Introductory lecture on topological insulators

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Outlines

- Introduction
  New phases of materials,
  Insulators

- Theory
  quantum Hall effect,
  edge modes,
  topological invariance

- Conclusion
Phases of matter

- In classical world we have solid, liquid and gas phases

- In quantum world we have metals, insulators, magnetisms, superconductors, etc: spontaneous symmetry breaking

Qi and Zhang PRB 2008
New phases of matter in quantum electron systems

- Quantum Hall effect  Nobel prize ’85,’98

- Super fluidity and superconductors Nobel prize ’72,’73,’87,’96’03

- Localization, disorder and quantum magnetism  Nobel prize ’70,’77,’94,’07

- Quantum phase transition, condensation  Nobel prize ’82,’01

Hasan and kane RMP, 2011
New phases of matter in quantum electron systems

- Quantum Hall effect
  QHE without magnetic field?
  Majorana fermions?

- Super fluidity and superconductors
  Sc without BCS-like paradigm?
  New routes to HTCS?

- Localization, disorder and quantum magnetism
  Spin-liquid,
  fractionalization

- Quantum phase transition, condensation
  Q-phase transition,
  New universality classes
How many insulators do we know?
Insulators

Band insulator

Peierls insulator
Insulators

Mott Insulator:
large Coulomb repulsion
electron cannot move

Anderson localization:
Impurity scattering
Insulators

What else? Topological Insulators!

TI is a band insulator characterized by a topological number and has gapless excitations at its boundaries.
Topological Insulators

TI electronic phases:
Kane & Mele  2005
Fu, Kane & Mele  2007
Moore & Balents  2007
Roy  2009

TI predictions:
Berneving, Hughes & Zhang  2006
Fu & Kane 2007

TI observations:
König et al, 2007
Hsieh et al 2008
All electronic states with an energy gap topologically equivalent to the vacuum?
Hall effect (Edwin Hall 1879)

Metals feature no gap: current can flow
Hall effect at high magnetic field

Electrons + high magnetic field $\rightarrow$ Discrete energy levels
Insulators vs IQHE

\[ \varepsilon_m = \hbar \omega_c (m + \frac{1}{2}) \]

Hall current

\[ E \neq 0 \rightarrow j_x = \sigma_{xy} (= N \frac{e^2}{h}) E_y \quad \text{NOT insulator!} \]

Hasan and Kane RMP, 2011
Bulk insulators: conduction through the edges

Conduction through the boundary

Finite Hall Conductivity

Quantum Hall Effect
K. von Klitzing et.al., (Nobel Prize, ‘85)
D.C. Tsui et.al., (Nobel Prize, ‘98)
What is the difference between a QHE states and an ordinary insulator?

\[ H(k)\{u_m(k)\} = E_m\{u_m(k)\} \]

Periodic conditions \quad \text{Sphere}

First Chern number \quad \begin{align*}
    c &= \frac{-1}{2\pi i} \int d^2k \nabla_k \times \frac{1}{N} \sum_{m=1}^{N} <u_k | \nabla_k | u_k > \\
\end{align*}

Topological invariant

\[
\sigma_{xy} = Ne^2/h \quad \begin{cases} 
    c = 0 & \text{Ordinary insulator} \\
    c = 1 & \text{IQHE} \\
\end{cases}
\]

Quantum Hall effect without magnetic field

Haldane model (1988)

- Alternating field but net magnetic field is zero
- $B(r) = 0$
- Zero gap, Dirac point
- $B(r) \neq 0$
- Energy gap
- $\sigma_{xy} = \frac{e^2}{h}$

Bulk band-structure
- Dirac point
- Massive Dirac particles

Edge structure
- Chiral fermions

Hall Transport is determined by band-structure
Edge $\leftrightarrow$ Bulk
**Edge states and the bulk boundary correspondence**

\[
H(q) = \hbar v_F \vec{q} \cdot \vec{\sigma} + m(y) \sigma_z
\]

\[
\psi_{q_x}(x, y) \propto e^{iq_x x} e^{-\int_0^y m(y') dy'/v_F} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[
\frac{dE(q_x)}{dq_x} = \hbar v_F > 0
\]

\[
\Delta n = N_R - N_L \quad \text{Bulk boundary correspondence}
\]

Kramers’ theorem

In QHE can only occur when time reversal symmetry is broken.

Spin-orbit interaction allows a different topological class when time reversal symmetry is preserved.

THEOREM: All eigenstates of a time reversal invariant Hamiltonian commute with TR operator are at least *twofold* degenerate

![Diagram showing conduction and valence bands](image)

Fu and Kane PRL, 2009

$Z_2$ Topological Insulator.
quantum spin Hall effect

\[ J_x^\uparrow - J_x^\downarrow = \sigma_{xy}^s \ E_y \]
2D topological insulator (HgTe/CdTe)

FIG. 2. (a) Bulk band structure of HgTe and CdTe; (b) schematic picture of quantum well geometry and lowest subbands for two different thicknesses. From Bernevig et al., 2006.
Band structures (HgTe/CdTe)

Effective edge Hamiltonian

Without SOI

\[ H_{\text{edge}} = A k_y \sigma^z \]

With SOI

\[ A \approx 3.6 \, \text{eV} \quad \dot{\hat{A}} \]
\[ \nu = \frac{A}{\hbar} = 5.5 \times 10^5 \, \text{m/s} \]

Bernevig, Hughes, and Zhang, Science 314, 1757 (2006)
Exp (HgTe/CdTe)

KÖnig, Wiedmann, Brne, Roth, Buhmann, Molenkamp, Qi and Zhang, Science 318, 766 (2007)
3D topological insulator

- Insulator in bulk
- Odd number of Dirac cone at surface
- Z2 topological insulator

3D topological insulator

3D TI ($\text{Bi}_{(1-x)}\text{Sb}_x$)

Hsieh, Qian, Wray, Xia, Hor, Cava and M. Z. Hasan, Nature 452, 970 (2008)
3D TI (Bi$_2$Sb$_3$)

Hsieh, Xia, Qian, Wray, Dil, Meier, Osterwalder, Patthey, Checkelsky, Ong, Fedorov, Lin, Bansil, Grauer, Hor, Cava and Hasan, Nature 460, 1101 (2009).
3D TI (different alloys)

Zhang, Liu, Qi, Dai, Fang, and Zhang, Nature Phys, 5, 438 (2009)
Theory

The quantum Hall state, a topologically non-trivial state of matter

\[ \sigma_{xy} = n \frac{e^2}{h} \]

- TKNN integer = the first Chern number.

\[ n = \int \frac{d^2 k}{(2\pi)^2} \varepsilon^{\mu\nu} F_{\mu\nu}(k) \]

- Topological states of matter are defined and described by topological field theory:

\[ S_{\text{eff}} = \frac{\sigma_{xy}}{2} \int d^2 x dt \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \]

- Physically measurable topological properties are all contained in the topological field theory, e.g. QHE, fractional charge, fractional statistics etc...

Theory

- Electromagnetic response of an insulator is described by an effective action:

\[ S_{\text{eff}} = \frac{1}{8\pi} \int d^3x dt \left( \varepsilon \vec{E}^2 - \frac{1}{\mu} \vec{B}^2 \right) \]

- However, another quadratic term is also allowed:

\[ S_{\theta} = \frac{\theta}{2\pi} \frac{\alpha}{2\pi} \int d^3x dt \vec{E} \cdot \vec{B} \]

- Physically, this term describes the magneto-electric effect. Under time reversal:

\[ \vec{E} \Rightarrow \vec{E} ; \quad \vec{B} \Rightarrow -\vec{B} \]
\[ \theta \Rightarrow -\theta \]

\[ 4\pi P = (\varepsilon - 1)E \]
\[ 4\pi M = (1 - 1/\mu)B \]

\[ 4\pi P = \alpha \theta / 2\pi B \]
\[ 4\pi M = \alpha \theta / 2\pi E \]
Effective model Hamiltonian

\[ H(k) = \epsilon_0(k)I_{4\times4} + \begin{pmatrix}
    \mathcal{M}(k) & A_1 k_z & 0 & A_2 k_-
    \\
    A_1 k_z & -\mathcal{M}(k) & A_2 k_- & 0
    \\
    0 & A_2 k_+ & \mathcal{M}(k) & -A_1 k_z
    \\
    A_2 k_+ & 0 & -A_1 k_z & -\mathcal{M}(k)
\end{pmatrix} + o(k^2) \]

\[ H_{\text{surf}}(k_x, k_y) = \begin{pmatrix}
    0 & A_2 k_-
    \\
    A_2 k_+ & 0
\end{pmatrix} \]
Conclusion

1. Many topological insulators (Strong SOI) are found.
2. Gapless spectrum in the surface (Dirac like or Majorana (SC)).
3. Modes on the surface are stable against perturbations and disorders.
4. Transport properties?
5. In the presence of any strain?
6. .......
Thanks for your attention