on Recent Progress in Condensed Matter Physics
IPM, June 2012

Outline

A bird's eye view on quasiparticles in CMP

Introduction to Majorana fermions (MF)

How to make MF

- Kitaev Toy model for MF
- Topological properties of systems hosting MF
- Topological superconductivity (TS)
- Possible physical realization of TS

How to use them

- Non-Abelian statistics and braiding
- Topological quantum computing

How to detect them

- Transport properties of MF

References

"Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect",
N. Read, D. Green, Phys. Rev. B (2000)

"Unpaired Majorana fermions in quantum wires",
Alexei Kitaev, cond-mat/0010440

"Topological phases and quantum computation",
Alexei Kitaev, Chris Laumann, arXiv:0904.2771

"New directions in the pursuit of Majorana fermions in solid state systems",
Jason Alicea, arXiv:1202.1293

"Introduction to topological superconductivity and Majorana fermions",
Martin Leijnse, Karsten Flensberg, arXiv:1206.1736

"Search for Majorana fermions in Superconductors",
Carlo Beenaker, arXiv:1112.1950

"Non-Abelian anyons and topological quantum computation",
Chetan Nayak et al., Rev. Mod. Phys. (2008)

Condensed matter physics (a bird's eye view)

Theory of Everything (TOE)

$$i\hbar \frac{\partial}{\partial t} |\Psi > \; = \mathcal{H} |\Psi >$$

$$\mathcal{H} = -\sum_{j}^{N_e} \frac{\hbar^2}{2m} \nabla_j^2 - \sum_{\alpha}^{N_i} \frac{\hbar^2}{2M_\alpha} \nabla_\alpha^2$$

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number of particles: N
single particle Hilbert space dimension: k
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N=30 k=2 (spin 1/2) $k^N=2^{30}\sim10^9$	**1 GB Memory to save single state**

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+
experimental input for guidance

- atomic conformations of small molecules,
- simple chemical reaction rates,
- structural phase transitions, ferromagnetism,
- superconducting transition temperatures

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They are not
first principle reductions
and
there are many failures for them

microscopic $\Longleftrightarrow$ macroscopic

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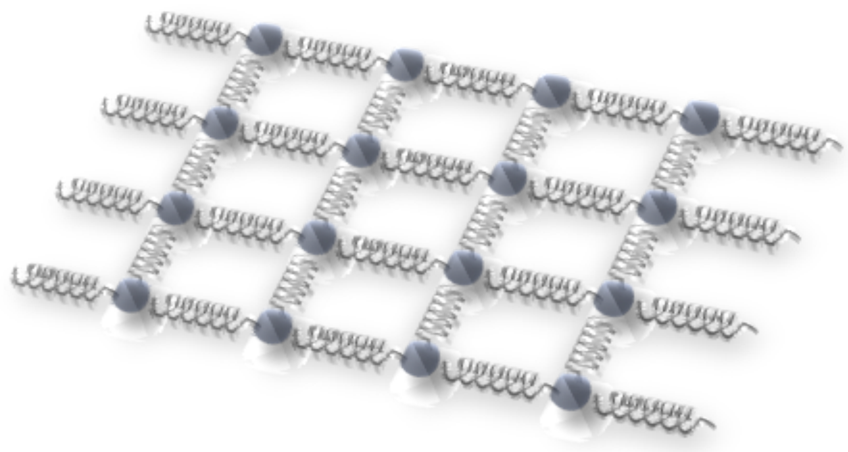
Condensed matter physics (a bird's eye view)

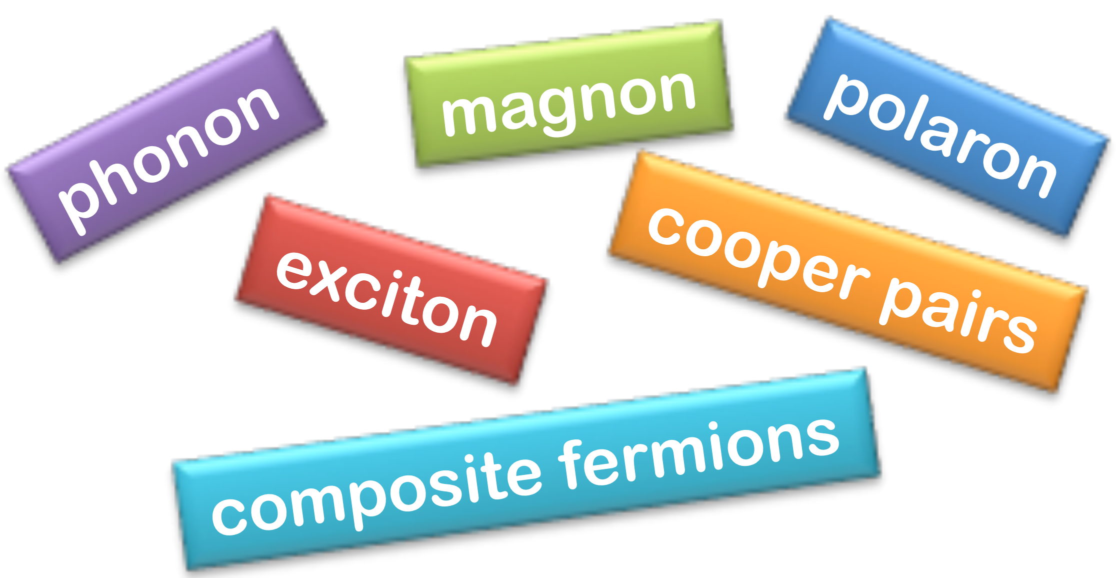
higher organizing principles in nature

The *emergent* **physical phenomena regulated by** *higher organizing principles* **have a property, namely their** *insensitivity to microscopics.*



One main purpose of CMP is to study

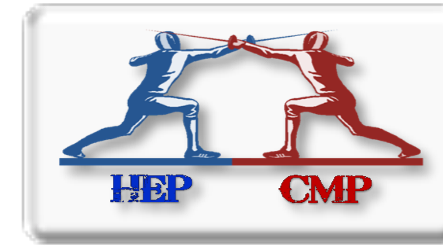
The elementary excitations



Condensed matter physics (a bird's eye view)

higher organizing principles in nature

More is Different



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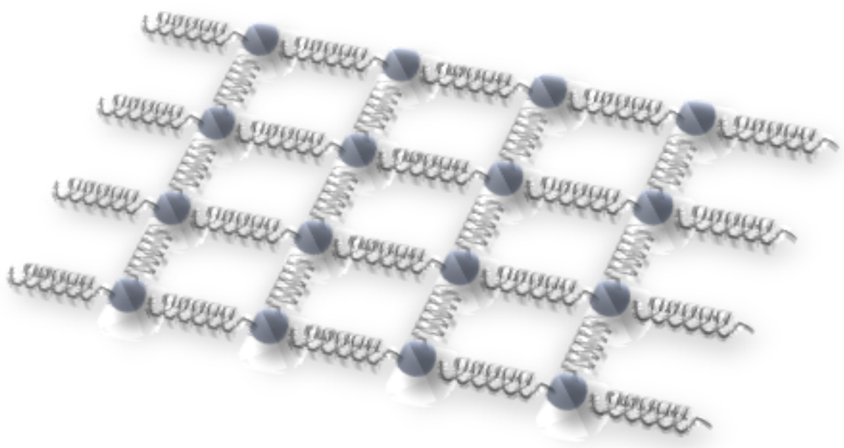

Emergence

Reductionism

One main purpose of CMP is to study

The elementary excitations

phonon

magnon

polaron

exciton

cooper pairs

composite fermions

Interactions
Correlations

We are searching new
Quasiparticles

Introduction to Majorana fermions (HEP vs CMP or LEP!)

A Majorana fermion is its own antiparticle $\gamma^\dagger = \gamma$

- $\diamond$ History: Neutral spin-$1/2$ particles as real solutions of Dirac eq.
- $\diamond$ Particle physics: Neutrinos might be Majorana fermions!

  Neutrinoless double beta decay (Not yet observed)

Ettore Majorana

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Emergence in condensed matter

- ◇ Prehistory: $d = (\gamma_1 + i\gamma_2)/\sqrt{2}$
- ◇ Isolated Majoranas are believed to exist in:

  Fractional quantum Hall state at $\nu = 5/2$

  Boundaries of topological superconductors

  (vortex cores and edges)

Ettore Majorana

Majorana's as protected qubits

A big challenge for Quantum Computer: Decoherence

Possible errors destroying the state of qubit (*spin system*)

Classical error: $\eta_{\rm cl}\,\sigma_j^x$

flips the *j*th qubit

Phase error: $\eta_{\rm ph}\,\sigma_j^z$

changes the relative sign

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1D spinless Fermionic lattice: No classical error (Fermion parity)

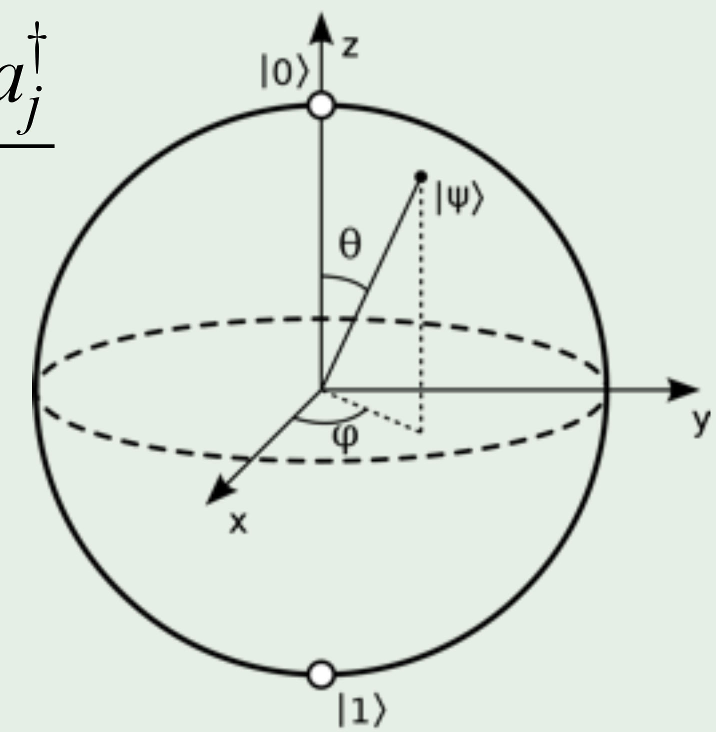
Phase error: $\eta_{\rm ph} a_j^\dagger a_j$

Majorana operators: $c_{2j-1} = a_j + a_j^\dagger, \qquad c_{2j} = \dfrac{a_j - a_j^\dagger}{\mathrm{i}}$

Phase error: $a_j^\dagger a_j = \dfrac{1}{2}(1 + \mathrm{i} c_{2j-1} c_{2j})$

Isolated Majorana is immune against any error

$a_1 \quad a_2 \quad \ldots \quad a_N$



Toy model: Kitaev proposal for Majorana fermions

Spinless p-wave superconductor (SPSC)

$$H_1 = \sum_j \left[-w(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) - \mu \left(a_j^\dagger a_j - \frac{1}{2} \right) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger \right]$$

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Majorana operators:

$$c_{2j-1} = \exp\left(\mathrm{i}\frac{\theta}{2}\right) a_j + \exp\left(-i\frac{\theta}{2}\right) a_j^\dagger \qquad c_{2j} = -\mathrm{i}\exp\left(\mathrm{i}\frac{\theta}{2}\right) a_j + \mathrm{i}\exp\left(-\mathrm{i}\frac{\theta}{2}\right) a_j^\dagger$$

$$H_1 = \frac{\mathrm{i}}{2}\sum_j \left[-\mu c_{2j-1} c_{2j} + (w + |\Delta|) c_{2j} c_{2j+1} + (-w + |\Delta|) c_{2j-1} c_{2j+2} \right]$$

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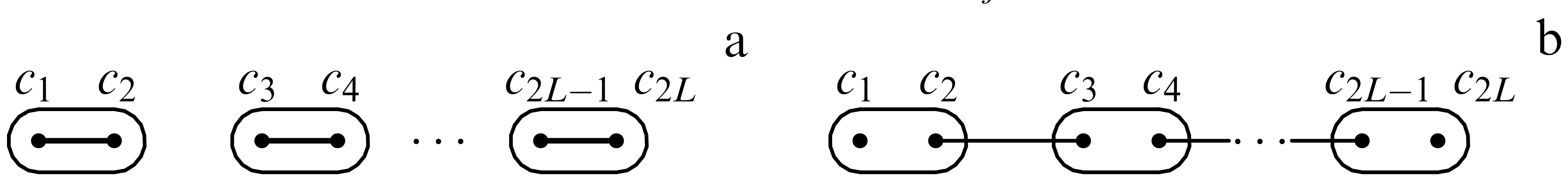
(a) $|\Delta| = w = 0,\ \mu < 0$ $\qquad H_1 = -\mu \sum_j \left(a_j^\dagger a_j - \frac{1}{2}\right) = \frac{\mathrm{i}}{2}(-\mu)\sum_j c_{2j-1} c_{2j}$

(b) $|\Delta| = w > 0,\ \mu = 0$ $\qquad H_1 = \mathrm{i}w \sum_j c_{2j} c_{2j+1}$



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(a) The Majorana operators from the same site paired, to form a ground state with occupation number 0;

(b) The Majorana operators at the ends remain unpaired, leading to a two-fold degenerate ground state,

$$-ic_1c_{2N}|\psi_0\rangle = |\psi_0\rangle \qquad -ic_1c_{2N}|\psi_1\rangle = -|\psi_1\rangle$$

$$c_1 = d + d^\dagger \ , \quad c_{2N} = (d - d^\dagger)/i \ \Rightarrow \ d^\dagger d|\psi_0\rangle = 0\,, \ \ d^\dagger d|\psi_1\rangle = |\psi_1\rangle$$

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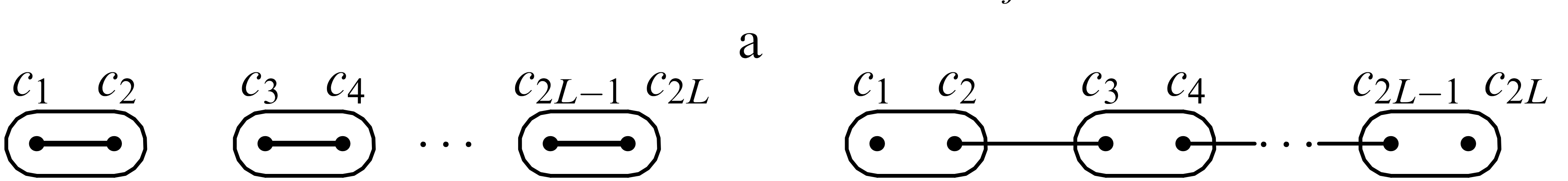

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Jordan-Wigner transformation

$$a_j = \left(\prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^+ \qquad a_j^\dagger = \left(\prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^- \qquad \boxed{\sigma_j^z = (-1)^{a_j^\dagger a_j}}$$

Transverse field Ising model (TFIM)

exercise 1

$$H_S = -J \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x - h_z \sum_{j=1}^{N} \sigma_j^z \qquad w = J, \quad \mu = -2h_z$$

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fermionic parity cannot be disturbed by local perturbation,
but the spin $\mathbf{Z}_2$ symmetry can be removed by h_x

Role of topology in Kitaev toy model

Kitaev Hamiltonian in k-space:

$$C_k^\dagger = [c_k^\dagger, c_{-k}] \qquad H = \frac{1}{2}\sum_{k\in BZ} C_k^\dagger \mathcal{H}_k C_k, \qquad \mathcal{H}_k = \begin{pmatrix} \epsilon_k & \tilde{\Delta}_k^* \\ \tilde{\Delta}_k & -\epsilon_k \end{pmatrix}$$

exercise 2

$$\epsilon_k = -t\cos k - \mu$$

$$\tilde{\Delta}_k = -i\Delta e^{i\phi}\sin k$$

$$\Longrightarrow \boxed{E(k) = \sqrt{\epsilon_k^2 + |\tilde{\Delta}_k|^2}}$$

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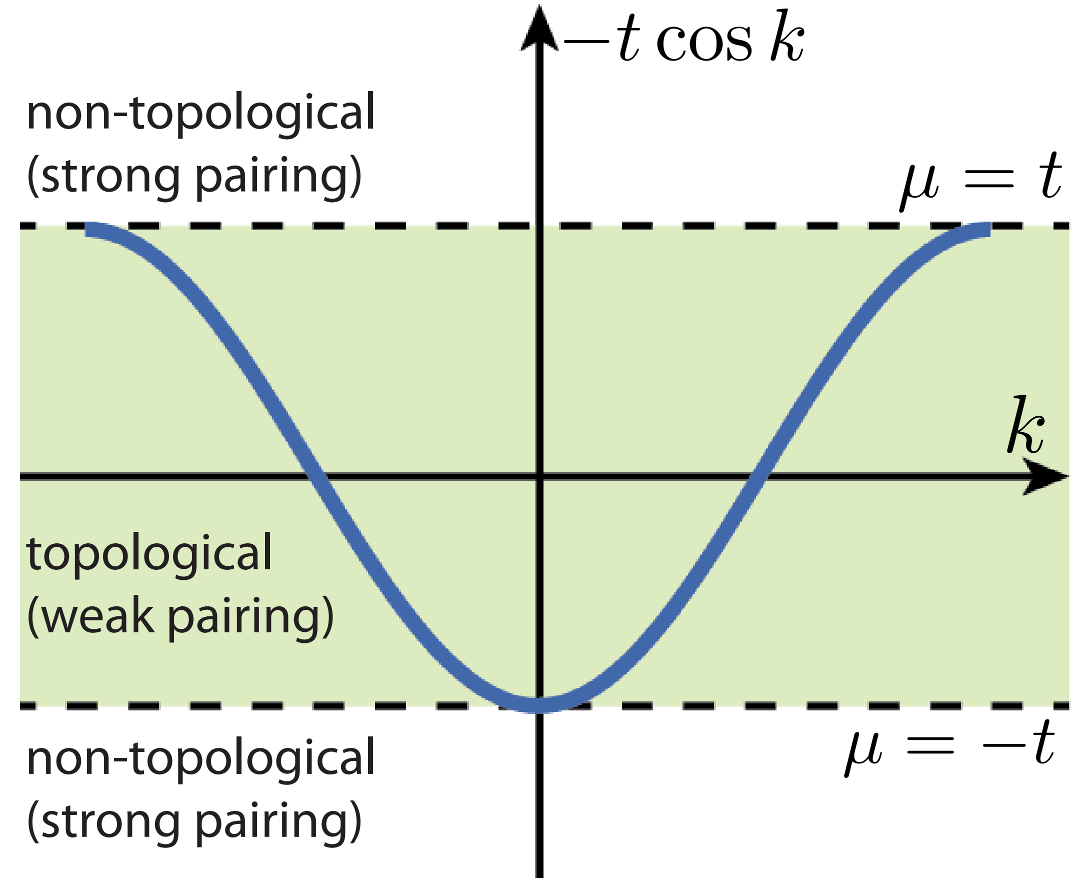
$$|\varphi_{\mathrm{C.p.}}(x)| \sim \begin{cases} e^{-|x|/\zeta}, & \mu < -t \\ \mathrm{const}, & |\mu| < t \end{cases}$$



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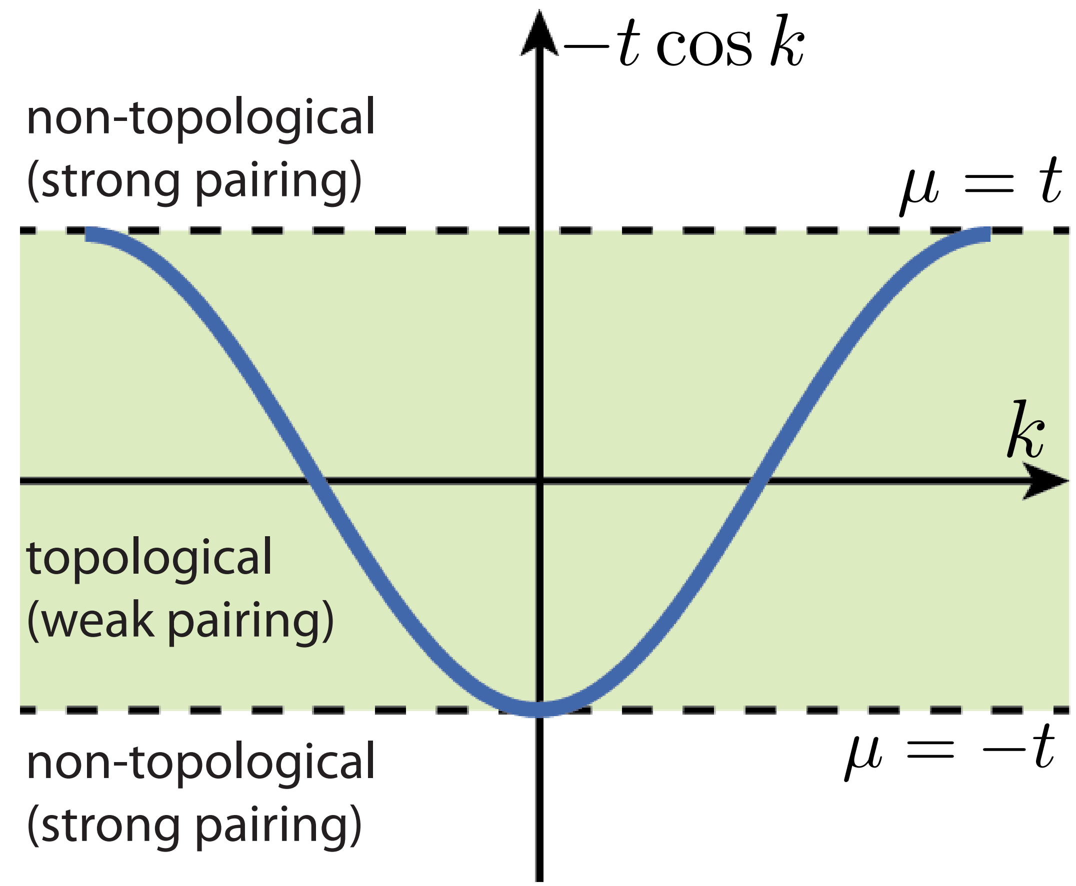
Strong ($|\mu| > t$) vs. weak pairing ($|\mu| < t$)

BCS-BEC crossover
(still no guarantee for
topologically distinct phases)



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General form: $\mathcal{H}_k = \mathbf{h}(k)\cdot\boldsymbol{\sigma}$ $\qquad \boldsymbol{\sigma} = \sigma^x\hat{\mathbf{x}} + \sigma^y\hat{\mathbf{y}} + \sigma^z\hat{\mathbf{z}}$

$$(C_{-k}^\dagger)^T = \sigma^x C_k \implies h_{x,y}(k) = -h_{x,y}(-k), \quad h_z(k) = h_z(-k)$$

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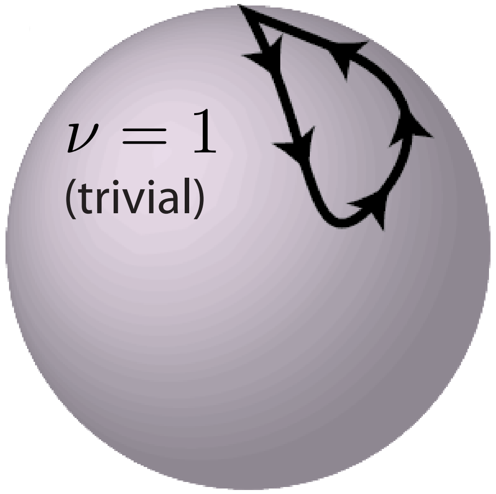
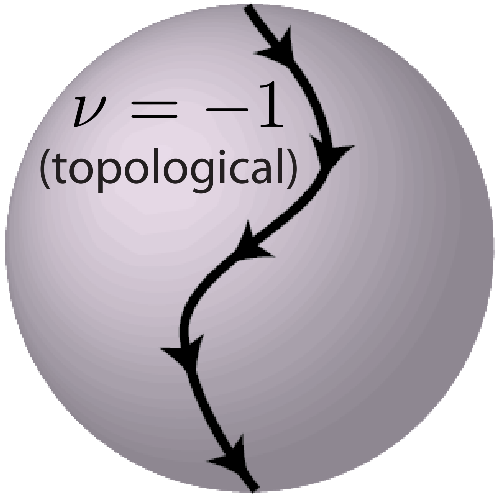
$$\hat{\mathbf{h}}(k) = \mathbf{h}(k)/|\mathbf{h}(k)|$$

$$\hat{\mathbf{h}}(0) = s_0\hat{\mathbf{z}}, \quad \hat{\mathbf{h}}(\pi) = s_\pi\hat{\mathbf{z}},$$

Z_2 topological invariant

$$\nu = s_0 s_\pi$$





Role of topology in Kitaev toy model

Kitaev Hamiltonian in k-space:

$$C_k^\dagger = [c_k^\dagger, c_{-k}] \qquad H = \frac{1}{2}\sum_{k\in BZ} C_k^\dagger \mathcal{H}_k C_k, \quad \mathcal{H}_k = \begin{pmatrix} \epsilon_k & \tilde{\Delta}_k^* \\ \tilde{\Delta}_k & -\epsilon_k \end{pmatrix}$$

General form: $\mathcal{H}_k = \mathbf{h}(k)\cdot\boldsymbol{\sigma} \qquad \boldsymbol{\sigma} = \sigma^x\hat{\mathbf{x}} + \sigma^y\hat{\mathbf{y}} + \sigma^z\hat{\mathbf{z}}$

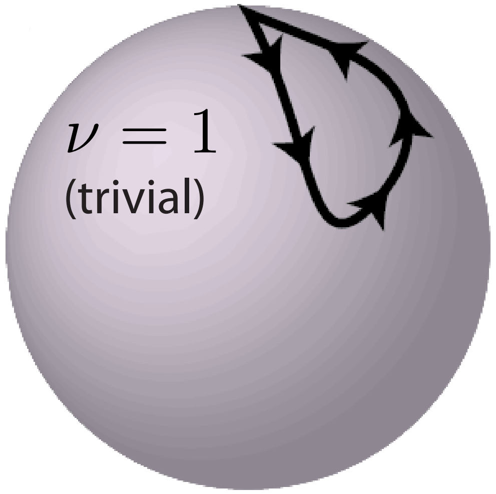
$$(C_{-k}^\dagger)^T = \sigma^x C_k \implies h_{x,y}(k) = -h_{x,y}(-k), \quad h_z(k) = h_z(-k)$$

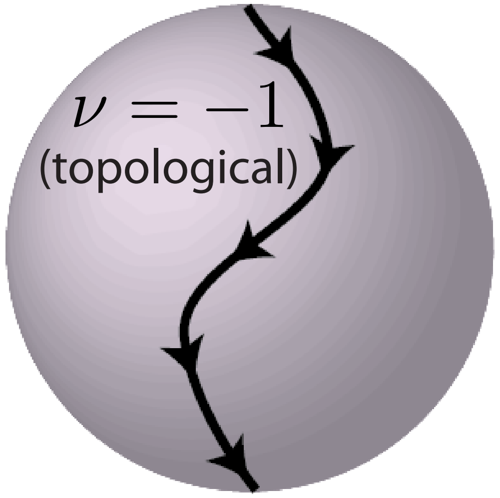
$$\hat{\mathbf{h}}(k) = \mathbf{h}(k)/|\mathbf{h}(k)|$$

$$\hat{\mathbf{h}}(0) = s_0\hat{\mathbf{z}}, \quad \hat{\mathbf{h}}(\pi) = s_\pi\hat{\mathbf{z}},$$

Z_2 topological invariant

$$\nu = s_0 s_\pi$$





Trivial: an even number of pairs of Fermi points

Topological: an odd number of pairs of Fermi points

Transition takes place when the gap closes;
i.e. in the boundaries where $\mathbf{h}(k)$ is ill-defined.

Topological superconductor

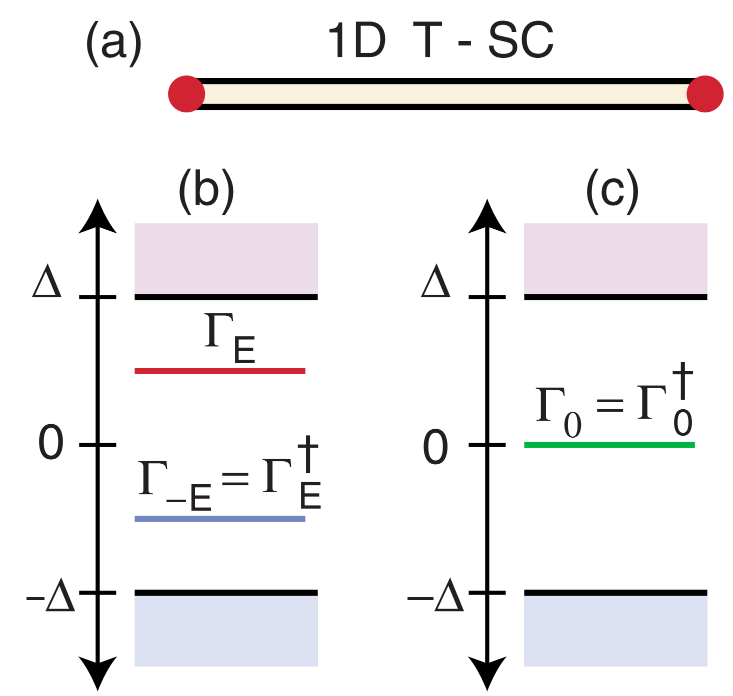
Spinless $p_x + ip_y$ superconductor

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = [\mathcal{H}_0(\mathbf{k}) - \mu]\tau_z + \Delta_0(k_x\tau_x + ik_y\tau_y)$$

particle-hole symmetry:

$$\Xi\mathcal{H}_{\mathrm{BdG}}(\mathbf{k})\Xi^{-1} = -\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k}) \quad (\Xi = \tau_x\mathcal{C})$$

$\Gamma_{-E} = \Gamma_E^\dagger$ ($E = 0$ if exists is protected)



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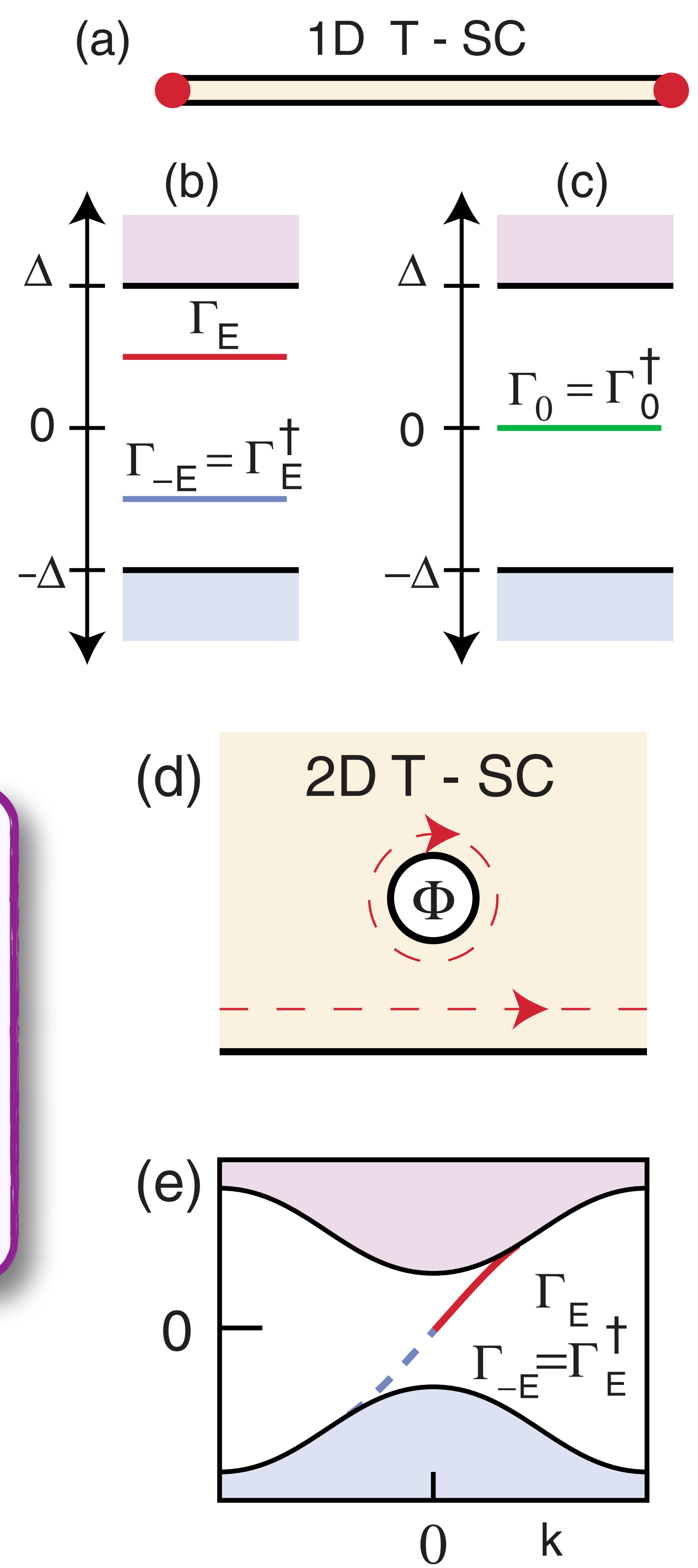
Protected states can exist in the boundaries:
Majorana edge states (2D)
Majorana bound states at the ends (1D)
Majorana bound to vortex core (2D)



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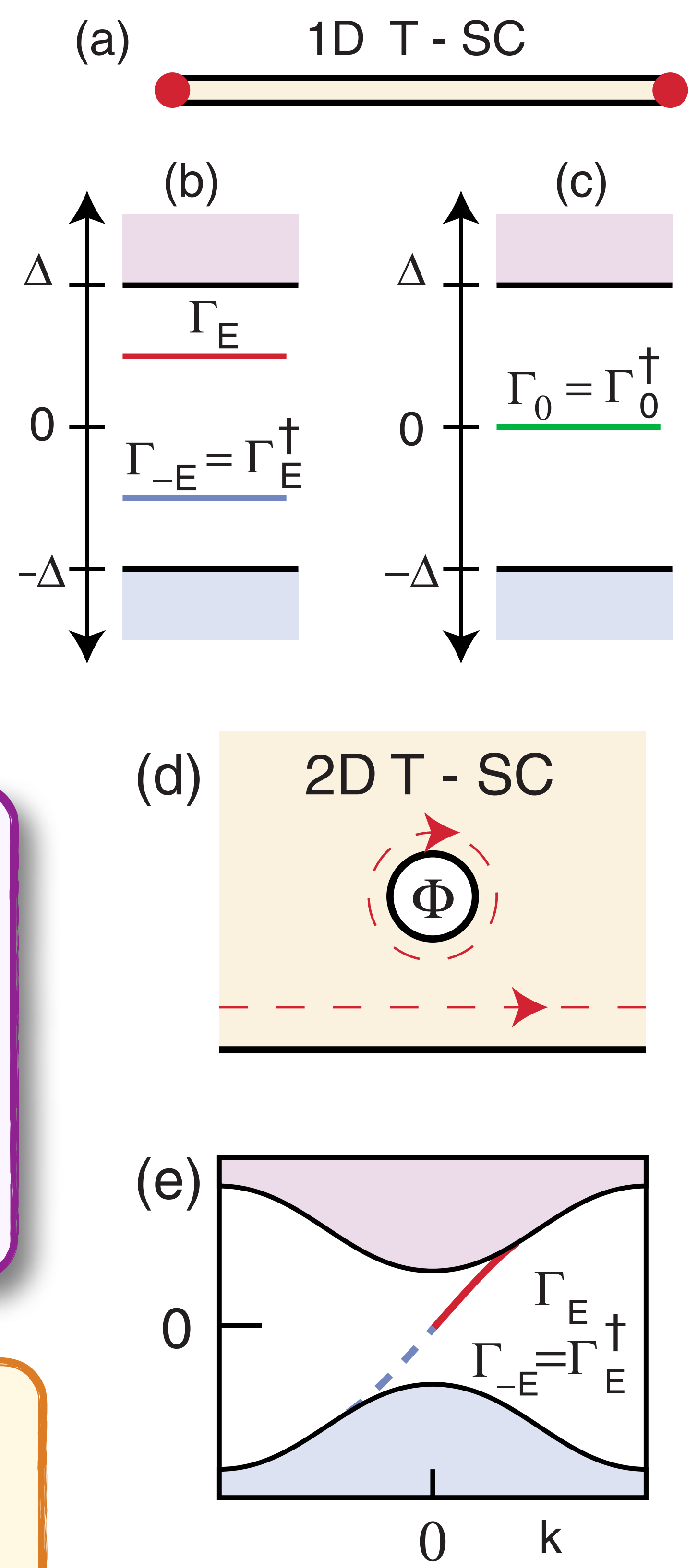

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Bulk-edge correspondence in topologically nontrivial state of matter

Role of topology in 2D spinless p-wave superconductor

Hamiltonian in k-space:

$$H = \frac{1}{2}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\Psi^\dagger(\mathbf{k})\mathcal{H}(\mathbf{k})\Psi(\mathbf{k}), \qquad \epsilon(k) = \frac{k^2}{2m} - \mu$$

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} \epsilon(k) & \tilde{\Delta}(\mathbf{k})^* \\ \tilde{\Delta}(\mathbf{k}) & -\epsilon(k) \end{pmatrix} \qquad \tilde{\Delta}(\mathbf{k}) = i\Delta e^{i\phi}(k_x + ik_y)$$

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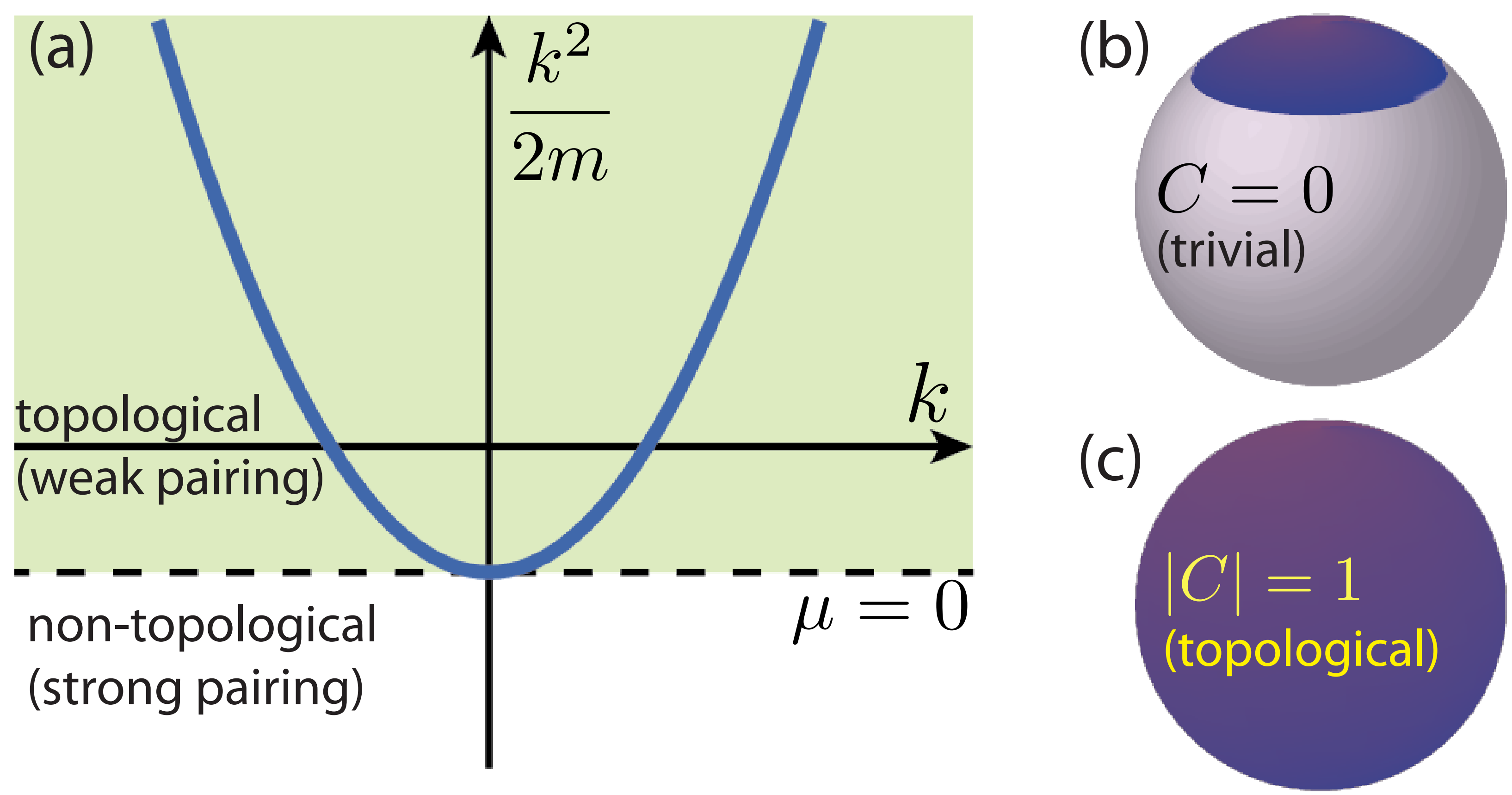
Chern number

$$C = \int \frac{d^2\mathbf{k}}{4\pi}[\hat{\mathbf{h}}\cdot(\partial_{k_x}\hat{\mathbf{h}}\times\partial_{k_y}\hat{\mathbf{h}})]$$

$C \neq 0$ nontrivial topology



Vortex core states in 2D spinless p-wave superconductor

$$H_{\text{edge}} = \frac{1}{2}\int d^2\mathbf{r}\Psi'^{\dagger}(\mathbf{r})\mathcal{H}(\mathbf{r})\Psi'(\mathbf{r}),$$

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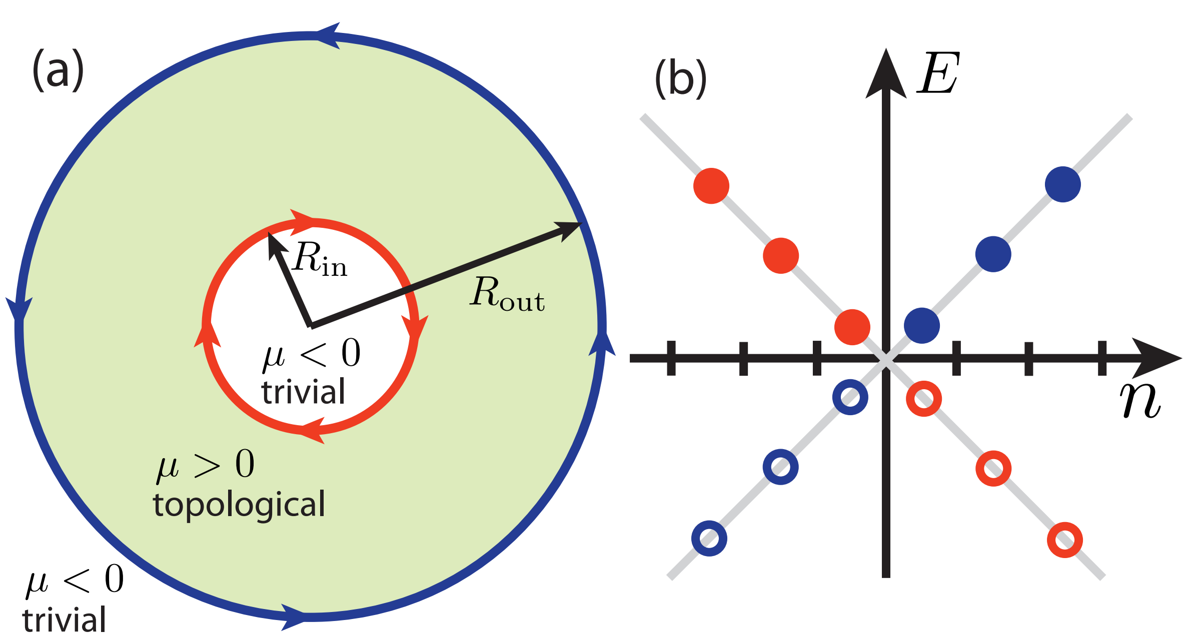
$E_{\rm out/in} = \pm n\Delta/R_{\rm out/in}$ for half-integer n (*anti-periodic boundary condition*)



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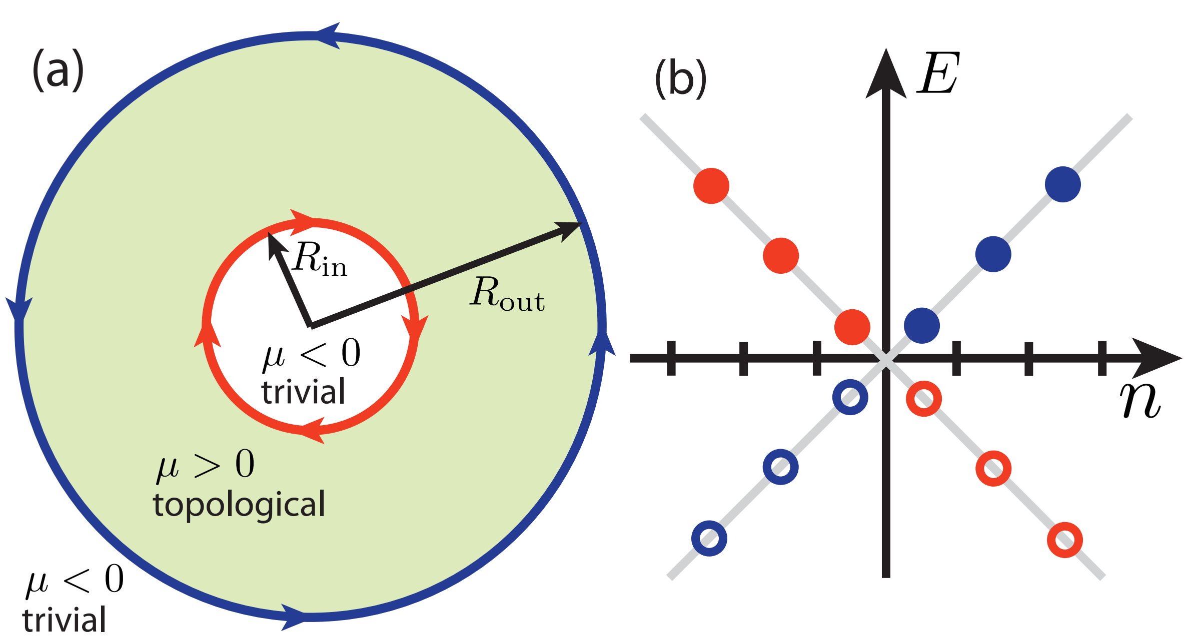
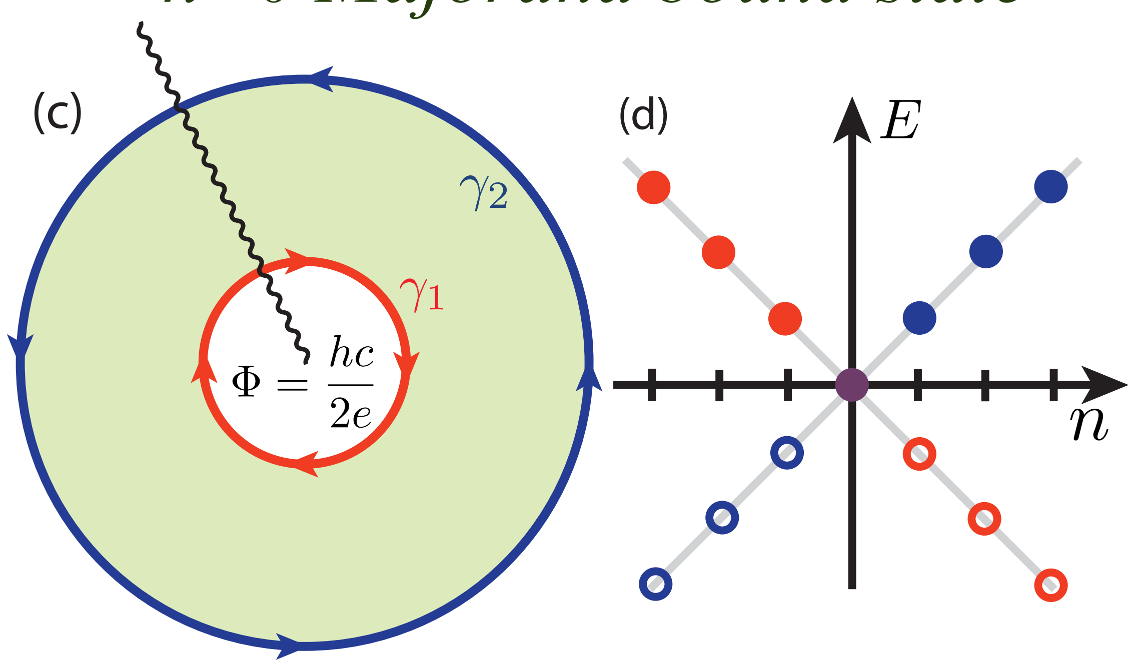
A vortex carrying a quantum flux $\Longrightarrow$ periodic boundary condition
energies with integer n
n=0 Majorana bound state





Physical realization: Artificial topological superconductor

Superconductor-topological insulator interface

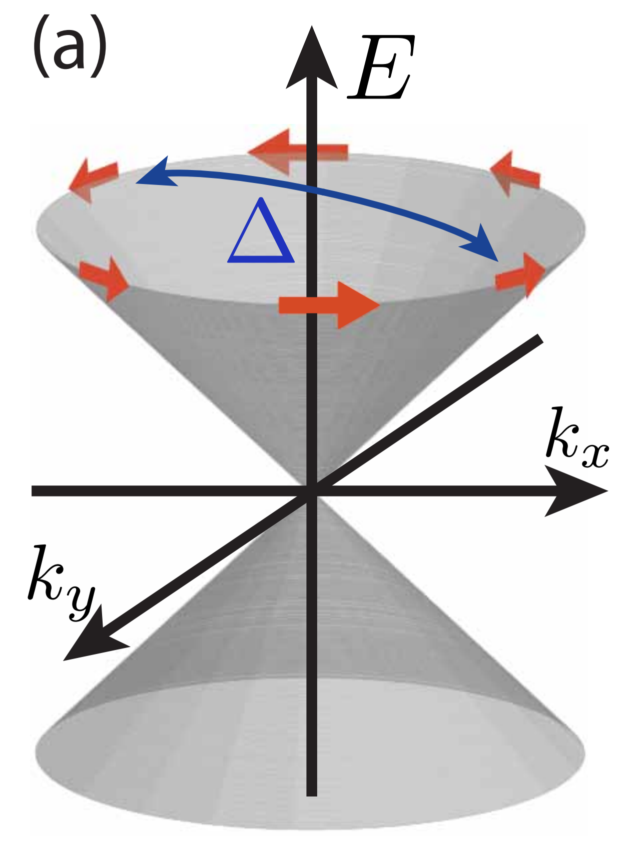
$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = (v_F \mathbf{k}\cdot\boldsymbol{\sigma} - \mu)\tau_z + \Delta_0$$

defining new fermionic operator: exercise 3

$$\psi_k = (c_{k\uparrow} + e^{i\theta_{\mathbf{k}}} c_{k\downarrow})/\sqrt{2} \qquad (\tan\theta_{\mathbf{k}} = k_y/k_x)$$

$$\tilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = (v_F|\mathbf{k}| - \mu)\tau_z + \Delta_0(e^{i\theta_{\mathbf{k}}}\psi_k^\dagger \psi_{-k}^\dagger + H.c.)$$

Similar to the spinless $p + ip$ superconductor



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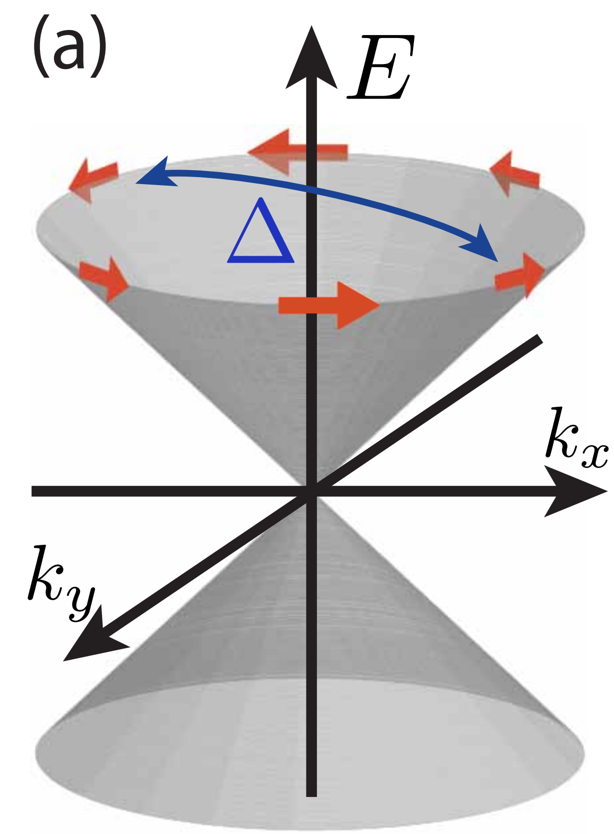
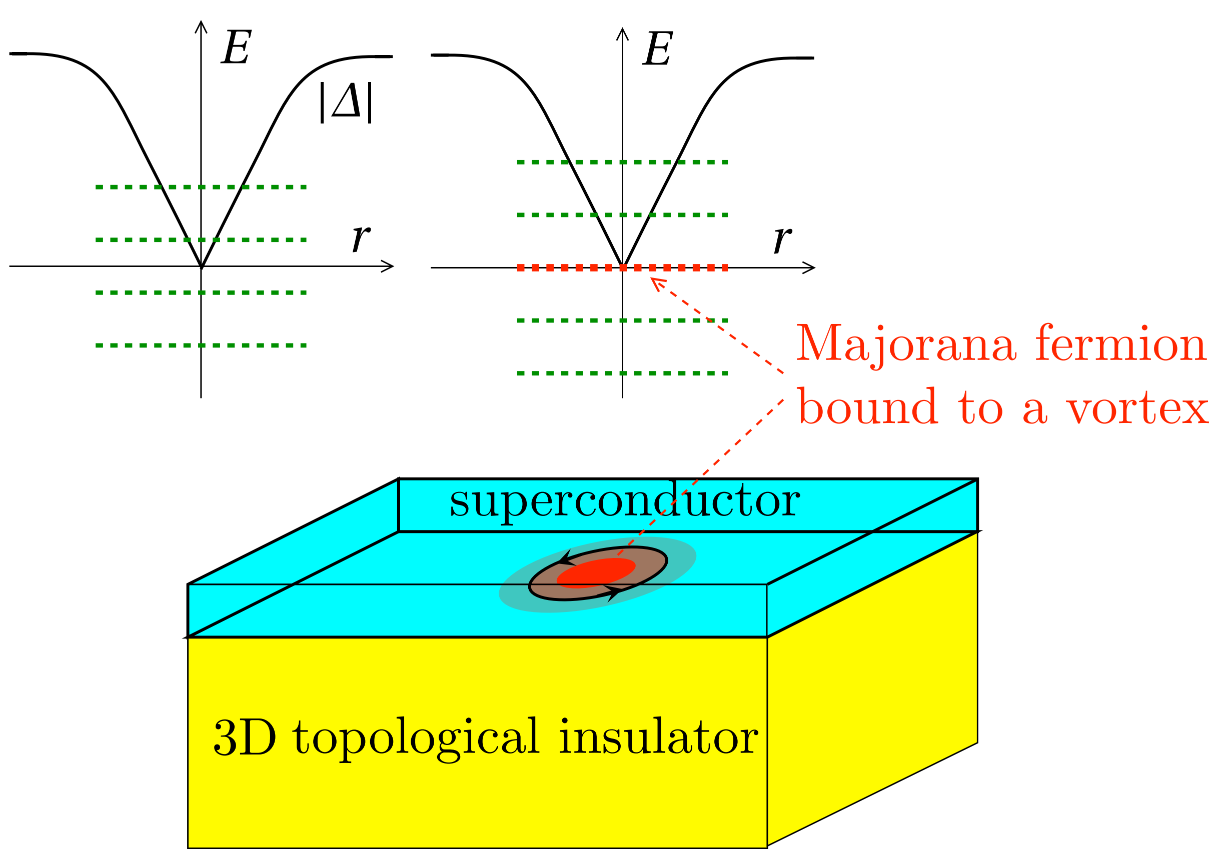
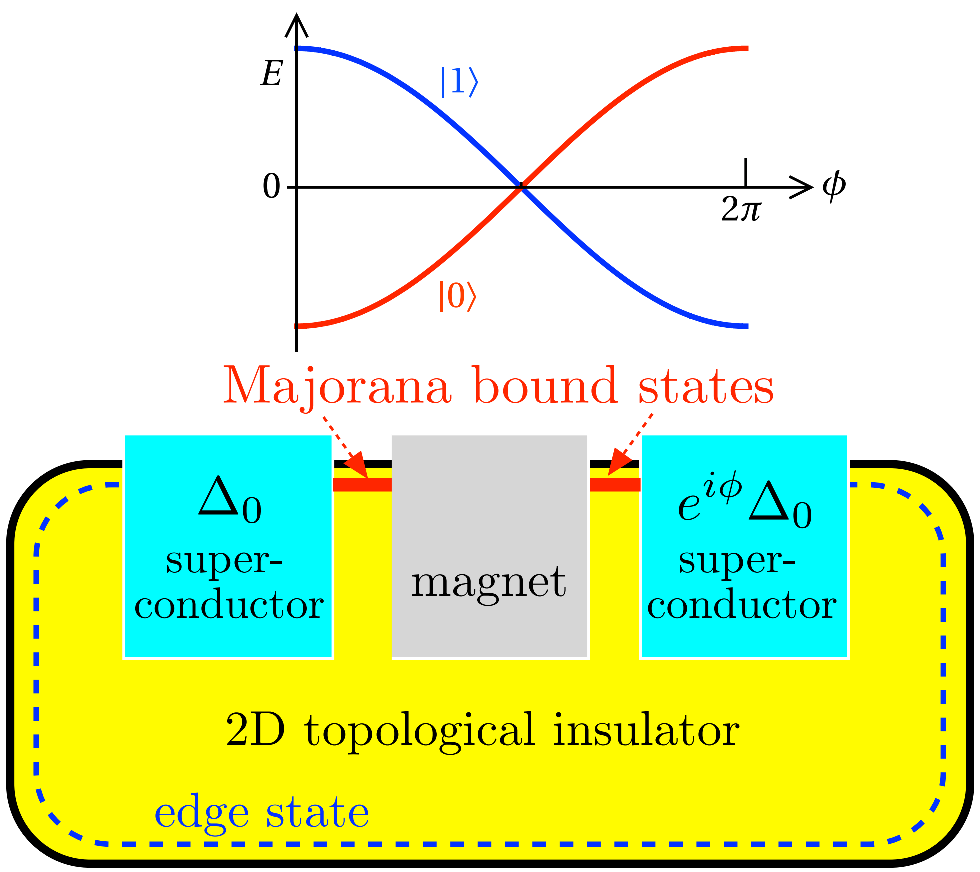




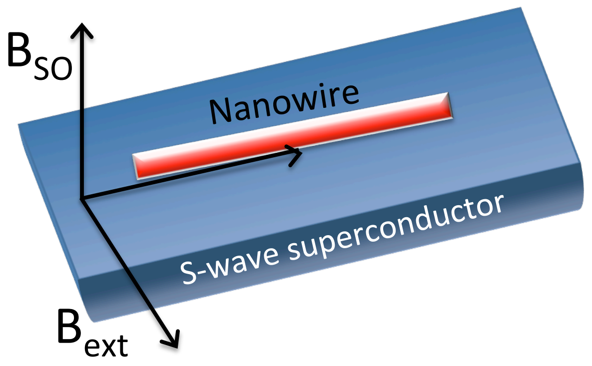


Physical realization: Nanowire with Rashba Spin-orbit

Nanowire on top of a superconductor

$$H = \int dy \Psi^\dagger(y)\mathcal{H}\Psi(y) \qquad \Psi^\dagger = (\psi^\dagger_\uparrow, \psi^\dagger_\downarrow, \psi_\downarrow, -\psi_\uparrow)$$

$$\mathcal{H} = [p^2/2m - \mu(y)]\tau_z + \alpha(y)p\sigma_z\tau_z + B(y)\sigma_x + \Delta(y)\tau_x$$



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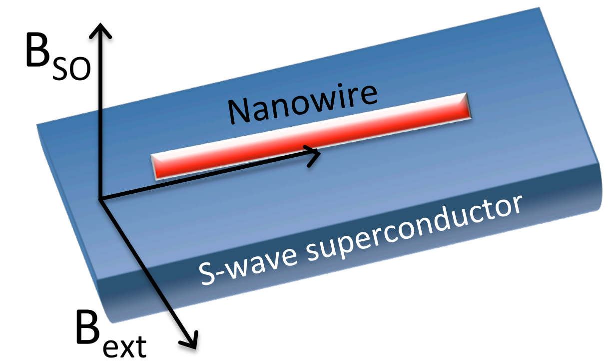
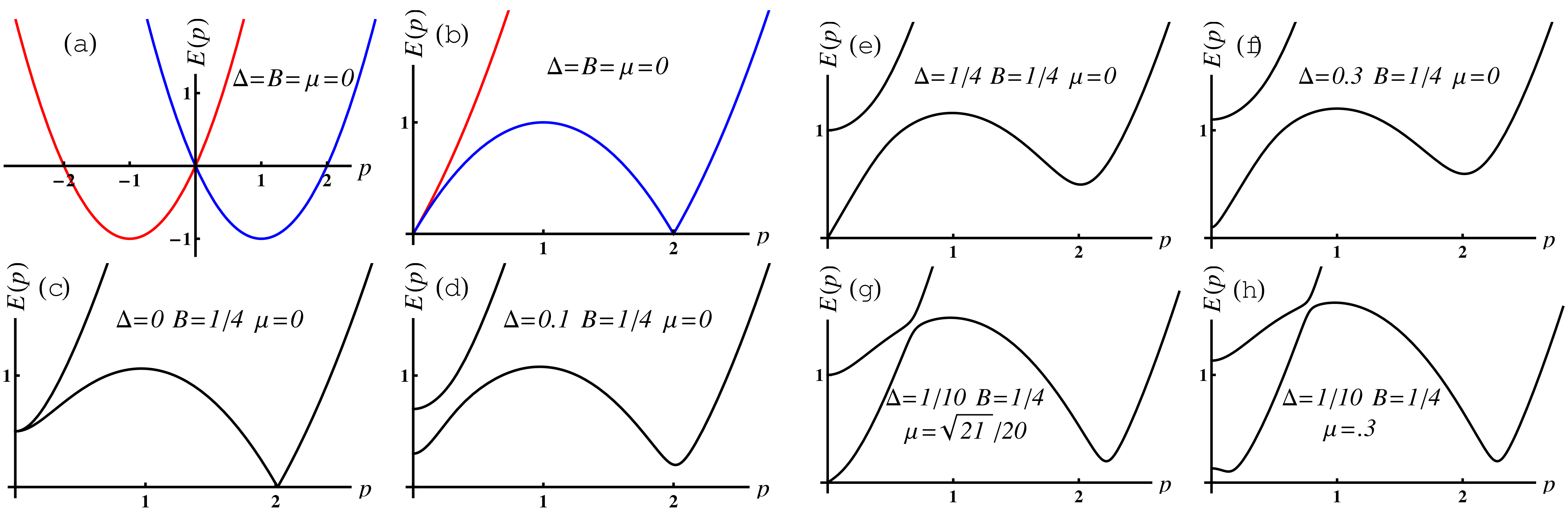


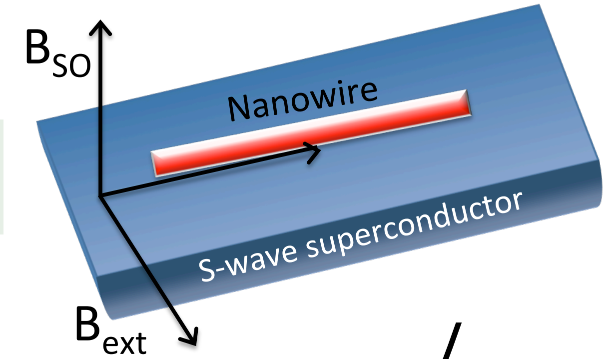
$$E_\pm^2 = B^2 + \Delta^2 + \xi_p^2 + (\alpha p)^2 \pm 2\sqrt{B^2(\Delta^2 + \xi_p^2) + (\alpha p)^2\xi_p^2}$$

exercise 4

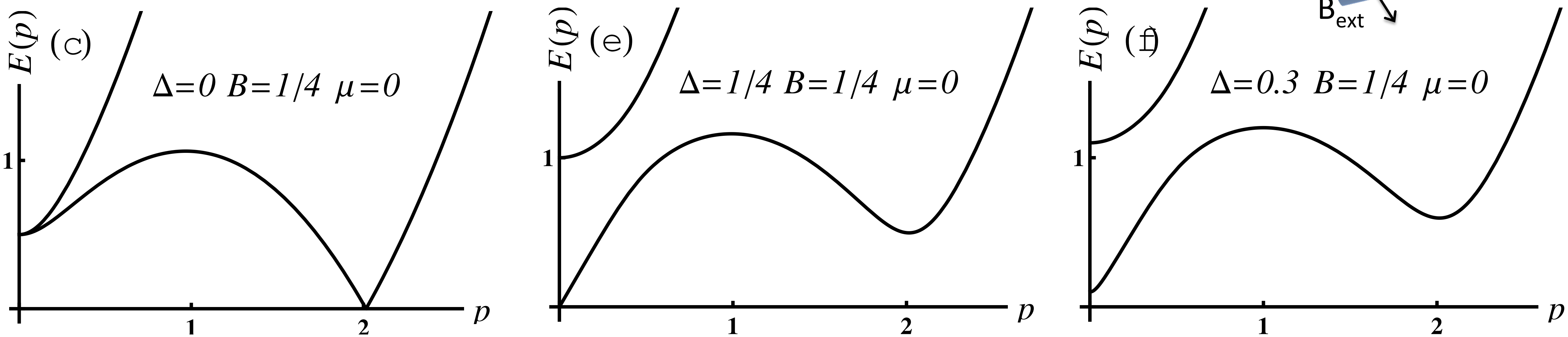


Physical realization: Nanowire with Rashba Spin-orbit

Two gaps (near $p = 0$ and $p = p_F$): E_0, E_1



$$E_0 = E(p=0) = |B - \sqrt{\Delta^2 + \mu^2}| \qquad \text{QPT at: } B^2 = \Delta^2 + \mu^2$$



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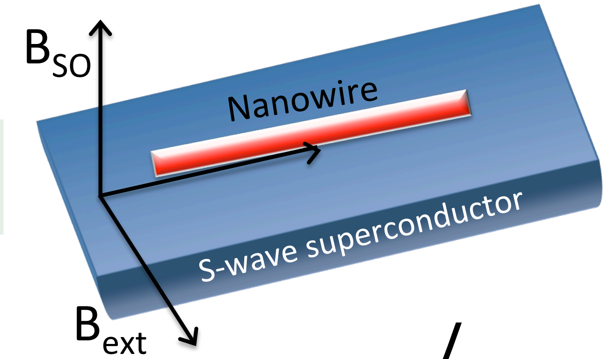

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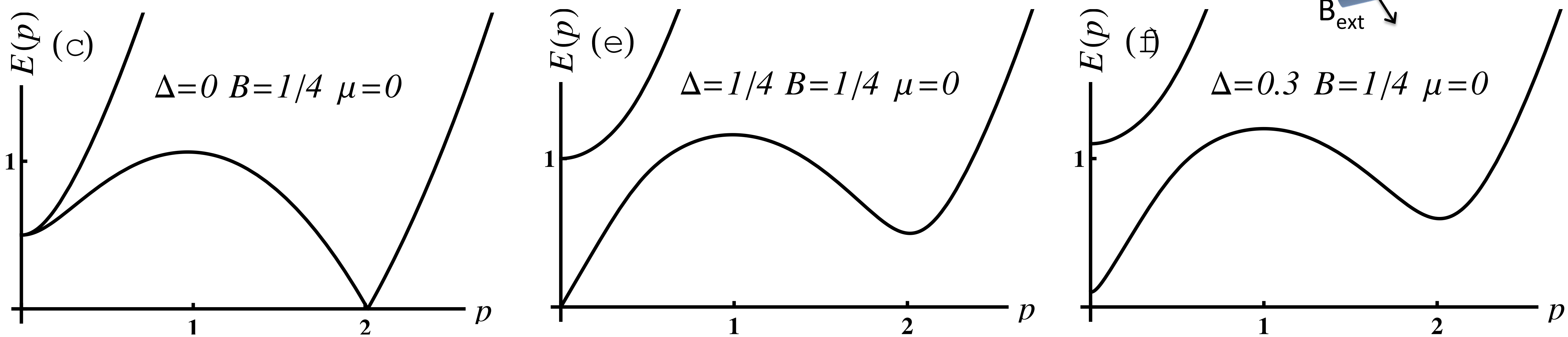
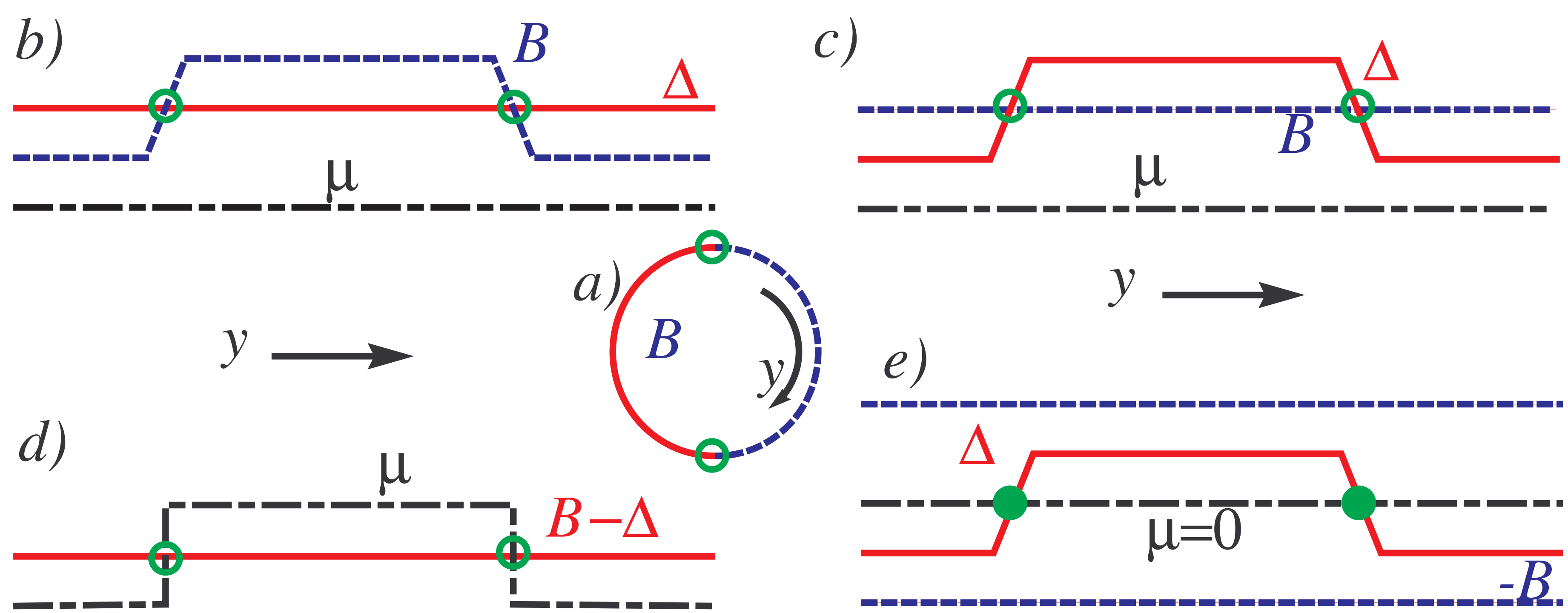


Spatially varying Δ

Spatially varying B

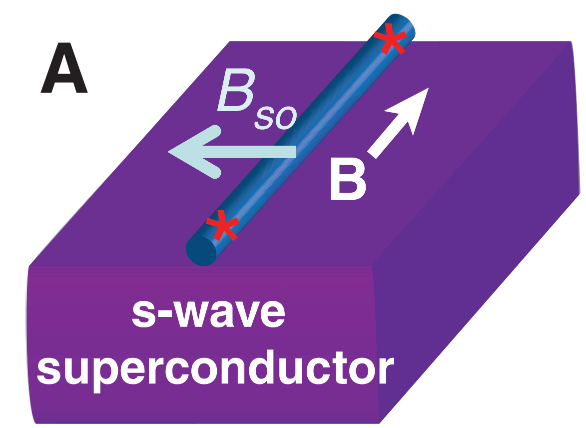
Spatially varying μ

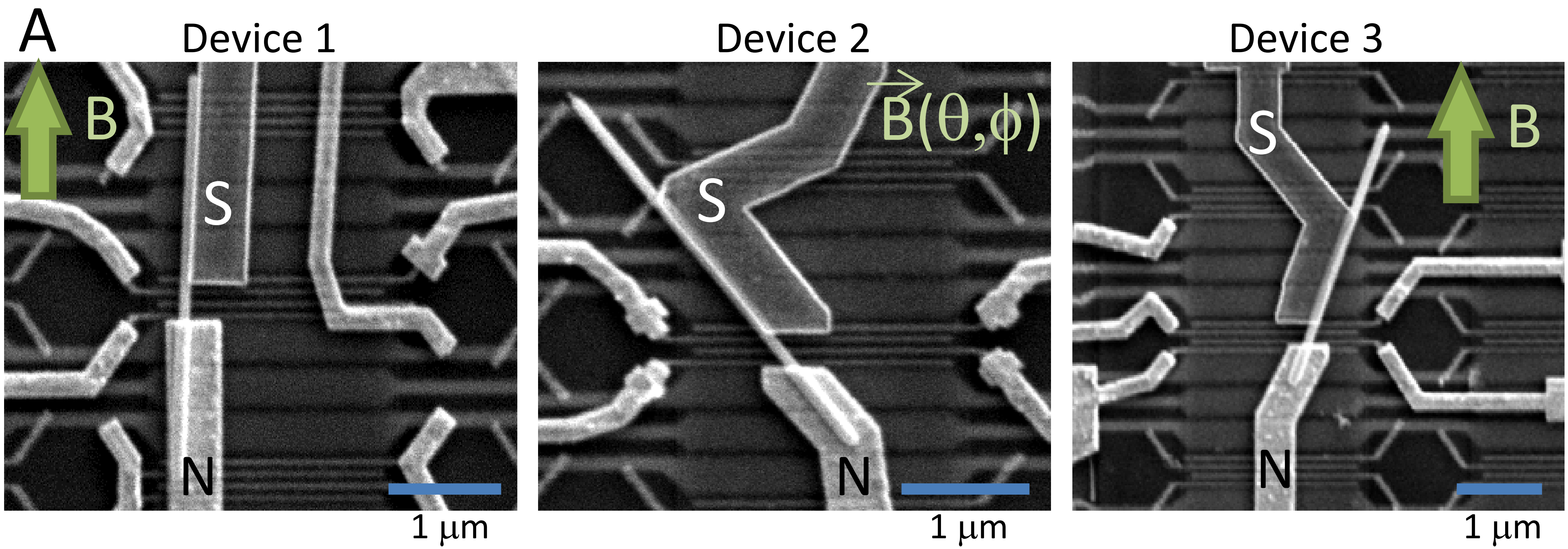


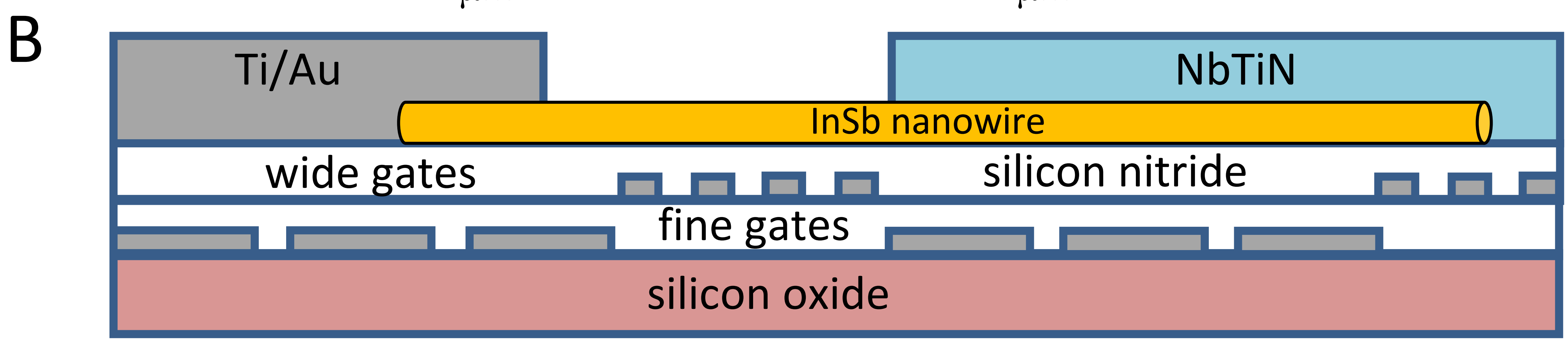
Physical realization: Experiment

Signatures of Majorana Fermions in Hybrid Superconductor-Semiconductor Nanowire Devices

V. Mourik,[1]* K. Zuo,[1]* S. M. Frolov,[1] S. R. Plissard,[2] E. P. A. M. Bakkers,[1,2] L. P. Kouwenhoven[1]†







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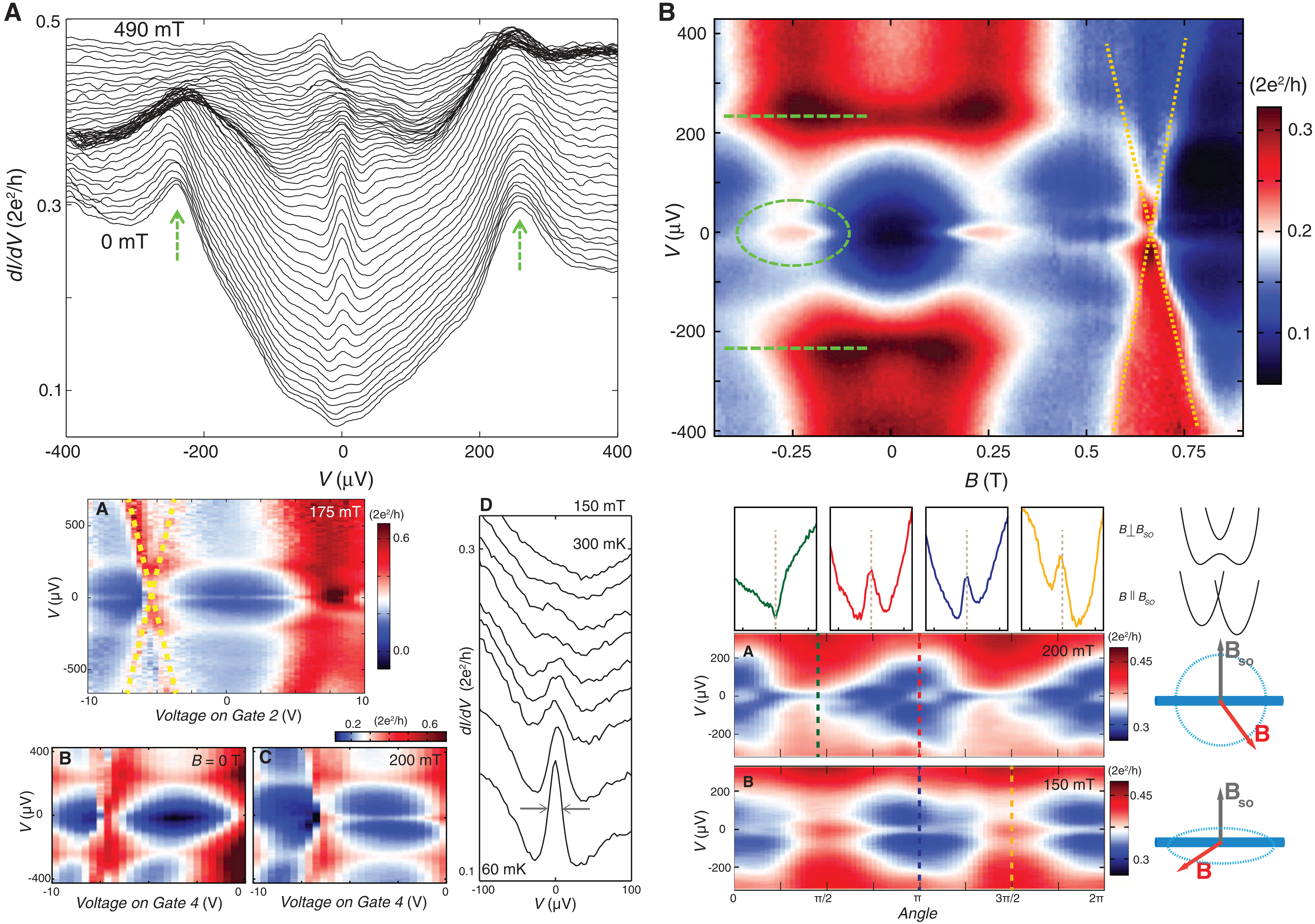

Properties: non-Abelian statistics

non-degenerate braiding: $\Psi \mapsto e^{i\alpha}\Psi$ degenerate braiding: $\Psi \mapsto U\Psi$

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Clockwise exchange of two MF

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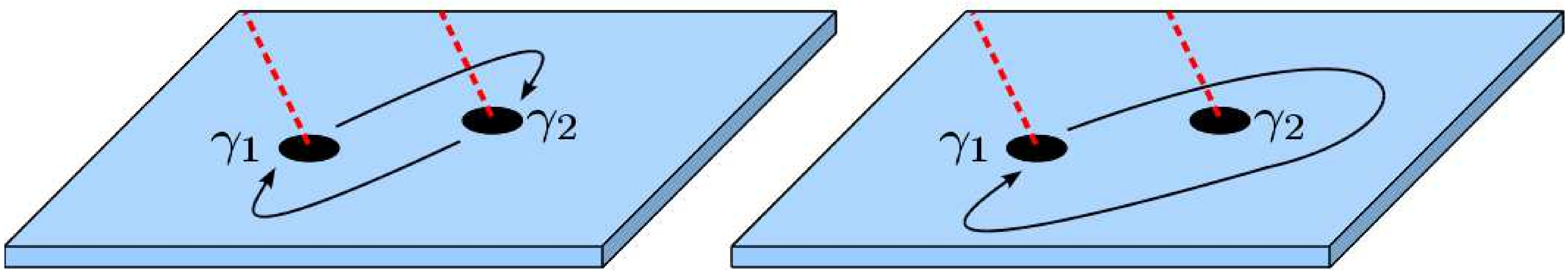
$$\gamma_1 \to -\gamma_2$$

$$\gamma_2 \to +\gamma_1$$

$$\gamma_i \to B_{12}\gamma_i B_{12}^\dagger$$

$$B_{12} = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)$$



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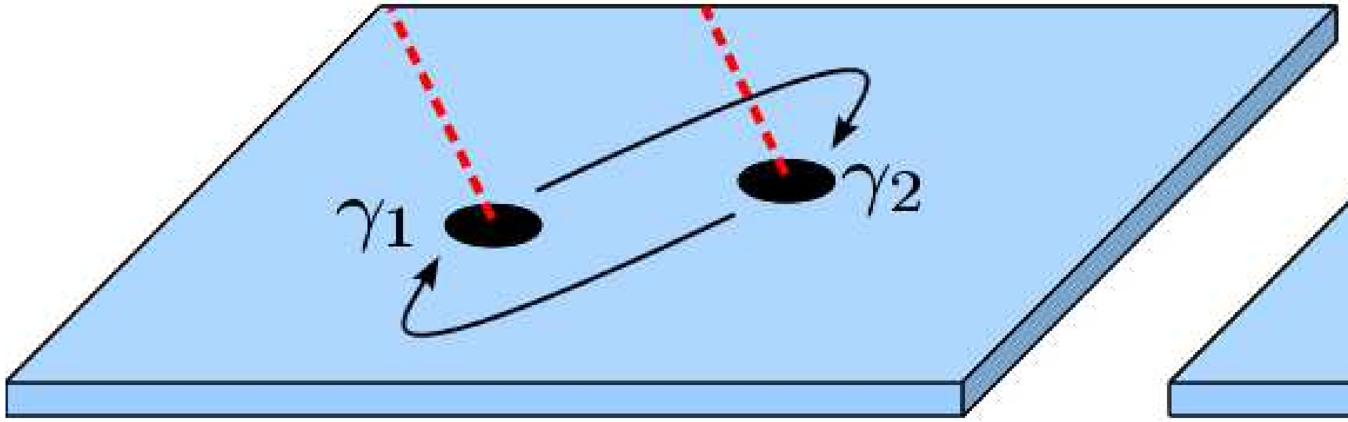
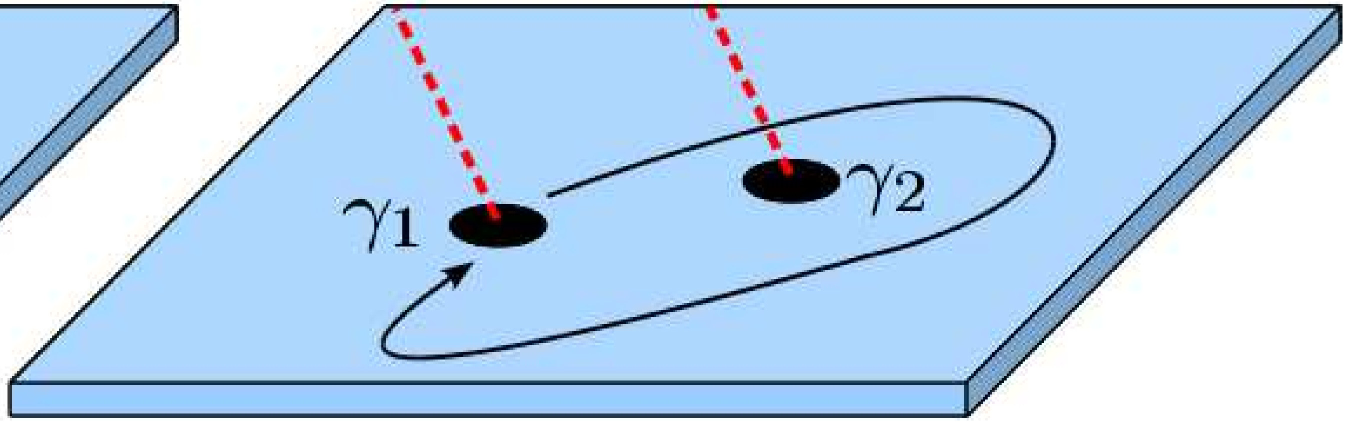
Rotating one MF around the other

$$B_{12}^2 = \gamma_1\gamma_2$$

$$\gamma_1 \to (\gamma_1\gamma_2)\,\gamma_1\,(\gamma_1\gamma_2)^\dagger = -\gamma_1$$

$$\gamma_2 \to (\gamma_1\gamma_2)\,\gamma_2\,(\gamma_1\gamma_2)^\dagger = -\gamma_2$$





Properties: non-Abelian statistics

$$B_{12}|0\rangle = \frac{1}{\sqrt{2}}(1+i)\,|0\rangle$$

$$B_{12}|1\rangle = \frac{1}{\sqrt{2}}(1-i)\,|1\rangle$$

$$|1\rangle = f_1^\dagger|0\rangle$$

$$f_1 = (\gamma_1 + i\gamma_2)/2$$

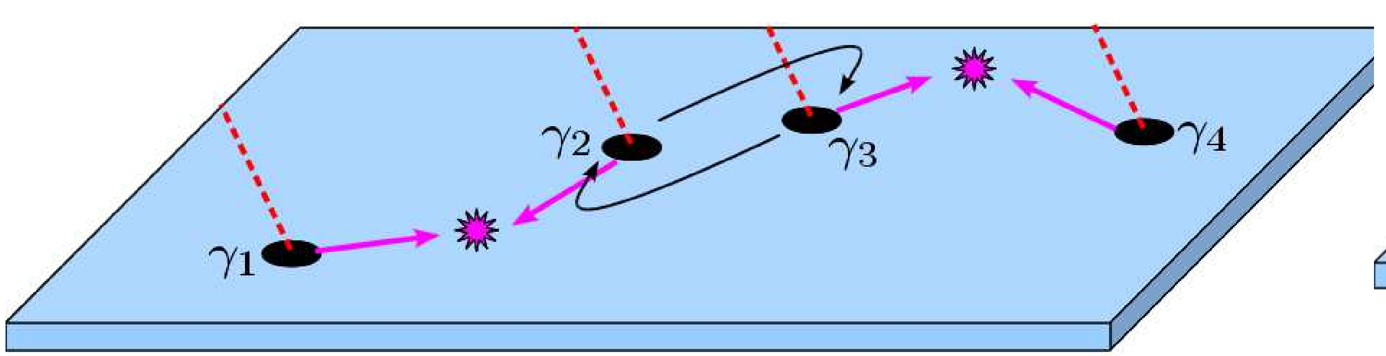
$$B_{12}|00\rangle = \frac{1}{\sqrt{2}}(1+i)\,|00\rangle,$$

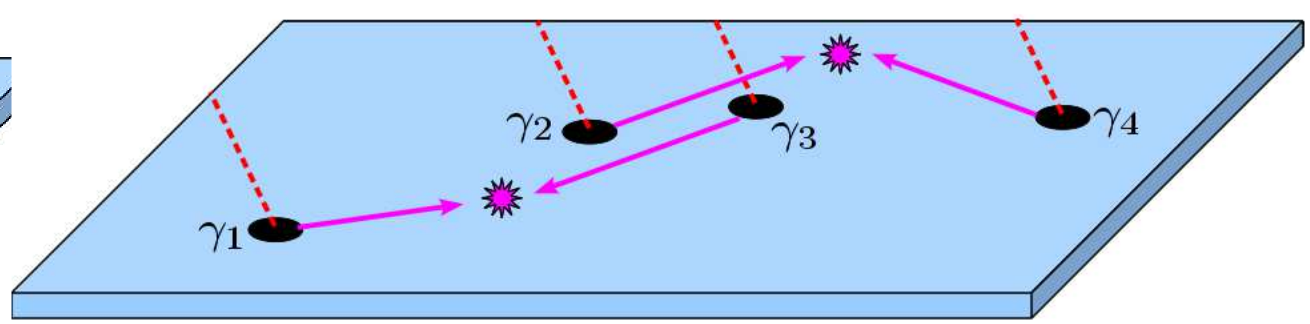
$$B_{23}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)\,,$$

$$B_{34}|00\rangle = \frac{1}{\sqrt{2}}(1+i)\,|00\rangle,$$

non-Abelian statistics
MF exchange

$$[B_{i-1,i}, B_{i,i+1}] = \gamma_{i-1}\gamma_{i+1}$$





Properties: topological quantum computing

$$|\bar{0}\rangle \equiv |00\rangle,\ |\bar{1}\rangle \equiv |11\rangle$$

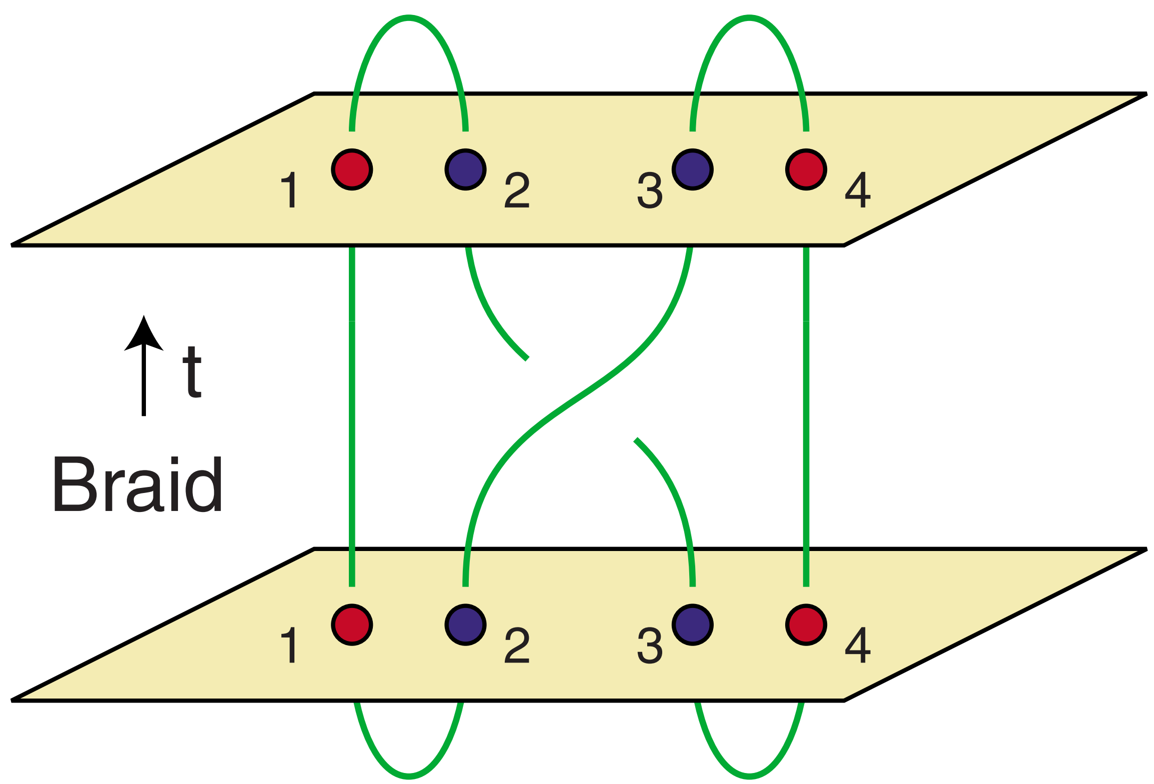
$$-i\gamma_1\gamma_2 = -i\gamma_3\gamma_4 = \sigma_z,$$

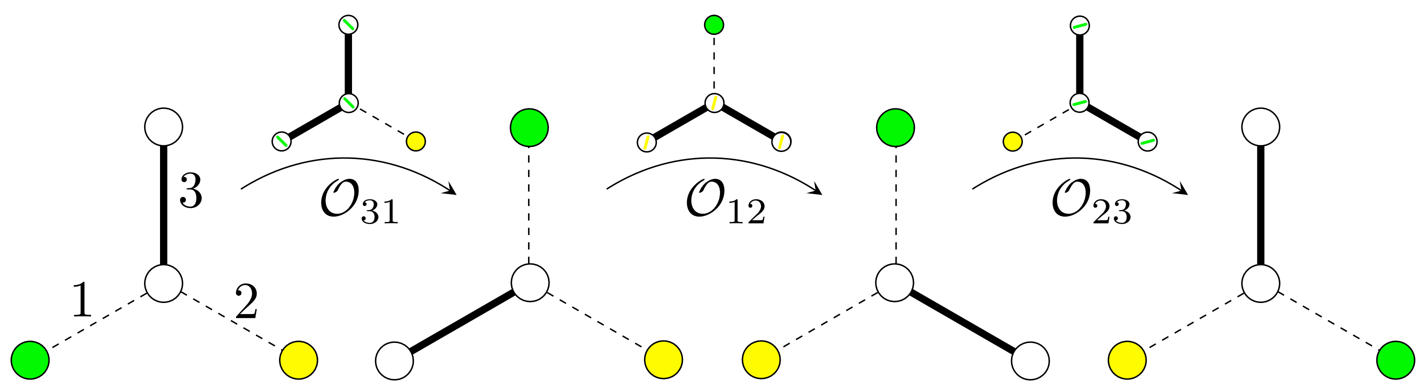
$$-i\gamma_2\gamma_3 = \sigma_x,$$

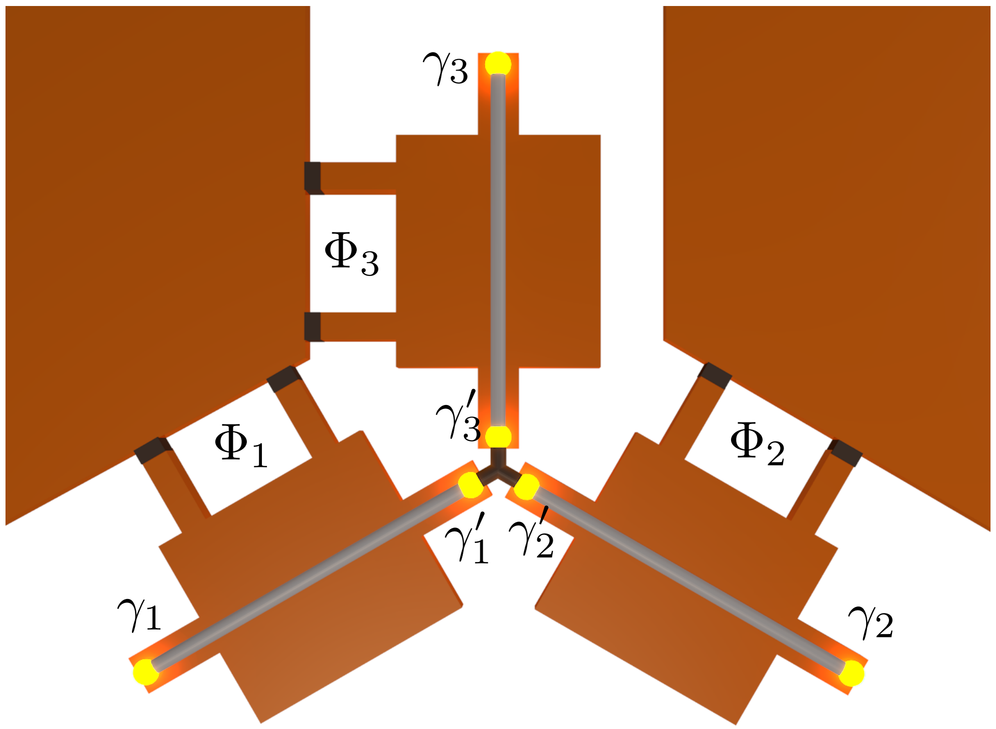
$$-i\gamma_1\gamma_3 = -i\gamma_2\gamma_4 = \sigma_y,$$

$$B_{12} = B_{34} = e^{-\frac{i\pi}{4}\sigma_z},$$

$$B_{23} = e^{-\frac{i\pi}{4}\sigma_x},$$







Majorana detection

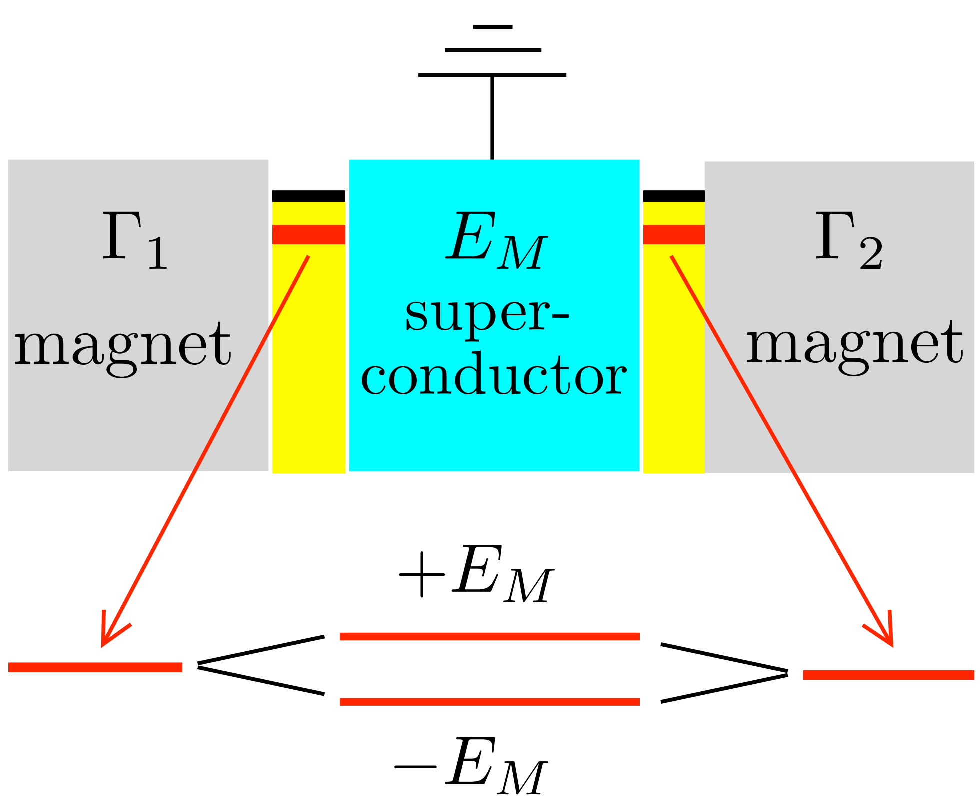
Peculiar characteristics (Tools for detection)

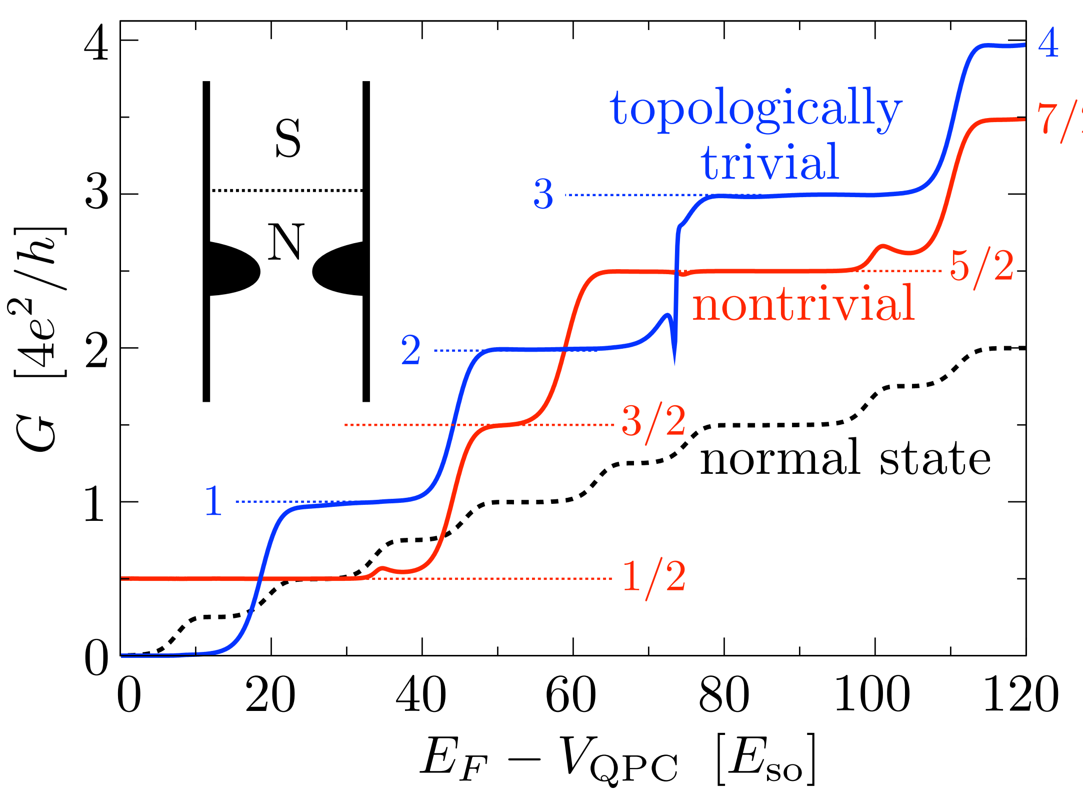
Nonlocal transport (through coupled Majoranas)

1. Direct coupling: $H_M = i\epsilon_M \gamma_1 \gamma_2$, $\epsilon_M \propto \exp(l/\xi)$
2. Coulomb interaction in mesoscopic TS islands

Fractional or 4π-periodic Josephson effect

Half-integer conductance quantization





What else?

Other detection mechanisms? (theory & experiment)

nonlocal transport and entanglement

Universal quantum computing (theory)

Non-topological operations

Coupling topological qubit with conventional qubits

Majorana in other systems (theory & experiment)

Topological superconductivity in $\mathbf{Sr_2RuO_4}$

FQHE at $\nu = 5/2$