INTRODUCTION : Observable

$\widehat{\Delta T}(\hat{n})$: CMB temp. fluctuation is a random function on the sky

Described in terms moments / N point correlations.

St. Homogeneity on S^2 $\Rightarrow \langle \widehat{\Delta T}(\hat{n}) \rangle = 0$

$$\langle \widehat{\Delta T}(\hat{n}_1)\widehat{\Delta T}(\hat{n}_2) \rangle \equiv C(\hat{n}_1 \cdot \hat{n}_2) = C(\theta) \qquad \hat{n}_1 \cdot \hat{n}_2 = \cos\theta$$

Analogy $\langle \phi(\vec{x})\phi(\vec{x}') \rangle = \xi(r)$

$r = |\vec{x} - \vec{x}'|$

Legendre expansion

$$C(\theta) \equiv C(\hat{n}_1 \cdot \hat{n}_2) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \, P_\ell(\hat{n}_1 \cdot \hat{n}_2)$$

Analogous to 3D $\quad \xi(r) = \int dk\, k^2 P(k)\, j_0(kr)$

In fact, in flat sky approx. $\ell \gg 1$, $\theta \ll 1$

$$C(\hat{n}_1 \cdot n_2) = \sum_\ell \ell \, C_\ell \, J_0\left((\ell + 1/2)\, 2\sin\tfrac{\theta}{2}\right)$$

$$P_\ell(\cos\theta) \xrightarrow{\ell \gg 1,\ \theta \ll 1} J_0\left((\ell+1/2)\, 2\sin\theta/2\right)$$

Flat sky polar coords around N pole

$$r = 2\sin\theta/2 \qquad \varphi \to \varphi, \quad k \doteq (\ell + 1/2)$$

$$\Rightarrow \quad C(r) \equiv \xi_{2D}(r) = \int dk\, k\, P_{2D}(k)\, J_0(kr)$$

"Fourier": Spherical Harmonic rep.

$$\widehat{\Delta T}(\hat{n}) = \sum \hat{a}_{\ell m} Y_{\ell m}(\hat{n})$$

S.I. (St. Homogeneous on S^2)

$$\langle \widehat{\Delta T}(\hat{n}_1)\, \Delta T(\hat{n}_2) \rangle = \langle \widehat{\Delta T}(R\hat{n}_1)\, \widehat{\Delta T}(R\hat{n}_2) \rangle$$

$$\langle \Delta T(\hat{n}_1)\, \Delta T(\hat{n}_2) \rangle \equiv C(\hat{n}_1 \cdot \hat{n}_2)$$

$$\sum_{\substack{\ell_1 m_1 \\ \ell_2 m_2}} \langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \rangle Y_{\ell_1 m_1}(\hat{n}_1) Y^*_{\ell_2 m_2}(\hat{n}_2) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell\, P_\ell(\hat{n}_1 \cdot \hat{n}_2)$$

$$= \sum_\ell C_\ell \sum_m Y_{\ell m}(n_1)\, Y^*_{\ell m}(\hat{n}_2)$$

Integrating over $\hat{n}_1$ & $\hat{n}_2$ after multiplying with $Y^*_{\ell_1 m_1}(n_1)\, Y_{\ell_2 m_2}(\hat{n}_2)$

$$\Rightarrow \quad \langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \rangle = C_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$$

C_ℓ Estimation given observed $\widetilde{\Delta T}(\hat{n})$

$$\tilde{a}_{\ell m} = \int d\Omega_{\hat{n}}\, \widetilde{\Delta T}(\hat{n})\, Y_{\ell m}(\hat{n})$$

Pseudo C_ℓ Estimator:

$$\tilde{C}_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$$

$$\mathrm{var}(\tilde{C}_\ell) = \frac{2}{2\ell+1} \left[C_\ell^S + C_\ell^N B_\ell^{-2} \right]^2$$

CMB : Phase-space distribution function

$f_A(\vec{x}, \vec{p}, \tau)$: No. of particles of species A in a given differential vol. in phase sp

$$dN = g_A f_A(\vec{x}, \vec{p}, \tau)\, dx_1 dx_2 dx_3\, dp_1 dp_2 p_3$$

Stress-Energy tensor for a given species (degeneracy g_A

$$T^{\mu}_{\nu}(\vec{x}, \tau) = g_A \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3 \sqrt{-g}} \frac{p^\mu p_\nu}{p_0} f_A(\vec{x}, \vec{p}, \tau)$$

remember, on-mass shell $g_{\mu\nu} p^\mu p^\nu = m^2$

Fluid approx. (no net mom. fluxes)

$$\rho \equiv T_{00} = g_A \int \frac{d^3p}{(2\pi)^3} f_A(\vec{x}, \vec{p}, \tau)\, E(p)$$

$p^0 = E(p) = \sqrt{p^2 + m^2}$

$$p \equiv T_{ii} = g_A \int \frac{d^3p}{(2\pi)^3} f_A(\vec{x}, \vec{p}, \tau) \frac{p^2}{3E(p)}$$

$f_A(p, \hat{p}, \vec{x}, \tau)$ $\quad$ $p = |\vec{p}| = \sqrt{E^2 - m^2}$ is equivalent label to Energy E

For homo. & isotropic distribution

$$f_A(p, \hat{p}, \vec{x}, \tau) \equiv f_A(p, \tau) = f_A(E, \tau)$$

$$f^{(0)}(E,\tau) \begin{cases} f_{BE} = 1/(e^{(E-\mu)/T} - 1) \\ f_{FD} = 1/[e^{(E-\mu)/T}+1] \end{cases}$$

CMB distribution is a black body ($\mu \simeq 0$)
(due to photon non conserving reactions)
$\mu \lesssim 9\times10^{-5}$ is also established expt. by FIRAS (Fixsen et al. 1996)

$$f^{(0)}_{CMB} = [e^{p/T}-1]^{-1} \qquad \| \text{ since } E = p \quad (m=0)$$

$$\rho_\gamma = 2\int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{p/T}-1} \times p = \frac{8\pi T^4}{(2\pi)^3} \underbrace{\int_0^\infty \frac{dx\, x^3}{e^x-1}}_{\pi^4/15}$$

$$= \frac{\pi^2}{15} T^4$$

$$\rho_\gamma \propto 1/a^4 \Rightarrow T \propto a^{-1}$$

$$T = p/\ln(f^{-1}-1)$$

In a perturbed universe, $\delta \ln a|_H = \delta T/T$

$$f(p, \hat{p}, \vec{x}, \tau) = f^{(0)}\left(\frac{p}{\cancel{T}(\cancel{T}+\Delta_T(\hat{p},\vec{x},\tau))}\right)$$

$$\vec{q} = a\vec{p}$$

$$T_0 = aT \qquad \vec{q} = q\hat{n}$$

$$f(q, \hat{n}, \vec{x}, \tau) = f_0\left(\frac{q}{1+\Delta}\right)$$

~~ΔT~~ $\Delta(\hat{n},\vec{x},\tau) \equiv \frac{\Delta T}{T}$

$$f(\vec{x}, p, \hat{p}, \tau) = \left[\exp\left\{ \frac{p}{T(\tau)\,(1 + \Delta_T(\vec{x}, \hat{p}, \tau))} \right\} - 1 \right]^{-1}$$

$$\delta f = f - f_{BE}$$

$$= \frac{\partial f_{BE}}{\partial T} (T \Delta_T)$$

$$\Rightarrow \quad \Delta_T = \left(\frac{\partial f_{BE}}{\partial \ln T} \right)^{-1} \delta f$$

Note $\quad \dfrac{\partial f_{BE}}{\partial \ln T} = - \dfrac{\partial f_{BE}}{\partial \ln p}$

$$\Rightarrow \quad \Delta_T = \left(\frac{\partial f_{BE}}{\partial \ln p} \right)^{-1} \delta f$$

Assumption

1. $\Delta_T(\hat{x}, \hat{p}, \tau)$ is independent of p is valid because Thompson scattering does not change photon momentum.
2. For $\delta \ln a$ fluctuations
   $\dfrac{\delta E}{E} = \dfrac{h\delta\nu}{\nu} = \delta \ln a \rightarrow$ change $\dfrac{\Delta T}{T}$ is independent of E.

Fourier transform

$$\hat{\Delta}_T(\vec{x}, \hat{n}, \tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \hat{\Delta}(\vec{k}, \hat{n}, \tau)$$

$$\cancel{\Delta(\vec{k}, \hat{n}, \tau) = \hat{\varphi}(\vec{k}) \frac{\hat{\Delta}(\vec{k}, \vec{k}\cdot\hat{n}, \tau)}{\varphi(\vec{k})}}$$

$$\cancel{= \hat{\varphi}(\vec{k})\, \tilde{\Delta}(k}$$

$$\hat{\Delta}(\vec{k}, \hat{n}, \tau) = \hat{\varphi}(\vec{k}) \frac{\Delta(\vec{k}, \hat{n}, \tau)}{\varphi(\vec{k}, \tau)}$$

The ratio $\dfrac{\Delta(\vec{k}, \hat{n}, \tau)}{\varphi(\vec{k}, \tau)} \equiv \tilde{\Delta}(k, \hat{k}\cdot\hat{n}, \tau)$ | Dodelson pg. 242.

The evolution of φ & Δ both depend only on $|\vec{k}|$ & $\hat{k}\cdot\hat{n}$. Two modes with same k & $\hat{k}\cdot\hat{p}$ evolve identically even if their initial amplitudes & phase are different.

$$\tilde{\Delta}(k, \hat{k}\cdot\hat{n}, \tau) = \sum_{l=0}^{\infty} (-i)^l (2l+1)\, \tilde{\Delta}_l(k, \tau)\, P_l(\hat{k}\cdot\hat{n})$$

$$= 4\pi \sum_{lm} (-i)^l\, \tilde{\Delta}_l(k, \tau)\, Y_{lm}(\hat{k})\, Y_{lm}(\hat{n})$$

$$\Theta_l = \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu)\, \Theta(\mu)$$

standard expansion in moments.

Observed $\frac{\Delta T}{T}(\hat{n}) = \Delta(\vec{x}=0, \hat{n}, \tau_0)$

$$= \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n})$$

$$\hat{\Delta}(x, \hat{n}, \tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \hat{\varphi}(\vec{k}) \times 4\pi \sum_{\ell m} (-i)^\ell \tilde{\Delta}_\ell(k,\tau) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{n})$$

$$\Rightarrow \quad a_{\ell m} = (-i)^\ell 4\pi \int \frac{d^3k}{(2\pi)^3} \hat{\varphi}(\vec{k}) \hat{\Delta}_\ell(k, \tau_0) Y^*_{\ell m}(\hat{k})$$

$$\langle a_{\ell m} a_{\ell' m'} \rangle = e^{-i\frac{\pi}{2}(\ell-\ell')} (4\pi)^2 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \hat{\varphi}(\vec{k}) \hat{\varphi}^*(\vec{k}') \rangle$$

$$\times \Delta_\ell(k, \tau_0) \Delta^*_{\ell'}(k', \tau_0) Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k}')$$

Use $\langle \varphi(\vec{k}) \varphi(\vec{k}') \rangle = P_\varphi(k) \delta(\vec{k} - \vec{k}')$

Stat. homo.

$$= e^{-i\frac{\pi}{2}(\ell-\ell')} (4\pi)^2 \int \frac{d^3k}{(2\pi)^3} P_\varphi(k) \Delta_\ell(k, \tau_0) \Delta^*_{\ell'}(k, \tau_0)$$
$$\times Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k})$$

$$= e^{-i\frac{\pi}{2}(\ell-\ell')} (4\pi)^2 \int \frac{dk}{(2\pi)^3} k^2 P_\varphi(k) \Delta_\ell(k, \tau_0) \Delta^*_{\ell'}(k, \tau_0)$$

$$\underbrace{\int d\Omega_{\hat{k}} Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k})}_{}$$

$$\langle a_{\ell m}\, a_{\ell' m'} \rangle = \left[4\pi \int \frac{dk}{k} \frac{k^3 P_\phi(k)}{2\pi^2} \left| \Delta_\ell(k, \tau_0) \right|^2 \right] \delta_{\ell\ell'} \delta_{mm'}$$

$$\Rightarrow \quad \underbrace{C_\ell}_{\text{Obeservable}} = 4\pi \int \frac{dk}{k} \underbrace{P_\phi(k)}_{\text{Early universe}} \underbrace{\left| \Delta_\ell(k, \tau_0) \right|^2}_{\text{Radiative Transport kernel}}$$

Obeservable

Early universe (Primordial power spectrum given by Inflation model)

Radiative Transport kernel (Post-recomb. universe given by Cosmological parameters)

Evolution of $\tilde{\Delta}_\ell(k, \tau)$

Need to get $\tilde{\Delta}_\ell(k, \tau_0)$ from $\tilde{\Delta}_\ell(k, \tau_s)$

$\tau_s \lesssim \tau_{rec}$

- Solve Boltzman eqn.: $\frac{df}{d\tau} = C[f]$ — Collision term.

  $f = f^{(0)} + \delta f$ (Linear order)

- However, a lot is learnt by ~~using~~ realizing that there are two distinct regimes with a "reasonably" sharp boundary

  $\tau < \tau_{rec}$ $C[f]$ is strong $\rightarrow$ fluid approx. for photon-baryon

  $\tau > \tau_{rec}$ $C[f] = 0$ — Free streaming of photons.

Fluid approx: Δ_0 & Δ_1 are non zero.

Monopole: Δ_0 ~~is di~~ $\rightarrow \frac{1}{4}\delta_\gamma$, density fluctuation

Dipole: $\Delta_1 \rightarrow$ velocity of baryons.

$\hat{n} \cdot \vec{v}_b$

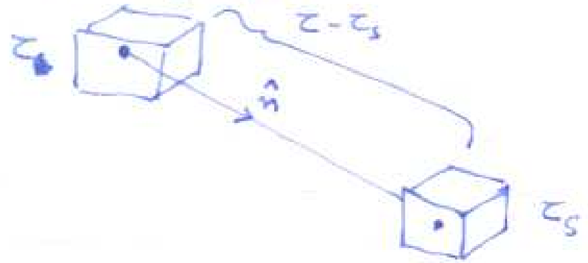
Solving the Boltzman Equation

Free streaming regime

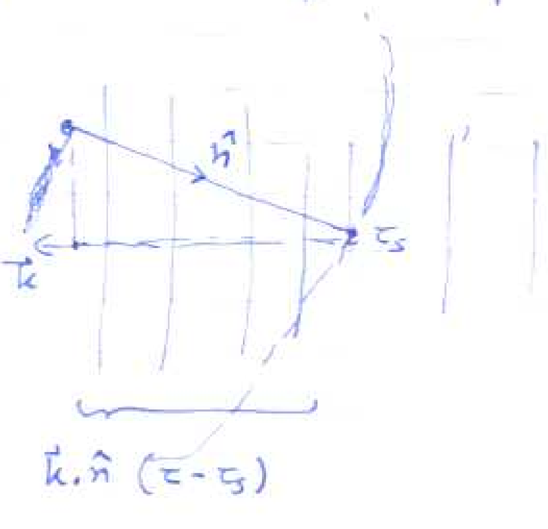
Consider $\Delta_\ell(k,\tau_s) \longrightarrow \Delta_\ell(k,\tau_0)$
where τ_s is well inside the free streaming regime. $\tau_0 < \tau < \tau_s$

$$\tilde{\Delta}(\vec{k},\hat{n},\tau) = e^{i\,\hat{n}\cdot\vec{k}\,(\tau-\tau_s)}\,\tilde{\Delta}(\vec{k},\hat{n},\tau_s)$$

Solution to B Eqn. when $C[f]=0$, but can understand this even without actually writing down the Boltz. Eqn.



$\Delta_\ell(\vec{x}_0,\hat{n},\tau)$ is built up by the same plane wave $e^{i\vec{k}\cdot\vec{x}}$ ~~sampled~~ that gives $\Delta_\ell(\vec{x}_s,\hat{n},\tau_s)$ but sampled at a different spatial point.



$$\tilde{\Delta}(|\vec{k}|, \hat{n}, \tau) = e^{i\hat{n}\cdot\vec{k}(\tau-\tau_s)} \tilde{\Delta}(|\vec{k}|\hat{k}\hat{n}, \tau_s)$$

$$\tilde{\Delta}(\vec{k}, \hat{k}\cdot\hat{n}, \tau) = \sum_\ell (-i)^\ell (2\ell+1)\, \hat{\Delta}_\ell(k,\tau)\, P_\ell(\hat{k}\cdot\hat{n})$$

$$\Rightarrow \quad \hat{\Delta}_\ell(k,\tau) = \frac{1}{2(-i)^\ell} \int_{-1}^{1} d\mu\, \tilde{\Delta}(k,\mu,\tau)\, P_\ell(\mu) d\mu$$

$\Delta\tau = \tau - \tau_s$

$$\boxed{e^{i\hat{n}\cdot\vec{k}(\tau-\tau_s)} = \sum_L (-i)^L (2L+1)\, j_L(k\Delta\tau)\, P_L(\hat{k}\cdot\hat{n})}$$

$$= \frac{1}{2(-i)^\ell} \int_{-1}^{-1} d\mu \Big[\sum_{\ell' L} (-i)^{\ell'+L} (2\ell'+1)(2L+1)\, j_L(k\Delta\tau)\, \tilde{\Delta}_{\ell'}(k,\tau_s)\, P_{\ell'}(\mu) P_L(\mu)\Big] P_\ell(\mu)$$

$$= \frac{1}{2} \sum_{\ell' L} (-i)^{\ell'+L-\ell} (2\ell'+1)(2L+1)\, j_L(k\Delta\tau)\, \tilde{\Delta}_{\ell'}(k,\tau_s) \times \int_{-1}^{1} d\mu\, P_\ell(\mu) P_{\ell'}(\mu) P_L(\mu)$$

$$\boxed{\int_{-1}^{1} d\mu\, P_\ell(\mu) P_{\ell'}(\mu) P_L(\mu) = \frac{2}{2L+1} \left[C^{L0}_{\ell 0 \ell' 0} \right]^2}$$

$$\Delta_\ell(k,\tau) = \sum_{\ell' L} (-i)^{\ell+\ell'-L} (2\ell'+1)\, j_L(k\Delta\tau)\, \tilde{\Delta}_{\ell'}(k,\tau_s) \left[C^{L0}_{\ell 0 \ell' 0}\right]^2$$

$$\Delta_\ell = \sum_{\ell'} A^{(\Delta\tau)}_{\ell\ell'} \tilde{\Delta}_{\ell'}(k,\tau_s) \qquad A_{\ell\ell'} = \sum_L (-i)^{\ell+\ell'-L} j_L(k\Delta\tau) \left[C^{L0}_{\ell 0 \ell' 0}\right]^2$$

Consider $\tilde{\Delta}_\ell(k,\tau_s) = \tilde{\Delta}_0(k,\tau_s)\,\delta_{\ell 0}$

$$\Rightarrow \tilde{\Delta}_\ell(k,\tau) = \sum_L (-i)^{\ell-L} j_L(k\Delta\tau)\,\tilde{\Delta}_0(k,\tau_s) \times \left[C^{L0}_{\ell 0 00}\right]^2$$

$$\left[C^{LM}_{\ell m 00} = \delta_{\ell L}\,\delta_{Mm} \Rightarrow C^{L0}_{\ell 0 00} = \delta_{\ell L} \right]$$ Var 8.5.1 (b)

$$\Rightarrow \tilde{\Delta}_\ell(k,\tau) = j_\ell(k\Delta\tau)\,\Delta_0(k,\tau_s)$$

$$\Rightarrow C_\ell = 4\pi \int \frac{dk}{k}\left(\frac{k^3 P_\varphi}{2\pi^2}\right) j_\ell^2(k\Delta\tau)\,|\Delta_0(k,\tau)|^2$$

After full form

Sach-Wolfe effect $\frac{\Delta \hat{T}}{T} = \frac{1}{3}\hat{\varphi}(\hat{n}, \tau_0 - \tau_{rec})$

$$\Rightarrow \Delta_0 = 1/3$$

$$\Rightarrow C_\ell = \frac{4\pi}{9}\int \frac{dk}{k}\left[\frac{k^3 P_\varphi}{2\pi^2}\right] j_\ell^2(k\Delta\tau)$$

[For HZ spectrum $\frac{k^3 P_\varphi}{2\pi^2} = A$]

$$C_\ell = \frac{4\pi}{9} A \underbrace{\int \frac{dk}{k} j_\ell^2(k\Delta\tau)}_{4/\pi} = \frac{16/9}{\ell(\ell+1)} A$$

In terms of $|\delta_k|^2$

$$C_\ell = \frac{4\pi}{9} \int \frac{dk}{k^2} |\delta_k|^2 j_\ell^2(k\Delta\tau)$$

$|\delta_k|^2 = A k^n$

$$C_\ell = \frac{4\pi A}{9} \frac{\Gamma(3-n)\,\Gamma\left((2\ell+n-1)/2\right)}{\Gamma^2\left((4-n)/2\right)\,\Gamma\left((2\ell+5-n)/2\right)}$$

$$\tilde{\Delta}_\ell(k,\tau_s) = \Delta_0\,\delta_{\ell 0} + \Delta_1\,\delta_{\ell 1}$$

$$\Rightarrow \tilde{\Delta}_\ell(k,\tau) = j_\ell(k\Delta\tau)\,\Delta_0 + \sum_L 3(-i)^{L-\ell+1} j_L(k\Delta\tau)\,\Delta_1 \left[C^{L0}_{\ell 010}\right]^2$$

$C^{L0}_{\ell 010}$ is nonzero for $\ell+1 < L < \ell-1$
& Also $L+\ell+1 = 2g$ should be even
$L = \ell-1$ & $\ell+1$

$$\tilde{\Delta}_\ell(k,\tau) = \Delta_0\, j_\ell(k\Delta\tau) + 3(-i)^2 j_{\ell+1}(k\Delta\tau)\left[C^{(\ell+1)0}_{\ell 010}\right]^2 \Delta_1 + 3(-i)^0 j_{\ell-1}(k\Delta\tau)\left[C^{(\ell-1)0}_{\ell 010}\right]^2 \Delta_1$$

Mathematica (myph.nb)

$$C^{(\ell+1)0}_{\ell 010} = (-1)^{2\ell}\sqrt{\frac{\ell+1}{2\ell+1}} \quad \& \quad C^{(\ell-1)0}_{\ell 010} = \sqrt{\frac{\ell}{2\ell+1}}$$

$$\tilde{\Delta}_\ell(k,\tau) = \Delta_0\, j_\ell(k\Delta\tau) + 3\Delta_1\left[\frac{\ell}{2\ell+1} j_{\ell-1} - \frac{(\ell+1)}{2\ell+1} j_{\ell+1}\right]$$

$$\boxed{\text{Use} \quad (2l+1)\frac{d j_l(x)}{dx} = l\, j_{l-1}(x) - (l+1)\, j_{l+1}(x)}$$

$$\Rightarrow \tilde{\Delta}_l(k,\tau) = \tilde{\Delta}_0(k,\tau_s)\, j_l(k\Delta\tau)$$
$$+ \tilde{\Delta}_1(k,\tau_s)\, \frac{3}{k}\frac{d j_l(k\Delta\tau)}{d\tau}$$

For $\tau_s \lesssim \tau_{dec}$, Baryon-photon fluid approx. holds
Hence, there is ~~at~~ only $\Delta_0(k,\tau_s)$ & $\Delta_1(k,\tau_s)$

$$\Delta_0(k,\tau_d) = \hat{\phi}(k)\left[\frac{(1+3R)}{3}\cos(k c_s \tau_d) - R\right]$$

$$\Delta_1(k,\tau_d) = \hat{\phi}(k)\frac{k c_s}{3}\left[(1+3R)\sin(k c_s \tau_d)\right]$$

Where $R = \dfrac{3}{4}\dfrac{\rho_b}{\rho_\gamma} \simeq \left(\dfrac{600}{1+z}\right)\left(\dfrac{\Omega_b h^2}{0.022}\right)$

$$C_l = 4\pi \int \frac{dk}{k}\, P_\phi(k) \left[\left(\frac{(1+3R_d)}{3}\cos(k c_s \tau_d) - R_d\right) j_l(k(\tau_0-\tau_d))\right.$$
$$\left. + (1+3R_d)\, c_s \sin(k c_s \tau_d)\, j_l'(k(\tau_0-\tau_d))\right]^2$$
$$\times e^{-k^2\sigma_d^2}$$

~~$j_\ell(k\Delta\tau)$ contribute $\ell \sim k x$~~

~~Base~~
Roughly
$j_\ell^2(x)$ contributes for $\ell \sim x$

For each range of ℓ, we can see what is important
$\ell \sim k\,\underbrace{(\tau_0 - \tau_d)}_{\Delta\tau}$

~~$\ell <<< k$~~ ℓ

* For small $\ell \lesssim 30 \sim \frac{\Delta\tau}{\tau_d}$, ~~k~~contribution comes from

$$k \sim 30(\Delta\tau)^{-1} << \tau_d^{-1}$$

$\Rightarrow \quad k c_s \tau_d << 1 \qquad \cos(k c_s \tau_d) \simeq 1 \quad \sin(k c_s \tau_d) \simeq 0$

We ~~m~~ recover the Sachs-Wolfe result.

~~* $\Delta\tau k c_s \tau_d$~~

* $\ell \gg \Delta\tau/\tau_d$ we have the acoustic oscillations.

$$\ell_p \simeq \frac{n\pi}{c_s \tau_d} \Delta\tau \quad \rightarrow \text{ density peaks \& troughs.}$$

* R is an offset creates unequal odd & even peaks

* $\ell >> \sigma_d^{-1}\Delta\tau$ Silk damping operates & damps the C_ℓ

4 — Solving the Boltzman Eqns (part 2)

$$\frac{df}{dt} = C[f]$$

- Change in no. density of ptls in a phase volume is given by the "collision" terms & "source" terms

- Consider only scalar perturbations in a flat univ. (conformal Newtonian gauge is sufficient. Note this can only do scalar perts.)

$\Psi(\vec{x},t)$ & $\Phi(\vec{x},t)$

$$ds^2 = (1+2\Psi)\,dt^2 - a^2 \delta_{ij} (1+2\Phi)\, dx^i dx^j$$

$$= a^2(\tau)\left[(1+2\Psi)d\tau^2 - (1+2\Phi)\,\delta_{ij}\,dx^i dx^j\right]$$

- Will consider f with fluctuation upto linear order in Ψ & Φ: $\Delta(\vec{x},\hat{p},\tau)$

$$f(\vec{x},p,\hat{p},\tau) = f^{(0)}(E,\tau) - p\frac{\partial f^{(0)}}{\partial p}\Delta$$

- $C[f]$ also upto linear order in f

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial x^i}}_{\text{only }1^{st}\text{ order}} \times \underbrace{\frac{dx^i}{dt}}_{(0^{th}\text{ order})} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \cancel{\underbrace{\frac{\partial f}{\partial \hat{p}}}_{\text{only }1^{st}\text{ order}}\underbrace{\frac{d\hat{p}}{dt}}_{\left(\substack{0^{th}\text{ order}\\ =0}\right)}}^{=0\ (1^{st}\text{ order})} = C[f]$$

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} = C[\Delta, f]$$

0^{th} order eqn.

$C[f^0] = 0$ by definition $f^{(0)}$ being an equilibrium distribution.
(no net change in phase-space density)

$$\frac{df^{(0)}}{dt} = 0$$

In FRW

$$\frac{dp}{dt} = -\frac{\dot{a}}{a} p$$

$$\Rightarrow \quad \frac{\partial f^{(0)}}{\partial t} = p \frac{\dot{a}}{a} \frac{\partial f^{(0)}}{\partial p}$$

$$\downarrow$$

$$\frac{\partial f^0}{\partial t} = \frac{\partial f^0}{\partial T}\frac{dT}{dt} = \left(T\frac{\partial f^0}{\partial T}\right)\frac{d\ln T}{dt}$$

for $f^{(0)}_{BE/FD}$: $T\frac{\partial f_0}{\partial T} = -p\frac{\partial f^0}{\partial p}$

$$\Rightarrow \quad \left(\frac{d\ln T}{dt} + \frac{d\ln a}{dt}\right) p \frac{\partial f^{(0)}}{\partial p} = 0$$

$$\Rightarrow \quad \frac{d\ln T}{dt} = -\frac{d\ln a}{dt}$$

$$\Rightarrow \quad T \propto 1/a$$

$\underline{1^{st} \text{ order}}$: Linear order pert.

$$f = f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Delta(\vec{x}, \hat{p}, \tau)$$

$$\left(\frac{df}{dt}\right)^{(1)} = -p \frac{\partial f^{(0)}}{\partial p} \Big[\underbrace{\frac{\partial \Delta}{\partial t} + \frac{\hat{p}}{a}\cdot\vec{\nabla}\Delta}_{\text{free streaming}} + \underbrace{\frac{\partial \Phi}{\partial t} + \frac{\hat{p}}{a}\cdot\nabla\Psi}_{\text{gravitational "source" terms}} \Big]$$

$$= -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T x_e \Big[\Delta_0 - \Delta + \hat{p}\cdot\vec{v}_b + (\underset{\substack{\uparrow \\ \text{Quadrupolar}}}{\ }) \Big]$$

n_e : electron density x_e : ionization fraction
$\bar{n}_e = n_e x_e$: free electron density, σ_T : Thomson crosssection

$$\Rightarrow \frac{\partial \Delta}{\partial t} + \frac{\hat{p}\cdot\vec{\nabla}\Delta}{a} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}\cdot\vec{\nabla}\Psi}{a} = \bar{n}_e \sigma_T \left[\Delta_0 - \Delta + \hat{p}\cdot\vec{v}_b + (\,)\right]$$

$t \to \tau$

$$\Rightarrow \frac{\partial \Delta}{\partial \tau} + \hat{p}\vec{\nabla}\Delta + \frac{\partial \Phi}{\partial \tau} + \hat{p}\vec{\nabla}\Psi = \bar{n}_e \sigma_T a \left[\Delta_0 - \Delta + \vec{p}\cdot\vec{v}_b\right]$$

optical depth to τ : $\kappa(t) = \int_t^{t_0} \bar{n}_e \sigma_T \, dt$

$$\Rightarrow \kappa(\tau) = \int_\tau^{\tau_0} d\tau \, \bar{n}_e \sigma_T a$$

$$\dot{\kappa}(\tau) \equiv \frac{d\kappa}{d\tau} = -\bar{n}_e \sigma_T a$$

visibility $g(\tau) = \dot{\kappa} e^{-\kappa}$
Prob. of photon scattering at τ

$$\Delta(\vec{x}, \hat{p}, \tau) \longrightarrow \Delta(\vec{k}, \hat{p}, \tau)$$

$$\Rightarrow \quad \dot{\Delta}(k,\tau) + i k \mu \Delta_k + \dot{\Phi}_k + i k \mu \Psi_k$$

$$= -\dot{\kappa} \left[\Delta_0 - \Delta + \mu \vec{v}_b - \frac{1}{2} \underbrace{\left(\frac{3\mu^2 - 1}{2} \right)}_{P_2(\mu)} \underbrace{(\Delta_2 + \Delta_{P2} + \Delta_{P0})}_{\Pi} \right]$$

1. $\mu = \hat{k} \cdot \hat{p}$

3. For $\bar{n}_e \sigma_T a = -\dot{\kappa} \gg H$ (small mean free path)

Can see the Δ will driven Δ_0, $\Delta_1 \to \frac{\vec{v}_b}{3}$

$\Delta_2 \to (\Delta_{P2} + \Delta_{P0})$

2. Notice evolution $\Delta(\vec{k}, \hat{p}, \tau)$ only depends on k & $\hat{k} \cdot \hat{p} \equiv \mu$

$$\Delta(\vec{k}, \hat{p}, \tau) = \hat{\Phi}(k) \, \tilde{\Delta}(k, \mu, \tau)$$

$$\tilde{\Delta}_\ell(k,\tau) = \frac{1}{2(-i)^\ell} \int d\mu \; P_\ell(\mu) \, \tilde{\Delta}(k, \mu, \tau)$$

Note $\quad \mu P_\ell(\mu) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(\mu) + \frac{\ell}{2\ell+1} P_{\ell-1}(\mu)$

Can similarly deal with $\mu^2 P_\ell(\mu)$ etc . .

Entire set of eqns.

photons

$$\dot{\Delta}_k + i k\mu \Delta = -\dot{\phi} - ik\mu\psi - \dot{\kappa}\left[\Delta_0 - \Delta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$$\Pi = \Delta_2 + \Delta_{P2} + \Delta_{P0}$$

$$\dot{\Delta}^P_k + ik\mu\Delta^P = -\dot{\kappa}\left[-\Delta^P + \frac{1}{2}(1 - P_2(\mu))\Pi\right]$$

DM

$$\dot{\delta} + ikv = -3\dot{\phi}$$

$$\dot{v} + \dot{a}/a\, v = -ik\psi$$

baryons

$$\dot{\delta}_b + ikv_b = -3\dot{\phi}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\psi + \frac{\dot{\kappa}}{R}\left[v_b + 3i\Delta_1\right]$$

Massless neutrinos (Free streaming part of photons)

$$\dot{\Delta}^\nu + ik\mu\Delta^\nu = -\dot{\Phi} - ik\mu\psi$$

Can ~~deduce~~ derive a set of coupled linear eqns. for $\breve{\Delta}_\ell(k,\tau)$ (for scalar perturbations)

$$\dot{\Delta}_0 = \overbrace{-k\,\Delta_1}^{fs} + \overbrace{\dot{\phi}}^{grav}$$

$$\dot{\Delta}_1 = \overbrace{\frac{k}{3}\left[\Delta_0 - 2\Delta_2\right]}^{fs} + \overbrace{\frac{k}{3}\psi}^{grav} + \overbrace{\dot{k}\left(\frac{v_b}{3} - \Delta_1\right)}^{source}$$

$$\dot{\Delta}_2 = \frac{k}{5}\left[2\Delta_1 - 3\Delta_3\right] + \dot{k}\left[\frac{\Pi}{10} - \Delta_2\right]$$

$\ell > 2$

$$\dot{\Delta}_\ell = \frac{k}{2\ell+1}\left[\ell\,\Delta_{\ell-1} - (\ell+1)\,\Delta_{\ell+1}\right] - \dot{k}\,\Delta_\ell$$

$$\dot{\Delta}_{p\ell} = \frac{k}{2\ell+1}\left[\ell\,\Delta_{p\,\ell-1} - (\ell+1)\,\Delta_{p\,\ell+1}\right] + \dot{k}\left[-\Delta_{p\ell} + \frac{1}{2}\Pi\left(\delta_{\ell 0} + \frac{\delta_{\ell 2}}{5}\right)\right]$$

Note: $-\dot{k} \gg H$: strong coupling.

$\ell > 2$ Δ_ℓ are damped $\ell > 2$ $\Delta_\ell \longrightarrow 0$

$\ell = 1$ $\Delta_1 \longrightarrow v_b/3$

$\ell = 2$ $\Delta_2 \longrightarrow \dfrac{\Pi}{10}$

In the absence of Π

$\Delta_{p\ell} \longrightarrow 0$

Solving Δ_ℓ as integral along the line of sight.

$$\dot{\Delta}_k + (ik\mu - \dot{\kappa})\Delta_k = \underbrace{-\dot{\Phi} - ik\mu\Psi - \dot{\kappa}[\Delta_0 + \mu v_b - \tfrac{1}{2}P_2(\mu)\Pi}_{\tilde{S}}$$

source function.

$$e^{-ik\mu\tau+\kappa}\frac{d}{d\tau}\left[\Delta_k e^{ik\mu\tau-\kappa}\right] = \tilde{S}$$

$$\Rightarrow \Delta_k(\tau_0) = \Delta_k(\tau_s)e^{-ik\mu(\tau_0-\tau_s)}e^{-\kappa(\tau_s)} + \int_{\tau_s}^{\tau_0} d\tau\, \tilde{S}(k,\tau)\, e^{ik\mu(\tau_0-\tau)-\kappa(\tau)}$$

If choose $\tau_s << \tau_{dec}$ $\kappa(\tau_s)$ is large, hence the first term vanishes.

$$\Rightarrow \Delta(k,\mu,\tau_0) = \int_0^{\tau_0} d\tau\, \tilde{S}(k,\mu,\tau)\, e^{ik\mu(\tau_0-\tau)-\kappa(\tau)}$$

Source $\tilde{S}$ has only μ dependent terms. only upto linear order in μ & μ^2

μ, μ^2 can be obtained $\mu \to \frac{1}{ik}\frac{d}{d\tau}e^{ik\mu\tau}$

So one can obtain

$$\Delta(k,\mu,\tau_0) = \int_0^{\tau_0} d\tau\, S(k,\tau)\, e^{ik\mu(\tau_0-\tau)-\kappa(\tau)}$$

$$S(k,\tau) = e^{-\kappa}\left[-\dot{\phi} - \dot{\kappa}\left(\Delta_0 + \frac{1}{4}\Pi\right)\right]$$

$$+ \frac{d}{d\tau}\left[e^{-\kappa}\left(\Psi - \frac{i\, v_b \dot{\kappa}}{k}\right)\right] - \frac{3}{4k^2}\frac{d^2}{d\tau^2}\left[e^{-\kappa}\dot{\kappa}\,\Pi\right]$$

$$\Rightarrow \quad \Delta_\ell(k,\tau_0) = \int_0^{\tau_0} d\tau\, S(k,\tau)\, j_\ell(k(\tau_0-\tau))$$

Integrate Sachs Wolfe : $S(k,\tau) \longrightarrow S(\tau) \quad (k \ll H)$
$(\Psi = -\phi)$

$$\Delta_\ell = -2\int_0^{\tau_0} d\tau\; \dot{\phi}\; j_\ell(k(\tau_0-\tau))$$

$$\phi(k,\tau) = \phi(k,\tau_{init})\, F(\tau)$$

$$C_\ell^{ISW} = 4\int \frac{dk}{k} P_\phi(k)\left[\int_0^{\tau_0} d\tau \left(\frac{dF}{d\tau}\right) j_\ell(k(\tau_0-\tau))\right]^2$$

$$C_\ell = C_\ell^{surface} + C_\ell^{ISW} + C_\ell^{surface-ISW} \leftarrow \text{neglect}$$

HZ spectrum

$$C_\ell = \frac{1}{\ell(\ell+1)}\left[1 + \frac{g(\Omega_i)}{\ell}\right] \qquad g(\Omega) = \int_0^{\tau_0} d\tau \left(\frac{dF}{d\tau}\right)^2$$