Deep Inelastic Scattering and Structure Functions

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Lecture 2

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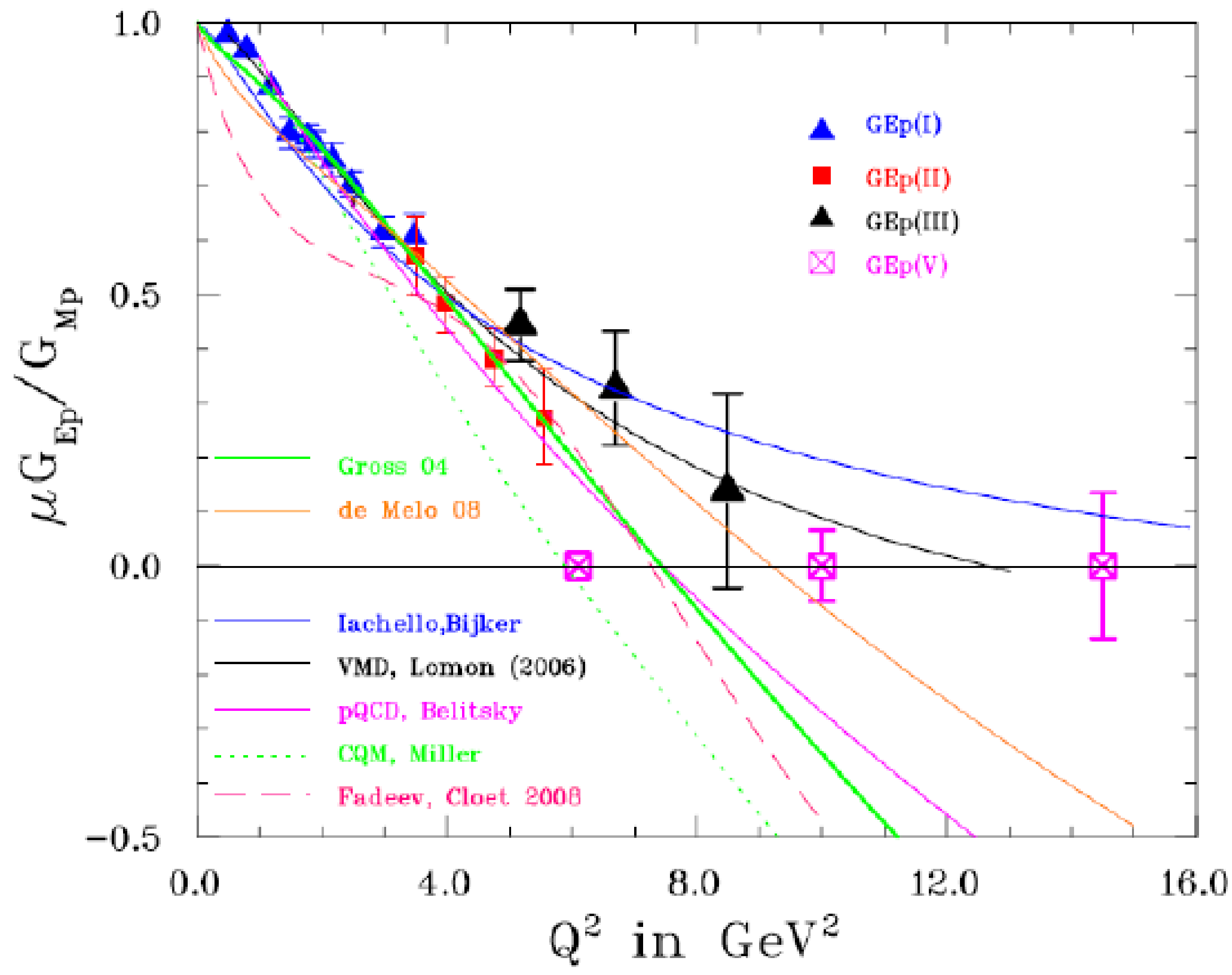

Charles F. Perdrisat, Jlab, 2010

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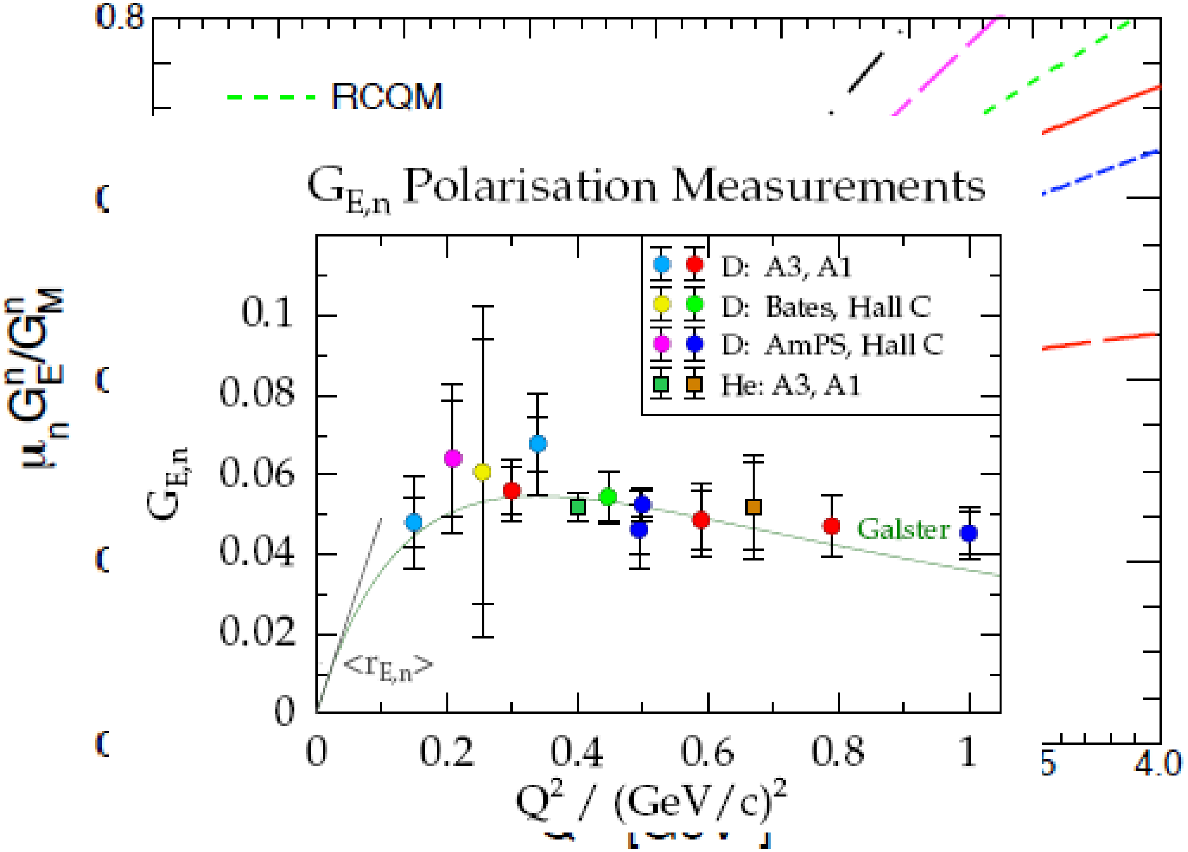
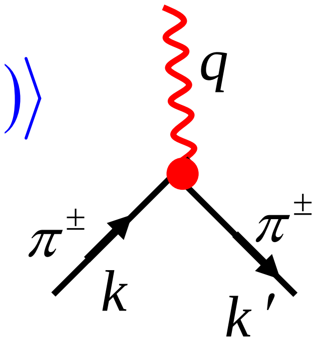


Solution to the exercise $T^{\mu}(k',k) = \langle \pi(k') | J^{\mu}(0) | \pi(k) \rangle$



The most jeneral Amplitude for interaction of charged pion with photon is

$$\pm e \varepsilon_{\mu}(q) T^{\mu}(k',k)$$

Pion spin =0 , so we have only k and k' . The most general 4- vectors that can be made are $q^{\mu} = (k'-k)^{\mu}$ and $(k'+k)^{\mu}$

The most general form of $T^{\mu} = A\,(k+k')^{\mu} + B\,(k'-k)^{\mu}$

Current conservation $q_{\mu} J^{\mu} = 0$

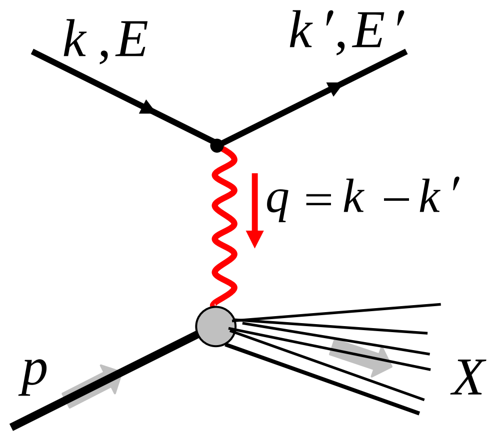
$$(k'-k)_{\mu}\left[A\,(k+k')^{\mu} + B\,(k'-k)^{\mu}\right] = 0$$

$$\Rightarrow \quad A\left(\underbrace{k'^2 - k^2}_{m_{\pi}^2 - m_{\pi}^2 = 0}\right) + B\,\underbrace{q_{\mu} q^{\mu}}_{= q^2 \neq 0} = 0 \quad \Rightarrow B = 0$$

$$\Rightarrow \quad T^{\mu}(k',k) = (k+k')^{\mu}\,\underbrace{F_{\pi}}_{A}(q^2)$$

II DEEP INELASTIC SCATTERING (DIS)

Incoming beam of lepton with Energy E scatters off a fixed hadron target. The Energy and the direction (scattering angle) of the scattered lepton is measured.



Basic diagram for DIS

No Final state hadron (denoted by X) is measured.

Lepton interacts with hadron through the exchange of a virtual photon, Z (or W, if lepton is a neutrino). The target hadron absorbs the virtual photon, to produce the final state hadrons X .

X may be the hadron itself (Elastic Scatt.) or an excited state of it. If q is large the initial hadron breaks up

Warning! various kinematic variables are used. I will chose z-axes along the incident lepton beam direction. The kinematic variables are:

Kinematic Variables

M: Mass of target

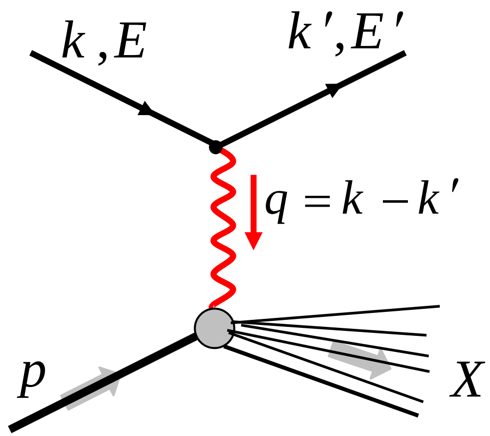
k: momentum of initial lepton

$k = (E, 0, 0, E)$ if lepton mass neglected.

Ω the solid angle into which final lepton scattered.

k' Momentum of scattered lepton

$$k' = (E', E'\sin\theta\cos\phi, E'\sin\theta\sin\phi, E'\cos\theta)$$



Basic diagram for DIS

p: Momentum of target. For fixed target, $p = (M, 0, 0, 0)$

q: momentum transfer, *i.e* the momentum of virtual photon $q = k - k'$

ν The energy loss of the lepton $\nu = E - E' = q \cdot p / M$

y: The fractional energy loss of lepton $y = \frac{\nu}{E} = \frac{p \cdot q}{p \cdot k}$

$$Q^2 \equiv -q^2 = 2EE'(1 - \cos\theta) = 4EE'\sin^2\frac{\theta}{2}$$

$$x = \frac{Q^2}{2M\nu} = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{2MEy}$$

$$\omega = \frac{1}{x}$$

x is called Bjorken variable. It is crucial in understanding of DIS, because QCD predicts that the structure functions are functions of x and independent of Q^2 in the leading order

DEFINITION: DIS is the study of lepton-hadron scattering in the Region of kinematics that $Q^2 \to \infty,\ \nu \to \infty,$ But x is Fixed and limited

The INVARIANT mass of final state hadronic system X is

$$M_X^2 = (p+q)^2 = M^2 + 2p.q + q^2 \qquad (1)$$

The invariant mass of X system must be at least equal to the mass of target nucleon. WHY?

$$\Rightarrow \quad M_X^2 \geq M^2 \ \Rightarrow\ M^2 + 2p.q - Q^2 \geq M^2 \ \Rightarrow\ \boxed{x \leq 1} \quad ☆$$

$$x = \frac{Q^2}{2\underbrace{p \cdot q}_{M\nu}}$$

Since Q^2 and ν are both positive, x must also be positive.

The lepton energy loss $\nu = E - E'$ is between zero and E, so, the physically allowed kinematic region is

$$0 \leq x \leq 1, \qquad 0 \leq y \leq 1$$

(1) can be written as

$$x = \frac{Q^2}{2p \cdot q} = 1 - \frac{M_x^2 - M^2}{2p \cdot q}$$

The value x=1 implies that $M_x^2 = M^2$ and so,

$x = 1$ corresponds to Elastic Scatt.

Any fixed hadron state X with invariant mass M_X^2 Contributes to the cross section at the value of x

$$x = \frac{1}{1+\left(M_X^2 - M^2\right)/Q^2}$$

In DIS limit $\left(Q^2 \to \infty\right)$ any state X with fixed mass M_X gets driven to $x = 1$

In particular, all nucleon resonances such as N* gets pushed to $x = 1$

The experimental measurements give the cross section as a function of final lepton energy and scattering angle $d^2\sigma/dE'd\Omega$ the results often are presented instead by giving the differential cross section as a function of $\left(x, Q^2, \phi\right)$ or $\left(x, y, \phi\right)$

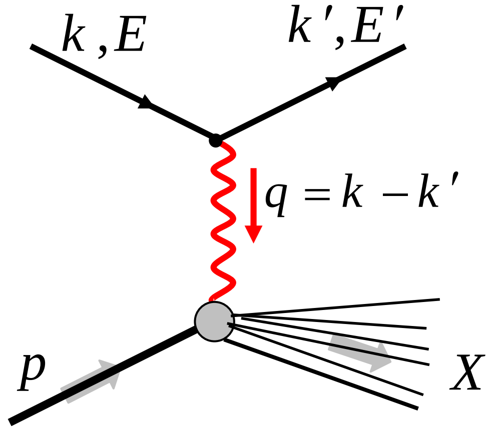
The Jacobian for converting between thses cases is

$$\frac{\partial\left(x, Q^2\right)}{\partial\left(x, y\right)} = \begin{vmatrix} 1 & 0 \\ 2MEy & 2MEx \end{vmatrix} = 2MEx = \frac{Q^2}{y}$$

$$\frac{\partial\left(x, y\right)}{\partial\left(E', \cos\theta\right)} \begin{vmatrix} 1 & \frac{-2EE'}{2M\nu} \\ -\frac{1}{E} & 0 \end{vmatrix} = \frac{E'}{M\nu}$$

The basic Feynman graph for DIS shown. The scattering Amplitude $\mathcal{M}$ is given by

$$\mathcal{M} = (-ie)^2 \left(\frac{-ig_{\mu\nu}}{q^2} \right) \langle k' | j_\ell^\mu(0) | k, s_\ell \rangle \langle X | j_h^\nu(0) | p, \lambda \rangle$$



s_ℓ : is the polarization of lepton and

λ : is the polarization of the initial hadron

For spin $1/2$ target, $\lambda = \pm 1/2$

The differential cross section is obtained from $\mathcal{M}$ by squaring it and multiplying by the phase space factors

λ : Can be chosen to be the value of spin in arbitrary direction, usually beam direction

$$d\sigma = \sum_X \int \frac{d^3k'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4(k + p - k' - p_x) \frac{|\mathcal{M}|^2}{(2E)(2M)(v_{rel} = 1)}$$

$$= \sum_x \int \frac{d^3k'}{(2\pi)^3 2E'} \frac{(2\pi)^4 \delta^4(k + p - k' - p_x)}{(2E)(2M)} \frac{e^4}{Q^4}$$

$$\times \langle p, \lambda | j_h^\mu(0) | X \rangle \langle X | j_h^\nu(0) | p, \lambda \rangle \langle k, s_\ell | j_{\ell\mu}(0) | k' \rangle \langle k' | j_{\ell\mu}(0) | k, s_\ell \rangle$$

$$d\sigma = \sum_X \int \frac{d^3k'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4(k + p - k' - p_x) \frac{|\mathcal{M}|^2}{(2E)(2M)(v_{rel} = 1)}$$

$$= \sum_x \int \frac{d^3k'}{(2\pi)^3 2E'} \frac{(2\pi)^4 \delta^4(k + p - k' - p_x)}{(2E)(2M)} \frac{e^4}{Q^4}$$

$$\times \langle p,\lambda | j_h^\mu(0) | X \rangle \langle X | j_h^\nu(0) | p,\lambda \rangle \langle k, s_\ell | j_{\ell\mu}(0) | k' \rangle \langle k' | j_{\ell\mu}(0) | k, s_\ell \rangle$$

I have used $\langle \alpha | j^\mu | \beta \rangle^* = \langle \beta | j^\mu | \alpha \rangle$ because the current is Hermitian, $j_\mu^\dagger = j_\mu$

Note that we summed over all final states X, since we do not measure them. The polarization of final state particles are also not measured and they are summed over as well. Usually we define leptonic tensor $\ell_{\mu\nu}$ and hadronic tensor $W_{\mu\nu}$

I am sure that you are familiar with the lepton tensor $\ell_{\mu\nu}$ when there is no polarization. It is

$$\ell_{\mu\nu} = \frac{1}{2} \sum_{\text{finalspin}} \left[\bar{u}(k') \gamma_\mu u(k) \right] \left[\bar{u}(k') \gamma_\mu u(k) \right]^*$$

$$= 2\left(k'_\mu k_\nu + k'_\nu k_\mu - \left(k' \cdot k - m_e^2 \right) g_{\mu\nu} \right) = 2\left(k'_\mu k_\nu + k'_\nu k_\mu + \left(q^2/2 \right) g_{\mu\nu} \right)$$

Averages over the spin states of the initial state lepton

With initial lepton polarized, we need to write it as

$$\ell^{\mu\nu} = \sum_{\text{final spin}} \bar{u}(k')\gamma^{\nu} u(k,s_{\ell})\bar{u}(k,s_{\ell})\gamma^{\mu} u(k')$$

No ½ and still easy to evaluate using

$$\sum_{\text{spin}} \bar{u}(k')\gamma^{\mu} u(k') = \not{k}' + m$$

The polarization of a spin ½ particle can be described by a spin vector, s_{ℓ} , defined in the rest frame of the particle by

$$2\mathbf{s}_{\ell} = \bar{u}(k,s_{\ell})\boldsymbol{\sigma} u(k,s_{\ell}) = \bar{u}(k,s_{\ell})\boldsymbol{\gamma}\gamma_5 u(k,s_{\ell})$$

The covarient spin 4- vector s_{ℓ}^{μ} is defined in an arbitrary Lorentz frame by boosting $s_{\ell} = (0,\mathbf{s}_{\ell})$ from the rest frame. In other words

$$2s_{\ell}^{\mu} = \bar{u}(k,s_{\ell})\gamma^{\mu}\gamma_5 u(k,s_{\ell})$$

Be ware! For spin ½ particle at rest with spin up along z axis, the spin vector is $\mathbf{s}_{\ell} = m\hat{z}$ This is a different normalization with an extra mass and it is very useful to avoid unnecessary factors of m appearing in the spin dependent cross-section.

In extereme relativistic case, mass of lepton is ignored, all formulae should be written in terms of relativistic spinors normalization 2E without any additional factor of m and thus, define a spin vector with dimension MASS which is m times the conventional definition.

With this normalization of s_ℓ, longitudinally polarized fermions in the extreme relativistic limit have $s_\ell = \mathcal{H}_\ell k$ where k is the lepton momentum and $\mathcal{H}_\ell = \pm$ is the lepton helicity.

The initial state spinor product can be written in terms of spin projection op.

$$u(k, s_\ell)\bar{u}(k, s_\ell) = (\not{k} + m)\frac{1+\gamma_5 \not{s}_\ell / m_\ell}{2}$$

And use for the final state lepton use $\sum_{\text{spin}} \bar{u}(k')\gamma^\mu u(k') = \not{k}' + m$ in $\ell^{\mu\nu}$

$$\ell^{\mu\nu} = \mathrm{Tr}(\not{k}' + m_\ell)\gamma^\nu(\not{k} + m_\ell)\frac{1+\gamma_5 \not{s}_\ell / m_\ell}{2}\gamma^\mu$$

$$= 2\left(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}\left(k \cdot k' - m_\ell^2\right)\right) - i\,\varepsilon^{\mu\nu\alpha\beta} q_\alpha s_{\ell\beta}$$

$$\approx 2\left(\underbrace{k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k \cdot k'} - i\,\varepsilon^{\mu\nu\alpha\beta} q_\alpha s_{\ell\beta}\right)$$

Spin independent part. Symmetric in $\mu\nu$

Spin dependent part. No mass dependence with our normalization. Antisymmetric in $\mu\nu$

$$\ell^{\mu\nu} = 2\left(k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - g^{\mu\nu}k \cdot k' - i\,\varepsilon^{\mu\nu\alpha\beta}q_{\alpha}s_{\ell\beta}\right)$$

Compare with unpolarized case

$$k \cdot k' = EE'(1-\cos\theta) = q^2/2$$

$$\ell_{\mu\nu} = 2\left[k'_{\mu}k_{\nu} + k'_{\nu}k_{\mu} + \left(q^2/2\right)g_{\mu\nu}\right]$$

$W_{\mu\nu}$: *The Hadronic Tensor* More complicated

We expect that strong interaction to have an important role in $W_{\mu\nu}$

$W_{\mu\nu}$ Describes the transition to all possible final states X

Amplitude for $N \to X$ transition is: $\langle X | j_{\mu} | N \rangle$

If no polarization envolved, we would average on the initial spin states and sum over the final spin states:

$$W^{\mu\nu}(N) = \frac{1}{2}\sum_{\substack{\text{final}\\ \text{spin}}}\ \sum_{\substack{\text{allstates}\\ X}} \langle X | J^{\mu} | N \rangle \langle X | J^{\nu} | N \rangle^{*} (2\pi)^4 \delta(p + q - p_X)$$

Phase space integral is included in $\sum_X$

Use Hermitian property and rewrite:

Note fore Future

The tensor

$$W^{\mu\nu}(N) = \frac{1}{2} \sum_{\substack{\text{final} \\ \text{spin}}} \sum_{\substack{\text{allstates} \\ X}} \langle X | J^{\mu} | N \rangle \langle X | J^{\nu} | N \rangle^{*} (2\pi)^{4} \delta(p + q - p_{x})$$

Equivalently can be written as

$$W^{\mu\nu}(p,q) = \frac{1}{2} \frac{1}{4\pi} \int d^{4}x \, e^{iq.x} \langle p | [J^{\mu}(x), J^{\nu}(0)] | p \rangle$$

The equivalence of these two expressions can be understood by inserting a complete set of states: $\sum_{X} |X\rangle\langle X|$

replacing $J^{\mu}(x) \to U(x) J^{\mu}(0) U^{-1}(x)$

$U(x)$ Is a translation by the vector x so that $U(x)|p\rangle = e^{ip.x}|p\rangle$

$$W^{\mu\nu}(p,q) = \frac{1}{2} \frac{1}{4\pi} \sum_{X} \int d^{4}x \, e^{iq.x} \begin{bmatrix} \langle p | J^{\mu}(x) | X \rangle \langle X | J^{\nu}(0) | p \rangle \\ -\langle p | J^{\nu}(0) | X \rangle \langle X | J^{\mu}(x) | p \rangle \end{bmatrix}$$

Now the second term does not contribute, due to energy-momentum conservation. To put it differently

Note fore Future

Translation invariance implies that

$$\langle p | J^{\mu}(x) | x \rangle = \langle p | J^{\mu}(0) | x \rangle e^{i(p-p_x)\cdot x}$$

$$\langle x | J^{\mu}(x) | p \rangle = \langle x | J^{\mu}(0) | p \rangle e^{i(p_x-p)\cdot x}$$

$$\Rightarrow$$

$$W^{\mu\nu}(p,q) = \frac{1}{2}\frac{1}{4\pi}\sum_x \left[\begin{array}{l} (2\pi)^4 \delta^4(q+p-p_x) \langle p | J^{\mu}(0) | x \rangle \langle x | J^{\mu}(0) | p \rangle \\ -(2\pi)^4 \delta^4(q+p_x-p) \langle p | J^{\nu}(0) | x \rangle \langle x | J^{\mu}(0) | p \rangle \end{array} \right]$$

Cannot be satisfied.

Now the generalization for the polarized case us trivial.

$$W^{\mu\nu}(p,q)_{\lambda'\lambda} = \cancel{\frac{1}{2}}\frac{1}{4\pi}\int d^4x\, e^{iq.x}\langle p,\lambda'|\left[J^\mu(x),J^\nu(0)\right]|p,\lambda\rangle$$

Write it it as

$$W^{\mu\nu}(p,q)_{\lambda'\lambda} = \frac{1}{4\pi}\sum_X \int d^4x\, e^{iq.x}\begin{bmatrix}\langle p,\lambda'|J^\mu(x)|X\rangle\langle X|J^\nu(0)|p,\lambda\rangle \\ -\langle p,\lambda'|J^\nu(0)|X\rangle\langle X|J^\mu(x)|p,\lambda\rangle\end{bmatrix}$$

$$\langle p,\lambda'|J^\mu(x)|x\rangle = \langle p,\lambda'|J^\mu(0)|x\rangle e^{i(p-p_x)\cdot x}$$

$$\langle x|J^\mu(x)|p\rangle = \langle x|J^\mu(0)|p\rangle e^{i(p_x-p)\cdot x}$$

Note fore Future

$$\Rightarrow$$

$$W^{\mu\nu}(p,q)_{\lambda'\lambda}$$

$$= \frac{1}{4\pi}\sum_x \begin{bmatrix}(2\pi)^4\,\delta^4(q+p-p_x)\langle p,\lambda'|J^\mu(0)|x\rangle\langle x|J^\mu(0)|p,\lambda\rangle \\ -(2\pi)^4\,\delta^4(q+p_x-p)\langle p,\lambda'|J^\nu(0)|x\rangle\langle x|J^\mu(0)|p,\lambda\rangle\end{bmatrix}$$

$$\langle x|J^\mu(x)|p\rangle = \langle x|J^\mu(0)|p\rangle e^{i(p_x-p)\cdot x}$$

$$W^{\mu\nu} = \frac{1}{2}\sum_{\substack{\text{allstates}\\ X}} \langle X|J^{\mu}|N\rangle \langle X|J^{\nu}|N\rangle^{*} (2\pi)^{4}\delta(p+q-p_{x})$$

No polarization

Summ incl. final spins

How to Parameterize $W^{\mu\nu}$

It is Lorentz Tensor. The most general Tensor should be constructed out of avalable 4- momenta q and p and available tensors $g_{\mu\nu}$ and anti-symmetric $\varepsilon_{\mu\nu\rho\sigma}$

For unpolarized case cannot use $\varepsilon_{\mu\nu\rho\sigma}$ Because, when multiplied by symm. $\ell^{\mu\nu}$ it gives zero. NO γ's Either, since summed over spins

We are left with only $p, q\ ,g_{\mu\nu}$

Out of p, q we can make: $q_{\mu}q_{\nu}$ $p_{\mu}p_{\nu}$, $p_{\mu}q_{\nu}$, $p_{\nu}q_{\mu}$ tensors only

$i\,\varepsilon^{\mu\nu\rho\sigma}p_{\rho}q_{\sigma}$ is also available but for the unpolarized case, no good

$$W_{\mu\nu} = -W_{1}g_{\mu\nu} + \frac{W_{2}}{m_{p}^{2}}p_{\mu}p_{\nu} + \frac{W_{4}}{m_{p}^{2}}q_{\mu}q_{\nu} + \frac{W_{5}}{m_{p}^{2}}\left(p_{\mu}q_{\nu} + q_{\mu}p_{\nu}\right)$$

Where did I get this? In a moment. But first a point

$$W_{\mu\nu} = -W_1 g_{\mu\nu} + \frac{W_2}{m_p^2} p_\mu p_\nu + \frac{W_4}{m_p^2} q_\mu q_\nu + \frac{W_5}{m_p^2}\left(p_\mu q_\nu + q_\mu p_\nu\right)$$

Where did I get this? In a moment. But first a point

From current conservation, we have

$$\partial^\mu j_\mu \sim q^\mu j_\mu \quad \Rightarrow \quad q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$$

Only the following tensors satisfy this condition:

$$\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right), \quad \left(p_a^\mu - \frac{(p_a \cdot q) q^\mu}{q^2}\right)\left(p_a^\nu - \frac{(p_a \cdot q) q^\nu}{q^2}\right), \quad i\,\varepsilon^{\mu\nu\rho\sigma} p_{a\rho} q_\sigma$$

Anti symmwtric, opposite parity with respect to the other two combinations. Since in E&M interaction conserves parity, this term is ignored

Let us get this tensorial combinations

Without it, we have 5 tensors left:

$$g_{\mu\nu} p_\mu p_\nu, \quad p_\mu q_\nu, \quad p_\nu q_\mu, \quad q_\mu q_\nu$$

$$W_{\mu\nu}^{(S)} = A\, g_{\mu\nu} + B\, q_\mu q_\nu + C\left(q_\mu p_\nu + q_\nu p_\mu\right) + D\, p_\mu p_\nu$$

Apply the condition $\partial^\mu j_\mu \sim q^\mu j_\mu \;\Rightarrow\; q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$

$$q^\mu W_{\mu\nu}^{(S)} = 0 \;\Rightarrow\; A q^\mu g_{\mu\nu} + B q^\mu q_\mu q_\nu + C q^\mu q_\mu p_\nu + C q^\mu q_\nu p_\mu + D q^\mu p_\mu p_\nu = 0$$
$$= A q_\nu + B q^2 q_\nu + C q^2 p_\nu + C\,(q\cdot p)\, q_\nu + D\,(q\cdot p)\, p_\nu = 0$$
$$= \left(A + Bq^2 + Cq\cdot p\right) q^\nu + \left(Cq^2 + Dq\cdot p\right) p_\nu = 0$$

p, q are independent variables $\Rightarrow$ each term must be separately zero

$\Rightarrow\Rightarrow$

$$Cq^2 + Dq\cdot p = 0 \quad\Rightarrow\quad C = -\frac{q\cdot p}{q^2}$$

$$A + Bq^2 + Cq\cdot p = 0 \;\Rightarrow B = -\frac{1}{q^2}A - \frac{1}{q^2}\left(-\frac{q\cdot p}{q^2}\right) q\cdot p\, D = -\frac{1}{q^2}A + \frac{\left(q\cdot p\right)^2}{q^4} D$$

Plug into

$$W_{\mu\nu}^{(S)} = \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] A + \left[p_\mu p_\nu - \frac{q\cdot p}{q^2}\left(p_\mu q_\nu + p_\nu q_\mu\right) + \frac{\left(q\cdot p\right)^2}{q^4} q_\mu q_\nu\right] D$$

With: $\quad \frac{A}{2m_p} \equiv -W_1, \quad \frac{D}{2m_p} \equiv \frac{1}{m_p^2} W_2$

Coefficients W_i are called STRUCTURE FUNCTIONS. They are functions of Lorentz scalars. And now we have two scalars: Q^2 and ν

Because, energy is transferred

Cress-section: Lab Frame:

Details in the write up

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{1}{2m_p}\frac{\alpha^2}{q^4}\frac{E'}{E}\ell^e_{\mu\nu}W^{\mu\nu}$$

$$\Rightarrow \frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{q^4}\frac{E'}{E}\left[4EE'\left(2W_1\sin^2\theta/2 + W_2\cos^2\theta/2\right)\right]$$

$$\Rightarrow \frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2\left(E'\right)^2}{q^4}\left(2W_1\sin^2\theta/2 + W_2\cos^2\theta/2\right)$$

Other choices of variables include $x \equiv Q^2/2m_p\nu,\ y = \nu/E$

Jacobian for going from $\left(\nu, Q^2\right)$ *to* $\left(x, y\right)$ is $dQ^2 d\nu = 2m_p E^2 y\, dx\, dy$

Physis is contained in the dependence of W_1 and W_2 on Q^2 and ν

Physics

How do W_1 and W_2 depend on Q^2 and ν ?

We have see 3 processes:

Let us write the cross sections with identical variables.
For details, see the write up

- Electron- muon elastic scattering

$$\frac{d\sigma}{dQ^2 d\nu} = \left(\frac{\alpha^2 \pi}{4E^2 \sin^4 \theta/2} \right) \frac{1}{EE'} \left\{ \cos^2 \theta/2 - \frac{q^2}{2m_\mu^2} \sin^2 \theta/2 \right\} \delta\left(\nu - \frac{Q^2}{2m_\mu} \right)$$

- Electron- nucleon elastic scattering

$$\frac{d\sigma}{dQ^2 d\nu} = \left(\frac{\alpha^2 \pi}{4E^2 \sin^4 \theta/2} \right) \frac{1}{EE'} \left\{ \frac{G_E^2\left(q^2\right) + \tau G_M^2\left(q^2\right)}{1+\tau} \cos^2 \theta/2 + 2\tau G_M^2\left(q^2\right) \sin^2 \theta/2 \right\} \delta\left(\nu - \frac{Q^2}{2m_p} \right)$$

- Deep inelastic scattering of electron- nucleon

$$\frac{d^2\sigma}{dQ^2 d\nu} = \frac{\alpha^2 \pi}{4E^2 \sin^4 \theta/2} \frac{1}{EE'} \left(2W_1 \sin^2 \theta/2 + W_2 \cos^2 \theta/2 \right)$$

- Electron- muon elastic scattering

$$\frac{d\sigma}{d\Omega}=\left(\frac{\alpha^{٢}}{٤E^{٢}\sin^{٤}\theta/٢}\right)\frac{E'}{E}\left\{\cos^{٢}\theta/٢-\frac{q^{٢}}{٢m_{\mu}^{٢}}\sin^{٢}\theta/٢\right\}$$

- Electron- nucleon elastic scattering

$$\frac{d\sigma}{d\Omega}=\left(\frac{\alpha^{٢}}{٤E^{٢}\sin^{٤}\theta/٢}\right)\frac{E'}{E}\left\{\frac{G_E^{٢}\left(q^{٢}\right)+\tau G_M^{٢}\left(q^{٢}\right)}{١+\tau}\cos^{٢}\theta/٢+٢\tau G_M^{٢}\left(q^{٢}\right)\sin^{٢}\theta/٢\right\}$$

Deep inelastic scattering of electron- nucleon

$$\frac{d^{2}\sigma}{d\Omega dE'}=\frac{4\alpha^{2}\left(E'\right)^{2}}{q^{4}}\left(2W_{1}\sin^{2}\theta/2+W_{2}\cos^{2}\theta/2\right)$$

RHS's have the same form, but LHS are different.

Remember that in DIS, E' and $\cos\theta$ are two independent variables. We can change them into Q^2 and ν .

Use the followings

$$q^{٢} = -٤EE'\sin^{٢}\theta/٢ \quad \Rightarrow \quad Q^{٢} = ٢EE'\left(١-\cos\theta\right)$$

$$\Rightarrow \quad dQ^{٢} = -٢EE'd\left(\cos\theta\right)$$

$$\nu = E - E' \quad \Rightarrow \quad dE' = -d\nu \quad \Rightarrow \quad d\left(\cos\theta\right)dE' = \frac{١}{٢EE'}dQ^{٢}d\nu$$

$$d\Omega_{\text{lab}} = ٢\pi\, d\left(\cos\theta\right) = ٢\pi\left(dq^{٢}/٢E'^{٢}\right) \quad \Rightarrow \quad d\Omega_{\text{lab}} = \pi\, dq^{٢}/E'^{٢}$$

That gives us

•Electron- muon elastic scattering

$$\frac{d\sigma}{dq^{٢}}=\left(\frac{\alpha^{٢}\pi}{٤E^{٢}\sin^{٤}\theta/٢}\right)\frac{١}{EE'}\left\{\cos^{٢}\theta/٢-\frac{q^{٢}}{٢m_{\mu}^{٢}}\sin^{٢}\theta/٢\right\}$$

•Electron- nucleon elastic scattering

$$\frac{d\sigma}{dq^{٢}}=\left(\frac{\alpha^{٢}\pi}{٤E^{٢}\sin^{٤}\theta/٢}\right)\frac{١}{EE'}\left\{\frac{G_{E}^{٢}\left(q^{٢}\right)+\tau G_{M}^{٢}\left(q^{٢}\right)}{١+\tau}\cos^{٢}\theta/٢+٢\tau G_{M}^{٢}\left(q^{٢}\right)\sin^{٢}\theta/٢\right\}$$

Deep inelastic scattering of electron- nucleon

$$\frac{d^{٢}\sigma}{dQ^{٢}d\nu}=\frac{\alpha^{٢}\pi}{٤E^{٢}\sin^{٤}\theta/٢}\frac{١}{EE'}\left(٢W_{١}\sin^{٢}\theta/٢+W_{٢}\cos^{٢}\theta/٢\right)$$

Now convert $d\sigma/dq^{2}$ into $d^{2}\sigma/dQ^{2}d\nu$

And this is the subtle point

And this is the subtle point

Multiply the first two by a $\delta-$function and convert it into a second order differential. That is, use $q^{٢} = -٢ m_p \nu$ or

$$\int d\nu\,\delta\left(\nu + q^2/2m_p\right) = 1$$

- Electron- muon elastic scattering

$$\frac{d\sigma}{dQ^2 d\nu} = \left(\frac{\alpha^2 \pi}{4E^2 \sin^4 \theta/2}\right)\frac{1}{EE'}\left\{\cos^2\theta/2 - \frac{q^2}{2m_\mu^2}\sin^2\theta/2\right\}\delta\left(\nu - \frac{Q^2}{2m_\mu}\right)$$

- Electron- nucleon elastic scattering

$$\frac{d\sigma}{dQ^2 d\nu} =$$

$$\left(\frac{\alpha^2 \pi}{4E^2 \sin^4 \theta/2}\right)\frac{1}{EE'}\left\{\frac{G_E^2\left(q^2\right) + \tau G_M^2\left(q^2\right)}{1+\tau}\cos^2\theta/2 + 2\tau G_M^2\left(q^2\right)\sin^2\theta/2\right\}\delta\left(\nu - \frac{Q^2}{2m_p}\right)$$

Deep inelastic scattering of electron- nucleon

$$\frac{d^{٢}\sigma}{dQ^{٢}d\nu}=\frac{\alpha^{٢}\pi}{٤E^{٢}\sin^{٤}\theta/٢}\frac{١}{EE'}\left(٢W_{١}\sin^{٢}\theta/٢+W_{٢}\cos^{٢}\theta/٢\right)$$

Physics: Bjorken Scaling

Compare cross sections for $e\mu \to e\mu$ and $eN \to eX$

$$\frac{d\sigma}{dQ^2 d\nu} = \left(\frac{\alpha^2\pi}{4E^2\sin^4\theta/2}\right)\frac{1}{EE'}\left\{\cos^2\theta/2 - \frac{q^2}{2m_\mu^2}\sin^2\theta/2\right\}\delta\left(\nu - \frac{Q^2}{2m_\mu}\right) \qquad ☆$$

And

$$\frac{d^2\sigma}{dQ^2 d\nu} = \frac{\alpha^2\pi}{4E^2\sin^4\theta/2}\frac{1}{EE'}\left(2W_1\sin^2\theta/2 + W_2\cos^2\theta/2\right) \qquad ☆☆$$

If the nucleon was a point particle ☆☆ should reduce to ☆
and we would have

$$W_2\left(\nu, Q^2\right)_{\text{point}} \to \delta\left(\nu - \frac{Q^2}{2m_p}\right), \qquad 2W_1\left(\nu, Q^2\right)_{\text{point}} \to \frac{Q^2}{2m_p^2}\delta\left(\nu - \frac{Q^2}{2m_p}\right)$$

Or using $\delta(x/a) = a\delta(x)$

$$\nu W_2\left(\nu, Q^2\right)_{\text{point}} = \delta\left(1 - \frac{Q^2}{2m_p\nu}\right) \quad , \quad 2m_p W_1\left(\nu, Q^2\right)_{\text{point}} = \frac{Q^2}{2m_p\nu}\delta\left(1 - \frac{Q^2}{2m_p\nu}\right)$$

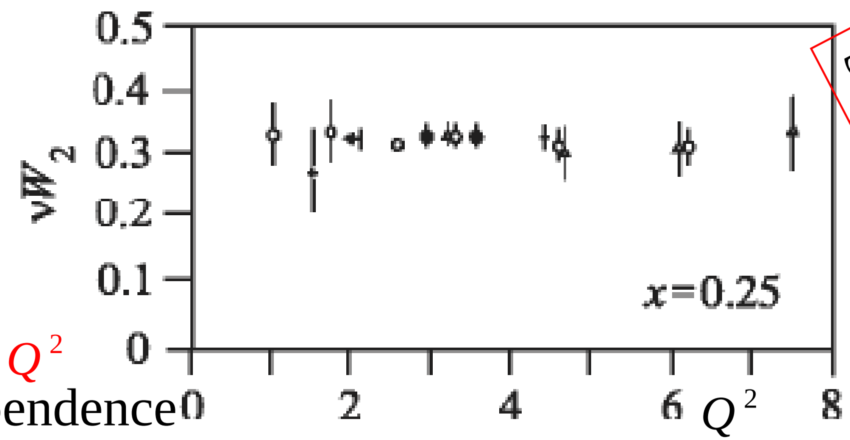
$$\nu W_2\left(\nu,Q^2\right)_{\text{point}}=\delta\left(1-\frac{Q^2}{2m_p\nu}\right)\quad,\quad 2m_pW_1\left(\nu,Q^2\right)_{\text{point}}=\frac{Q^2}{2m_p\nu}\delta\left(1-\frac{Q^2}{2m_p\nu}\right)$$

x

Physics: we see that the RHS depends only on the ratio Q^2/ν

But not individually on Q^2 and ν

The same property does not emerge for elastic electron- proton scattering! FF

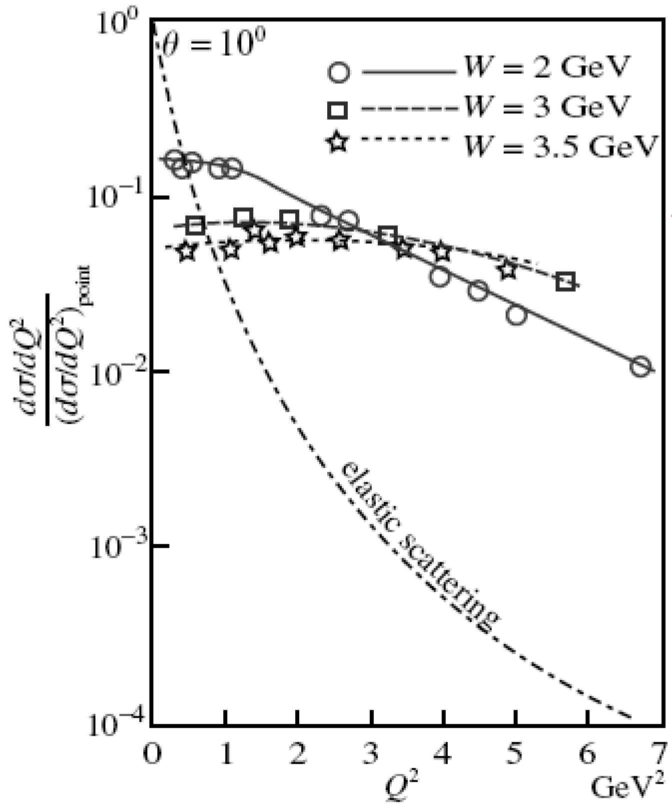


Scaling

No Q^2 dependence

Friedmann, J.I and Kendall, H.W, *Ann*.Rev. *Nucl Sci*. **22**, 203(1972).

Experimentally we measure the cross section.
Then at each fixed Q^2 plot the results as a funct
of $\tan^2\theta/2$ that determines W_1 and W_2



M.Breidenbach *et al*., *Phys*.Rev.*Lett*., 23:935,1969.

Physics: Scaling

The tensor $W_{\mu\nu}$ is dimensionless, as are the structure functions. The structure functions are dimensionless functions of Lorentz invariant variables $p^2, p \cdot q$ and q^2. It is conventional to write them as functions of $x = Q^2/2p \cdot q$ and $Q^2 = -q^2$ in order to be able to write them (SF) as dimensionless functions of dimensionless variables of x and Q^2/m_p^2. In the elastic scattering there is a strong dependence on Q^2/m_p^2, and the elastic form factors fall off like a power of Q^2/m_p^2. It was thought that the same behavior will persist for the DID structure functions. The scale m_p is a typical hadronic scale at which the confinement effects (and other non-perturbative effects) become important. Bjorken was the first to point out that if the constituents of hadron were essentially pointlike objects, then the hadronic scale should be irrelevant and the structure functions then only depend on x, and must be independent of Q^2

This is the famous SCALING

In the limit $Q^2 \to \infty$, and $\nu \to \infty$, but x is finite $W_{1,2}(x, Q^2)$ scale

$$m_p W_1(Q^2, \nu) \to F_1(x), \quad \nu W_2(Q^2, \nu) \to F_2(x)$$

QCD effect at high energies, or large Q^2

Hadronic scale irrelevance:

We now know that QCD is an asymptotically free theory, and the strong interaction coupling constant α_s becomes small at short distances. Thus at large Q^2 non-perturbative effects (such as the hadronic mass scale) are irrelevant, and QCD is described by a dimensionless coupling constant. Hence, we recover the prediction of scaling in QCD, since there is NO dimensionful scale in the problem.

However, in Quantum Field Theory, scale invariance is broken, because of quantum corrections. This introduces scaling violations, because of anomalous dimensions and the RUNING of α_s . Since at high energies, α_s is small, these scaling vilations can be reliably computed in PQCD.

Physics

Early experiments with elastic *ep* showed that the form factors factors $G_{E,M}\left(Q^2\right)$

Fall off rapidly as Q^2 increses. These experiments show that proton is not a point particle but it has structure. So, it must be a bound state. But the bound state of what? And what is the nature of the binding force?

DIS gives us two key observations:

- Structure functions are approximately independent of Q^2
- More importantly, as Q^2 increases, SF DO NOT go to zero.

$$\frac{d^2\sigma}{dQ^2 d\nu} = \frac{\pi\sigma_{\text{Mott}}}{m_p EE'}\left(2W_1 \tan^2\theta/2 + W_2\right)$$

Presence of $\tan^2\theta/2$ says as $W_1\left(x,Q^2\right) \to F_1\left(x\right) \neq 0$

First point indicates that when the nucleon is probed by photon, the photon "sees" it as the weakly bound state of point particles, partons

Why point particle ?

If it was the bound state of say, $n\pi^+$ or Λk^+ then $W_{1,2}$ would strongly depend on Q^2 and $F_1\left(x\right)$ would be zero

Second point says that the partons are FERMIONS, Spin ½ objects

Why spin ½?

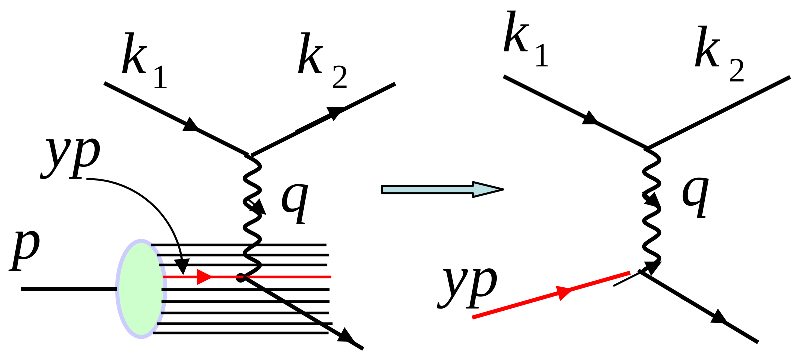
If it was spinless, we would not have tan $\tan^2\theta/2$ term

Also Callan-Gross relation $2xF_1(x) = F_2(x)$

So, nucleon is composed of point particles $\left(F_1(x) \neq 0\right)$

And they are Fermions

Feynman used the kinematics of Elastic Scattering off a point particle, for example $e+\mu \rightarrow e+\mu$ and explained the Scaling behavior of the structure Functions:



Each parton carries a fraction f of energy and momentum of the nucleon it is easy to show that $f=x$

$$p_i^\mu = f\ p^\mu,\quad m_i = f\ m_p$$

$$(p_i+q)^2 = (f\ p+q)^2 = m_i^2 = (f\ m_p)^2$$

$$\Rightarrow f^2 m^2 = q^2 + 2f\ p\cdot q = f^2 m_p^2$$

$$\Rightarrow f = -\frac{q^2}{2p.q} = x$$

Uncertainty principle: $\Delta Q \Delta\lambda \approx \hbar \quad , \quad \Delta\nu \Delta t \approx \hbar$

Short distances

Short time to deliver energy to parton: parton is almost Free! At high energy partons are free and at low energy they

Condition for elastic scattering of electron and parton leads to scaling:

We can use $e\mu \to e\mu$ Cross section formula with, $m_\mu \to m_i$, and $e \to e_i$

$$\frac{d\sigma^i}{dQ^2 d\nu} = \left(\frac{\alpha^2 \pi}{4E^2 \sin^4 \theta/2}\right) \frac{1}{EE'} \left\{ e_i^2 \cos^2 \theta/2 - e_i^2 \frac{q^2}{2m_i^2} \sin^2 \theta/2 \right\} \delta\left(\nu - \frac{Q^2}{2m_i}\right)$$

Compare with general inelastic scattering cross section:

$$\frac{d^2\sigma}{dQ^2 d\nu} = \left(\frac{\alpha^2 \pi}{4E^2 \sin^4 \theta/2}\right) \frac{1}{EE'} \left(2W_1 \tan^2 \theta/2 + W_2\right)$$

We get the contribution of each parton i in the structure functions

$$w_1^i = e_i^2 \frac{Q^2}{4m_i^2} \delta\left(\nu - \frac{Q^2}{2m_i}\right) = e_i^2 \frac{Q^2}{4m_p^2 x_i^2} \delta\left(\nu - \frac{Q^2}{2m_p x_i}\right)$$

$$w_2^i = e_i^2 \delta\left(\nu - \frac{Q^2}{2m_i}\right) = e_i^2 \delta\left(\nu - \frac{Q^2}{2m_p x_i}\right)$$

If the structure function of nucleon is in fact sum of contributions of each parton, it must be an incoherent sum. so, to get W_1 and W_2 of nucleon ,we must multiply the contribution of each parton type i by the probability of finding that parton in the nucleon and then sum over. Moreover, each parton can have any value of x, which is a continues variable, hence need to integrate over x.

$$W_1(Q^2,\nu)=\sum_i\int_0^1 dx_i f_i(x_i) w_1^i=\sum_i\int_0^1 dx_i f_i(x_i)\left[e_i^2\frac{Q^2}{4m_p^2x_i^2}\delta\left(\nu-\frac{Q^2}{2m_px_i}\right)\right]$$

$$W_2(Q^2,\nu)=\sum_i\int_0^1 dx_i f_i(x_i) w_2^i=\sum_i\int_0^1 dx_i f_i(x_i)\left[e_i^2\delta\left(\nu-\frac{Q^2}{2m_px_i}\right)\right]$$

Rewrite the delta function

$$\delta\left(\nu-\frac{Q^2}{2m_px_i}\right)=\delta\left[\frac{\nu}{x_i}\left(x_i-\frac{Q^2}{2m_p\nu}\right)\right]=\frac{x_i}{\nu}\delta(x_i-x)$$

We get

$$W_1\left(Q^2,\nu\right)=\sum_i\int_0^1 dx_i\, f_i\left(x_i\right)\left[e_i^2\frac{Q^2}{4m_p^2x_i^2}\frac{x_i}{\nu}\delta\left(x_i-x\right)\right]$$

$$=\sum_i\int_0^1 dx_i\, f_i\left(x_i\right)\left[e_i^2\frac{1}{m_p}\frac{x}{2x_i}\delta\left(x_i-x\right)\right]=\frac{1}{2m_p}\sum_i e_i^2 f\left(x\right)$$

$$W_2\left(Q^2,\nu\right)=\sum_i\int_0^1 dx_i\, f_i\left(x_i\right)w_2^i=\sum_i\int_0^1 dx_i\, f_i\left(x_i\right)\left[e_i^2\delta\left(\nu-\frac{Q^2}{2m_px_i}\right)\right]$$

$$=\sum_i\int_0^1 dx_i\, f_i\left(x_i\right)\left[e_i^2\frac{x_i}{\nu}\delta\left(x_i-x\right)\right]=\frac{1}{\nu}\sum_i e_i^2 x\, f_i\left(x\right)$$

$$m_pW_1\left(Q^2,\nu\right)=\sum_i\frac{e_i^2}{2}f_i\left(x\right)\equiv F_1\left(x\right)$$

$$\nu W_2\left(Q^2,\nu\right)=\sum_i e_i^2 x f_i\left(x\right)\equiv F_2\left(x\right)$$

So, they are functions of only x

We also see the parton model main equation, Callan-Gross Equation

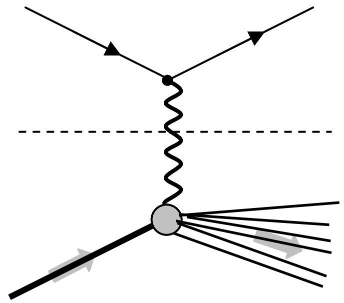
$$2xF_1\left(x\right)=F_2\left(x\right)=\sum_i e_i^2 x\, f_i\left(x\right)$$

Callan – Gross relation $\Rightarrow$ quark has spin 1/2 $F_2 = 2xF_1$

DIS can be viewed as a 2 step process:

1. $e^-(k) \to \gamma^*(q) + e^-(k')$ $\Leftarrow$ Only provides γ^*
2. $\gamma^*(q) + N$ (or p_a) $\to X$ p_a = parton

Look at the step 2: below the dashed line.



W_1, W_2 (or F_1, F_2)
Represent the photoproduction of hadrons (X) from nucleon: $\sigma_{tot}\left(\gamma^* + N \to x\right)$

In the nucleon rest frame ν is the photon's energy $\nu = q_0$. since phton is virtual, $q^2 = m_{\gamma^*} \neq 0$, then photon has two transver polarization, $\varepsilon_T^\mu(q)$ i.e. $\lambda = \pm 1$ and also one longitudinal (or scalar) polarization $\varepsilon_L^\mu(q)$ with $\lambda = 0$

So, to calculate $\sigma_{tot}\left(\gamma^* + N \to x\right)$ we need to consider both

Callan – Gross relation $\Rightarrow$ quark has spin 1/2 $F_2 = 2xF_1$

Take the z axes along the direction of photon momentum, then

$$q_\mu = (q_0, 0, 0, q_3)$$

Virtual photon is like a massive boson, but with spacelike $(q^2 < 0)$ four momentum.

Polarization vector should be perpendicular to q therfore

$$q \cdot \varepsilon(\lambda) = q_\mu \varepsilon^\mu(\lambda) = 0 \quad ; \quad \lambda = \pm 1, 0$$

For a free (real) massive spin-1 particle, 4- momentum is timelike $(q^2 > 0)$ and all three polarization vectors satisfy the above condition. But for virtual, spacelike photon, one of its polarization vectors must be timelike. So we define them as:

$$\varepsilon_T^\mu = \varepsilon^\mu(\lambda = \pm 1) = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \quad \varepsilon^2(\lambda = \pm 1) = -1$$

$$\varepsilon_L^\mu = \varepsilon^\mu(\lambda = 0) = \frac{1}{\sqrt{Q^2}}(q_3, 0, 0, q_0) = \frac{1}{\sqrt{Q^2}}\left(\sqrt{\nu^2 + Q^2}, 0, 0, \nu\right), \quad \varepsilon^2(\lambda = 0) = +1$$

Callan – Gross relation $\Longrightarrow$ quark has spin 1/2 $F_2 = 2xF_1$

Remark: we could choose a frame (Breit Frame) in which $\nu = 0$ and write $\varepsilon^{\mu}\left(\lambda = 0\right) = \left(1, 0, 0, 0\right)$ that is why we call $\lambda = 0$ state scalar.

We also write the normalization conditions, in a single orthonormality condition

$$\varepsilon\left(\lambda\right)\cdot\varepsilon\left(\lambda'\right) = \varepsilon_{\mu}\left(\lambda\right)\cdot\varepsilon^{\mu}\left(\lambda'\right) = \left(-1\right)^{\lambda}\delta_{\lambda\lambda'}$$

With the complementary relation

$$\sum_{\lambda}\left(-1\right)^{\lambda+1}\varepsilon^{\mu}\left(\lambda\right)\varepsilon^{\nu}\left(\lambda\right) = -\left(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^{2}}\right)$$

Exercise : Show that polarization vector of scalar photon can be written as

$$\varepsilon^{\mu}\left(\lambda = 0\right) = \eta\left(p_a^{\mu} - \frac{\left(p_a\cdot q\right)}{q^2}q^{\mu}\right) \;;\; \eta = \left(p_a^2 - \frac{\left(p_a\cdot q\right)^2}{q^2}\right)^{-1/2}$$

p_a : Parton momentum

Callan – Gross relation $\Rightarrow$ quark has spin 1/2 $F_2 = 2xF_1$

To calculate $\sigma_{tot}\left(\gamma^* + N \to x\right)$ besides the scalar photon state we also need to define the flux K of the photon. Let me skip the details and just give the conventional definition

$$K = \left(W^2 - m_a^2\right)/2m_a = \nu - Q^2/2m_a$$

Now, we are ready to calculate $\sigma_{tot}\left(\gamma^* + N \to x\right)$

The transition amplitude is $T_{fi} = (2\pi)^4 \delta^4\left(q + p - p_X\right)\varepsilon_\mu(\lambda)\langle p_X | J^\mu(0) | p\rangle$

The absorption cross section in the lab frame is

$$\frac{d\sigma_\gamma}{d\Omega} = \frac{e^2}{4Km_p}\sum_X (2\pi)^4 \delta^4\left(\varepsilon + p - p_X\right) \times \frac{1}{2}\sum_\lambda \langle p_X | \varepsilon_\mu(\lambda) J^\mu | p\rangle \langle p_X | \varepsilon_\nu(\lambda) J^\nu | p\rangle^*$$

$$= \frac{e^2}{4Km_p}\sum_\lambda \varepsilon_\mu(\lambda)\varepsilon_\nu(\lambda)^* \underbrace{\frac{1}{2}\sum_X \langle p | J^\mu | p_X\rangle \langle p_X | J^\nu | p\rangle^* (2\pi)^4 \delta^4\left(\varepsilon + p - p_X\right)}_{\equiv W^{\mu\nu}}$$

$$\Rightarrow \quad \sigma_{tot}^{\gamma^*} = \frac{4\pi^2\alpha}{K}\sum_\lambda \varepsilon_\mu(\lambda)\varepsilon_\nu(\lambda)^* W^{\mu\nu}$$

Parameterized already in terms of W_1, W_2

Callan – Gross relation $\Longrightarrow$ quark has spin 1/2 $\quad F_2 = 2xF_1$

Now we can determine the longitudinal σ_L and transversal σ_T cross sections. Recall that

$$\frac{1}{2m_p}W^{\mu\nu} = \left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right]W_1 + \frac{1}{m_p^2}\left[\left(p_a^\mu - \frac{q\cdot p_a}{q^2}q^\mu\right)\left(p_a^\nu - \frac{q\cdot p_a}{q^2}q^\nu\right)\right]W_2$$

And use $\varepsilon^\mu(\lambda=\circ) = \eta\left(p_a^\mu - \frac{(p_a\cdot q)}{q^2}q^\mu\right)$; $\eta = \left(p_a^2 - \frac{(p_a\cdot q)^2}{q^2}\right)$ To rewrite it as

$$\frac{1}{2m_p}W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)W_1 + \frac{\varepsilon^\mu(0)\varepsilon^\nu(0)}{\eta^2 m_p^2}W_2$$

$$\sigma_{\mathrm{tot}}^{\gamma^*} = \frac{8m_p\pi^2\alpha}{K}\sum_\lambda \varepsilon_\mu(\lambda)\varepsilon_\nu(\lambda)^*\left[\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)W_1 + \frac{\varepsilon^\mu(0)\varepsilon^\nu(0)}{\eta^2 m_p^2}W_2\right]$$

Callan – Gross relation $\Rightarrow$ quark has spin 1/2 $F_2 = 2xF_1$

Using $\varepsilon_\mu(\pm 1)\varepsilon^\mu(0)=0$, $\varepsilon_\mu(\pm 1) g^{\mu\nu}\varepsilon_\nu(\pm 1)^* = -1$ we get

$$\sigma_T^{\gamma^*}(\lambda=\pm 1)=\frac{8m_p\pi^2\alpha}{K}\sum_{\lambda=\pm}\varepsilon_\mu(\lambda)\varepsilon_\nu(\lambda)^*\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)W_1=\boxed{\frac{8m_p\pi^2\alpha}{K}W_1}$$

$$\sigma_L^{\gamma^*}=\frac{8m_p\pi^2\alpha}{K}\left[\varepsilon_\mu(0)\varepsilon_\nu(0)^*\left(-g^{\mu\nu}W_1+\frac{\varepsilon^\mu(0)\varepsilon^\nu(0)}{\eta^2 m_p^2}W_2\right)\right]$$

$$=\boxed{\frac{8m_p\pi^2\alpha}{K}\left(-W+\frac{1}{\eta^2 m_p^2}\right)W_2=\frac{8m_p\pi^2\alpha}{K}\left[\left(1+\frac{\nu^2}{Q^2}\right)W_2-W_1\right]}$$

These relations also shows why W_1 and W_2 are positive

The combination defined as $\left(1+\nu^2/Q^2\right)W_2-W_1\equiv W_L$

Callan – Gross relation $\Longrightarrow$ quark has spin 1/2 $F_2 = 2xF_1$

In the scaling region, we arrive at

$$\sigma_T^{\gamma^*} \to \frac{4\pi^2\alpha}{Km_p} F_1, \quad \sigma_L^{\gamma^*} \to \frac{4\pi^2\alpha}{Km_p} \frac{1}{2x}\left(F_2 - 2xF_1\right) = \frac{4\pi^2\alpha}{Km_p} F_L$$

Callan- Gross relation $F_2 = 2xF_1$ correspomds to

$$\frac{\sigma_L^{\gamma^*}}{\sigma_T^{\gamma^*}} \to 0$$

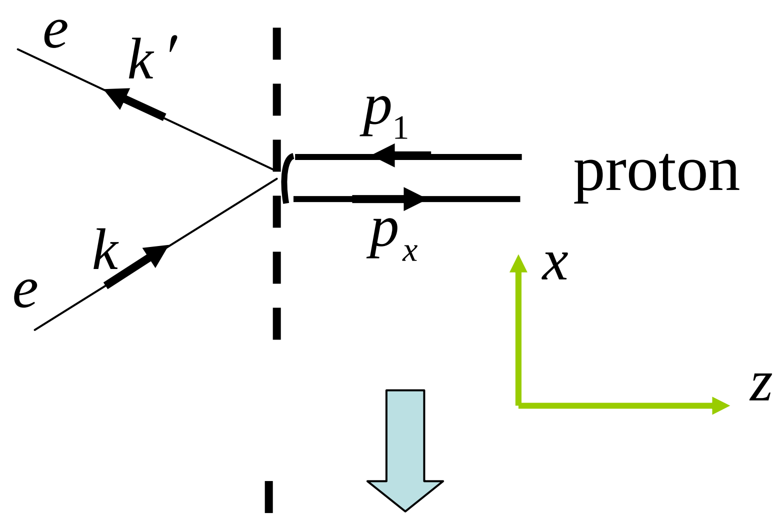
And valid for spin ½ target, if spin of target was zero, we wouldn't have magnetic scattering and F_1 would be zero or equivalently

$$\frac{\sigma_T^{\gamma^*}}{\sigma_L^{\gamma^*}} \to 0$$ Inverse of

We can understand both relations in the Breit frame using angular momentum (helicity) conservation

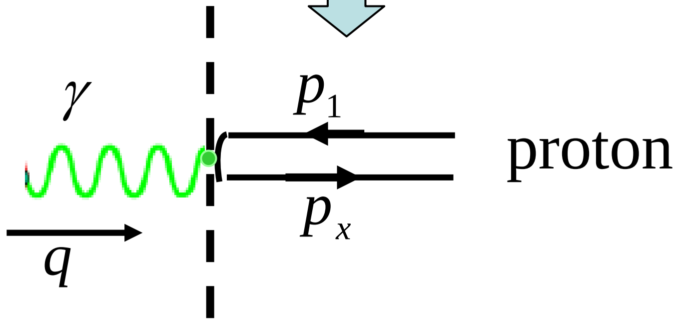
Callan – Gross relation ⇒ quark has spin 1/2 $F_2 = 2xF_1$

Breit Frame: photon and parton move along the same line, opposite directions. After collision, partonexactly reversed its 3- momentum. suppose electron moves in x-y plane.



After collision, only p_z electron changes, while p_x remains unchanged.

⇒ No energy transfer to the target only recoils after absorbing.

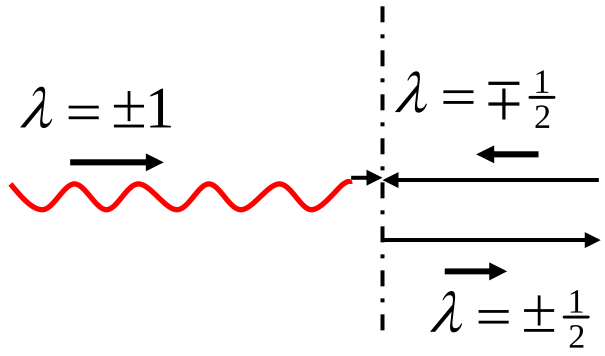


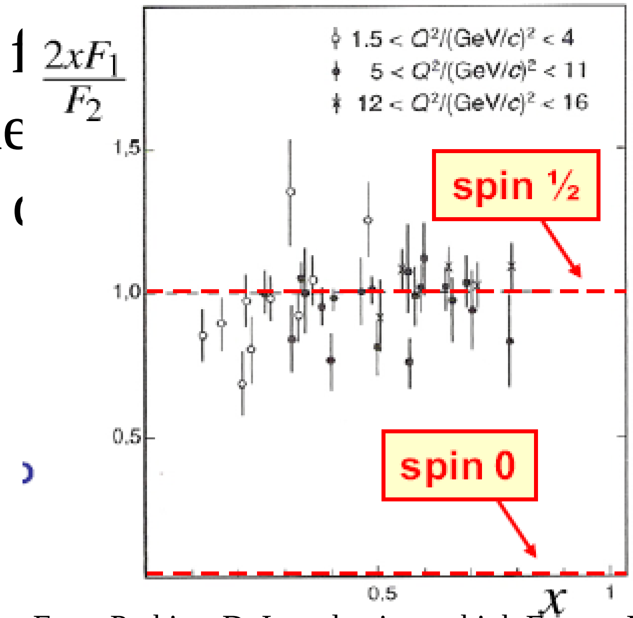
Now, photon gets absorbed by the parton of the nucleon. parton with spin zero can absorb a scalar photon. No problem ther. Initial state total helicity is zero and so is the final state helicity.⇒ $\sigma_L^{\gamma^*} \neq 0$ But a spin zero parton cannot absorb transvere $\lambda = \pm 1$ photon. Angular momentum conservation.

Callan – Gross relation ⇒ quark has spin 1/2 $F_2 = 2xF_1$

$\Rightarrow \dfrac{\sigma_T^{\gamma^*}}{\sigma_L^{\gamma^*}} \to 0$ (if the parton is a spin 0 particle)

For spin ½ parton, when it heads back, helicity is changed by 1 unit. This can be accomodated by only by transversally polarized photon. Longitudinally polarized photon cannot provide th~ needed ± 1 unit of helicity change f parton. $\Rightarrow \sigma_L^{\gamma^*} / \sigma_T^{\gamma^*} \to 0$ Which is the Callan- Gross relation. What does (show?





From Perkins, D. Introduction to high Energu Physics, 3rd ed. Addison-Wesley Publication Co.

Deep Inelastic Scattering and Structure Functions

IPP11, September 3-7, 2011

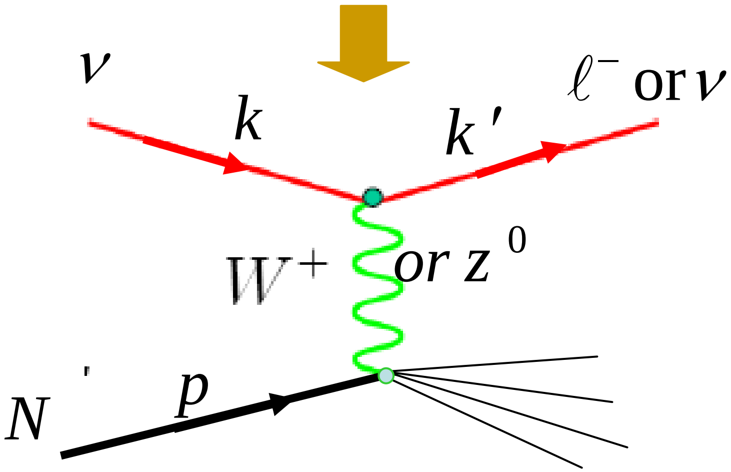
Lecture 3

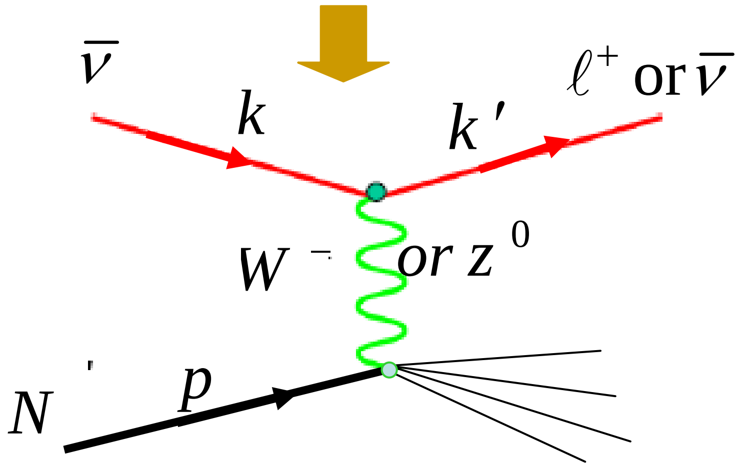
Firooz Arash

Tafresh University

$\nu\ (\bar{\nu})$ Probe

$$\nu + N \to \ell^- \text{ or} \left(\nu\right) \qquad \bar{\nu} + N \to \ell^+ \text{ or} \left(\bar{\nu}\right)$$





Requires gauge boson coupling to the partons through a current which has a mixture current of vector (V) and axial vector (A)

Many times you heard this week that such a current exists:
All we need is to replace γ^μ by $\gamma^\mu\left(1-\gamma^5\right)$ in the upper part of the right graph and $q^2 \to q^2 - M_W^2$, $e^2 \to \frac{G}{\sqrt{2}} M_W^2$

$$\ell^{\mu\nu}\left(ee\right) \to 8\ell^{\mu\nu}\left(\nu\right)$$

Recall that for a particle of mass m_i and charge e_i

$$w_1^i = e_i^2 \frac{Q^2}{4m_i^2}\delta\left(\nu - \frac{Q^2}{2m_i}\right) = e_i^2 \frac{Q^2}{4m_p^2 x_i^2}\delta\left(\nu - \frac{Q^2}{2m_p x_i}\right)$$

$$w_2^i = e_i^2 \delta\left(\nu - \frac{Q^2}{2m_i}\right) = e_i^2 \delta\left(\nu - \frac{Q^2}{2m_p x_i}\right)$$

A Reminder

And we summed over all such contribution to get W_1 and W_2

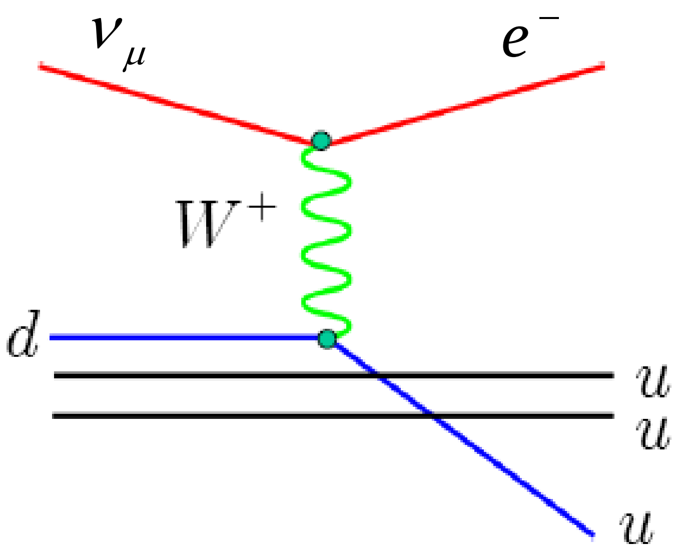
$$W_{1,2} = \sum_i W_{1,2}^i \Rightarrow \begin{cases} m_p W_1\left(Q^2,\nu\right) = \sum_i \frac{e_i^2}{2} f_i(x) \equiv F_1(x)_1 \\ \nu W_2\left(Q^2,\nu\right) = \sum_i e_i^2 x f_i(x) \equiv F_2(x) \end{cases}$$

Also, when we parameterized $W_{\mu\nu}^e$ we ignorred the antisymmetric Tensor $i\,\varepsilon^{\mu\nu\rho\sigma} p_{\alpha\rho} q_\sigma$. NO More, an analogous term also appears in the lepton tensor due to V-A interaction.: That is, Now we have a 3rd Structur function, W_3 which is a little more subtle than $W_{1,2}$

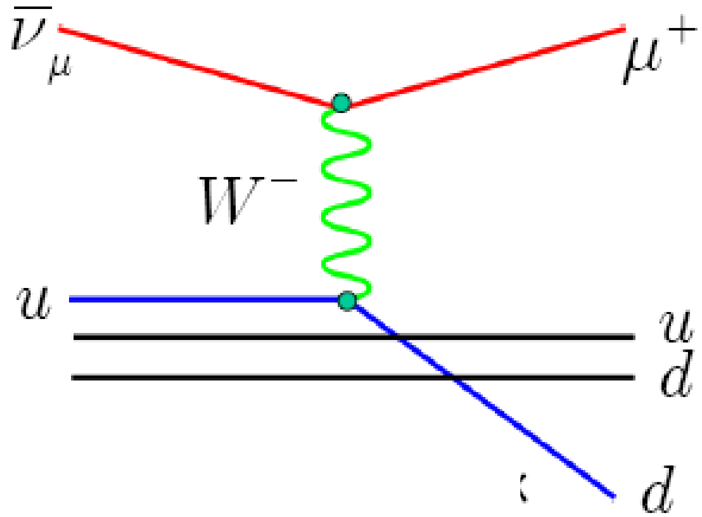
With neurrino (anti- neutrino) probe

$$W_{1,2,3} = \begin{cases} m_p W_1\left(Q^2,\nu\right) = \sum_i \frac{e_i^2}{2} f_i(x) \equiv F_1(x) \\ m_p W_1\left(Q^2,\nu\right) = \sum_i \frac{e_i^2}{2} f_i(x) \equiv F_1(x) \\ \nu W_2\left(Q^2,\nu\right) = \sum_i e_i^2 x f_i(x) \equiv F_2(x) \\ \frac{W_3}{m_p} = \sum_i \frac{W_3^i}{m_i} \Rightarrow \nu W_3^{\nu,\bar{\nu}} \equiv -F_3^{\nu,\bar{\nu}}(x) = 2\sum_i \left[f_i(x) - \bar{f}_i(x)\right] \end{cases}$$

With ν and $\bar{\nu}$ interaction, they select the quark flavor in the nucleon



$f\,(x)$ Is prob. of finding a d quark in the target



$f\,(x)$ Is prob. of finding a u quark in the target

We Also learned one more thing:

There is anti-quark (as well as quark) inside the nucleon!

Where did we learn this?!

An exchanged photon treats all charged partons equally. Thus the E&M structure functions of proton is the sum over all partons. However, in neutrino induced interactions, the reaction must proceed by charged current scattering for the final state electron or muon. Therefore the DIS reactions are, e.g.

$\nu_e p \to e^- x$ With an exchange of W^+ ,

$\bar{\nu}_e p \to e^+ x$ With an exchange of W^-

Not all quarks and anti-quarks couple to W^+, W^-

In reaction $\nu_e p \to e^- x$ the incoming ν_e decays to $\nu_e \to W^+ e^-$ then

$W^+ d \to u, W^+ s \to c, W^+ \bar{u} \to \bar{d}$, and $W^+ \bar{c} \to \bar{s}$

⇨ In proton, W^+ couples to $d, s, \bar{u}$ and c quarks

Similarily, in reaction $\bar{\nu}_e \to W^- e^+$ and W^- couples to $\bar{d}, \bar{s}$, and c quarks

So, for a nutrino beam with W^+ exchange the structure functions of the proton is

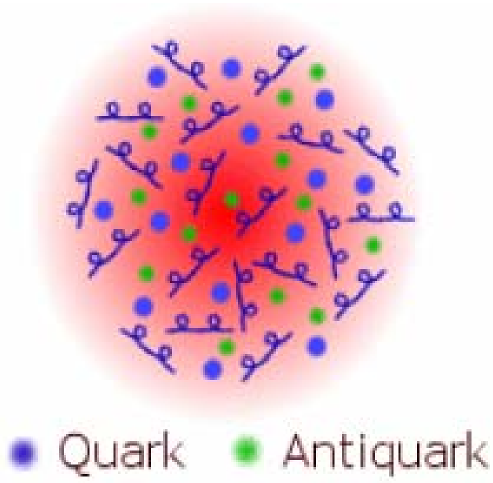
$$F_2^{W^+p}(x) = 2x\left[d_p(x) + s_p(x) + b_p(x) + \bar{u}_p(x) + \bar{c}_p(x)\right]$$

$$F_3^{W^+p}(x) = 2\left[d_p(x) + s_p(x) + b_p(x) - \bar{u}_p(x) - \bar{c}_p(x)\right]$$

Likewise, the structure functions determined by anti-neutrino beam with W^- exchange are

$$F_2^{W^-p}(x) = 2x\left[\bar{d}_p(x) + \bar{s}_p(x) + \bar{b}_p(x) + u_p(x) + c_p(x)\right]$$

$$F_3^{W^-p}(x) = -2\left[\bar{d}_p(x) + \bar{s}_p(x) + \bar{b}_p(x) - u_p(x) - c_p(x)\right]$$



In neutron:

$$u_p \to d_n$$

$$d_p \to u_n$$

What do structure functions tell us ?

We start with $F_2^{e^- p}$ structure function

The probability f(x) for the quark f to carry a fraction x of the nucleon momentum is an intrinsic property of the nucleon and is process independent.

The probability f(x) for the quark f to carry a fraction x of the nucleon momentum is an intrinsic property of the nucleon and is process independent.

$x\,f\,(x)$: amount of momentum carried by a particular parton

Parton momentum dist.

Momentum carried by all partons has to add up to one:

Momentum Sum Rule

$$\int x[u(x)+\bar{u}(x)+d(x)+\bar{d}(x)+s(x)+\bar{s}(x)+\ldots+g(x)]dx = 1$$

The u and d quarks are already present in the proton, since in the constituent quark model (CQM), the proton is a uud state where. They carry the bulk of proton momentum at large x . The *sea* quarks do not contribute to the baryon number of the hadron since they are always produced in a virtual $q-\bar{q}$ pair. So, for every $\bar{u}_p$ there is a $u_{p,sea}$ the u, and d quark densities include both valence and sea quark densities.

So,

$$u_p(x) = u_p^v(x) + u_{p,sea},$$

$$d_p(x) = d_p^v(x) + d_{p,sea}$$

Quantum numbers have to be right: So, for proton we must have

$$\int\left[u_p(x)-u_{p,sea}(x)\right]dx=2\quad(\#u_{val}),\qquad \int\left[d_p(x)-d_{p,sea}(x)\right]dx=1\quad(\#d_{val})$$

$$\int\left[s(x)-\bar{s}(x)+\cdots\right]dx=0\quad \text{"sea" quark contribution}$$

Take separate contributions of the valence and the "sea"

$$u(x)=u_{\mathrm{val}}(x)+u_{sea}(x),\qquad d(x)=d_{\mathrm{val}}(x)+d_{sea}(x)$$

For now, All sea contributios are "equal"

$$\rightarrow u_s(x)=d_s(x)=\bar{u}_s(x)=\bar{d}_s(x)\equiv S(x)$$

Arash. Tomio, PLB 410,207 (1997).

$$F_2^{ep}(x)=x\left[\frac{4}{9}u_{\mathrm{val}}(x)+\frac{1}{9}d_{\mathrm{val}}(x)+\frac{10}{9}S\right]$$

Grossly violated

$$F_2^{en}(x)=x\left[\frac{4}{9}d_{\mathrm{val}}(x)+\frac{1}{9}u_{\mathrm{val}}(x)+\frac{10}{9}S\right]$$

$$F_2^{ep}(x)-F_2^{en}(x)=x\left[\frac{1}{3}u_{\mathrm{val}}-\frac{1}{3}d_{\mathrm{val}}\right]$$

Gottfreid Sum Rule.

$$\Rightarrow\int_0^1\tfrac{1}{x}\left(F_2^{ep}-F_2^{en}\right)dx=\tfrac{1}{3}\int_0^1\left(u_{\mathrm{val}}(x)-d_{\mathrm{val}}(x)\right)dx=\frac{1}{3}=\frac{1}{N_{\mathrm{val}}}$$

Area under F_2 , neglecting the strange sea quark

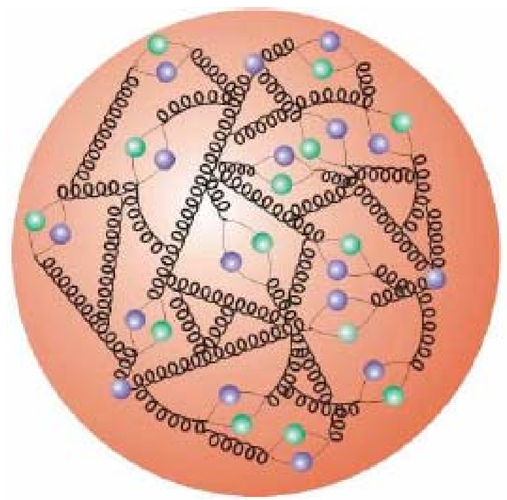
$$\int_0^1 F_2^{ep}(x)dx = \tfrac{4}{9}\int_0^1 dx\, x\left[u(x)+\bar{u}(x)\right]+\tfrac{1}{9}\int_0^1 dx\, x\left[d(x)+\bar{d}(x)\right]=\tfrac{4}{9}f_u+\tfrac{1}{9}f_d$$

$$\int_0^1 F_2^{en}(x)dx = \tfrac{4}{9}\int_0^1 dx\, x\left[d(x)+\bar{d}(x)\right]+\tfrac{1}{9}\int_0^1 dx\, x\left[u(x)+\bar{u}(x)\right]=\tfrac{4}{9}f_d+\tfrac{1}{9}f_u$$

Add and subtract $\Longrightarrow$ $f_u = 0.36$ and $f_d = 0.18$

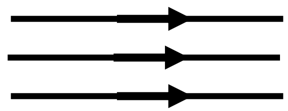
Quarks and anti-quarks carry only half of the momentum.

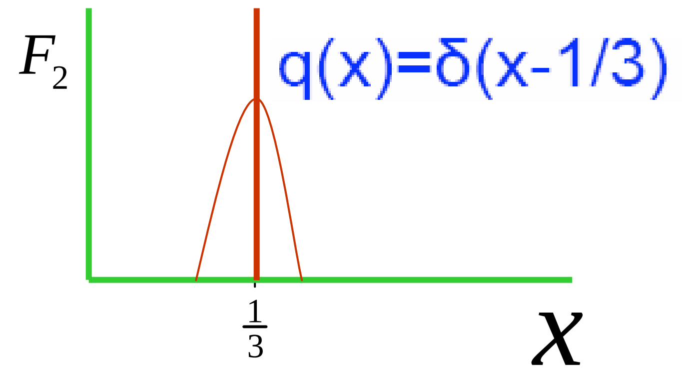
First direct evidence of GLUON!

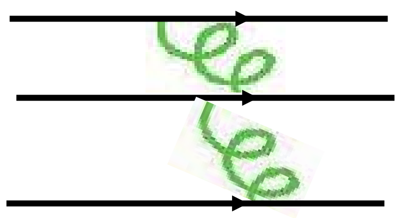


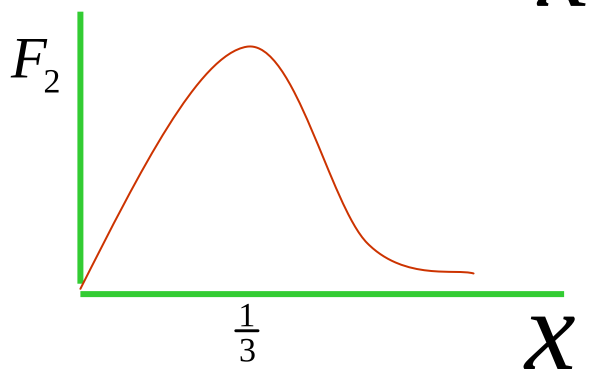
I used “sea” quark and “valence” quark. They will be clarified in a moment. For Now, If There is No sea quark, then only 3 non-interacting quarks

$$F_2 = x\sum_q e_q^2 q(x)$$





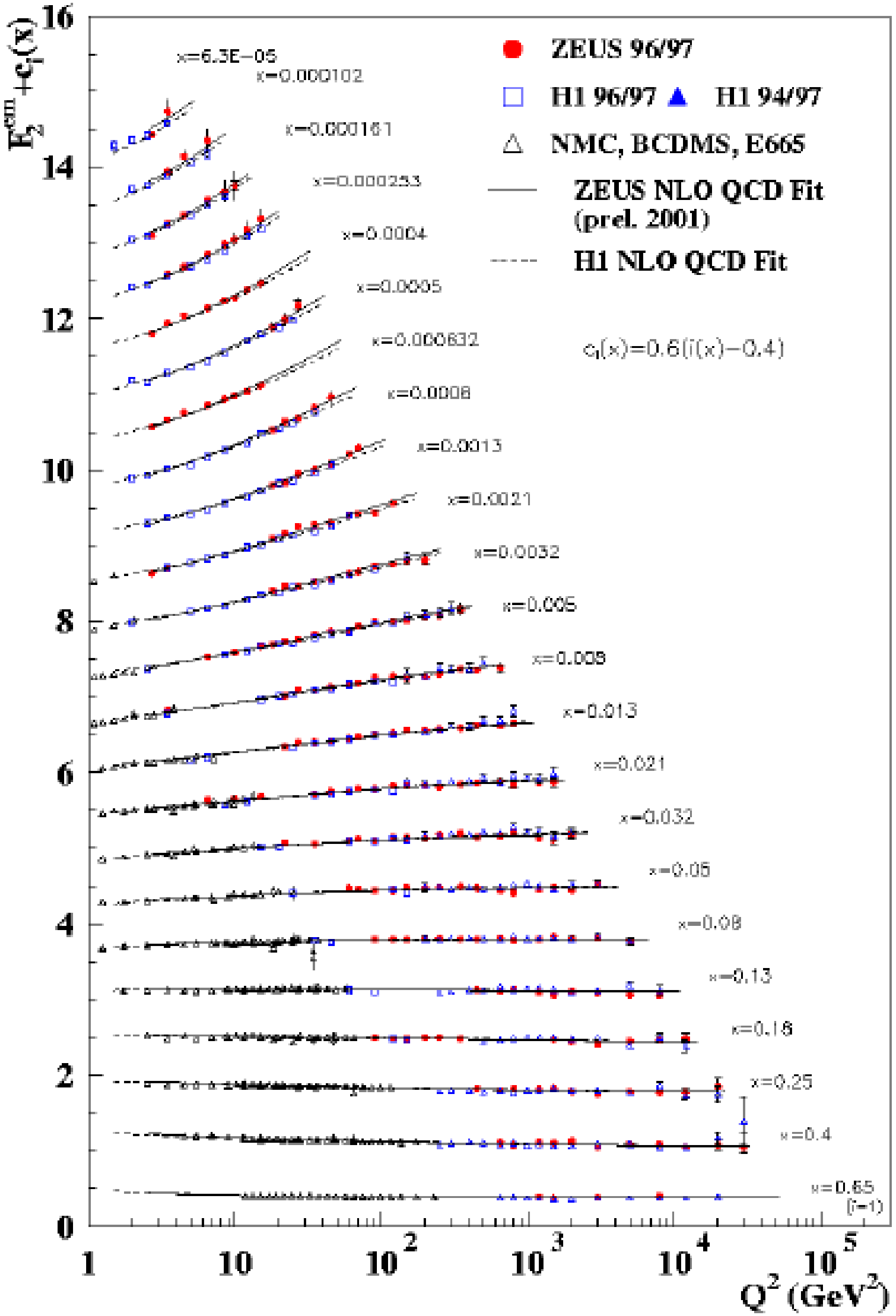


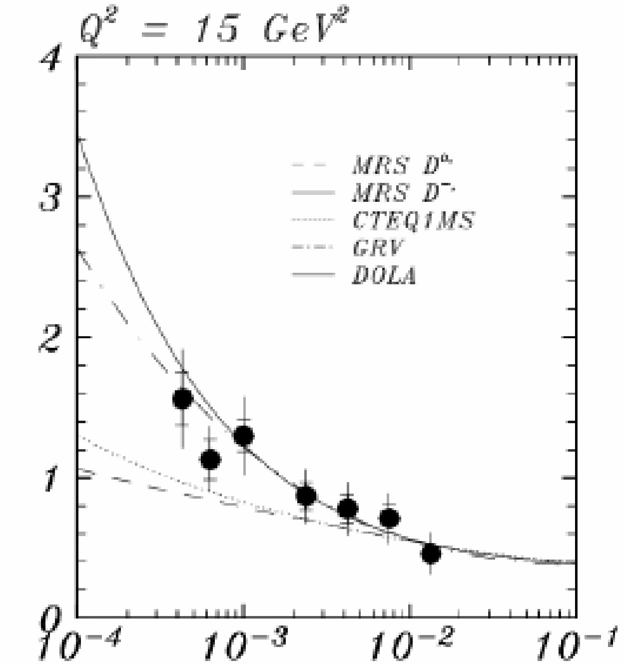


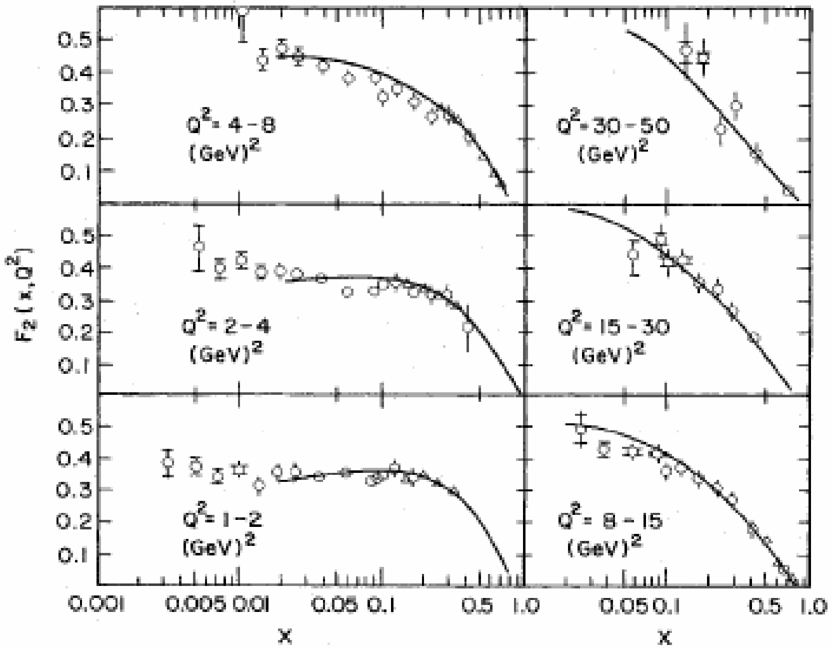
What we have learned so far?

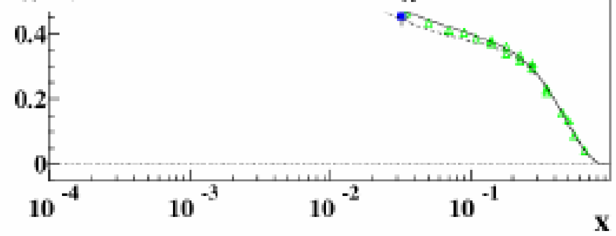
The probe sees putative objects (free pointlike) partons inside the proton- that is the scaling property. Does data supports it?

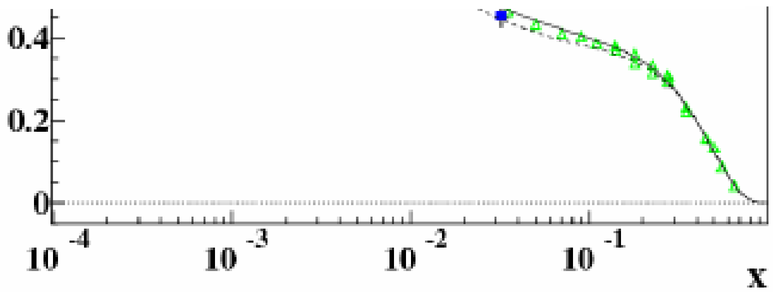
Scaling property. Does it hold ? See some real Data



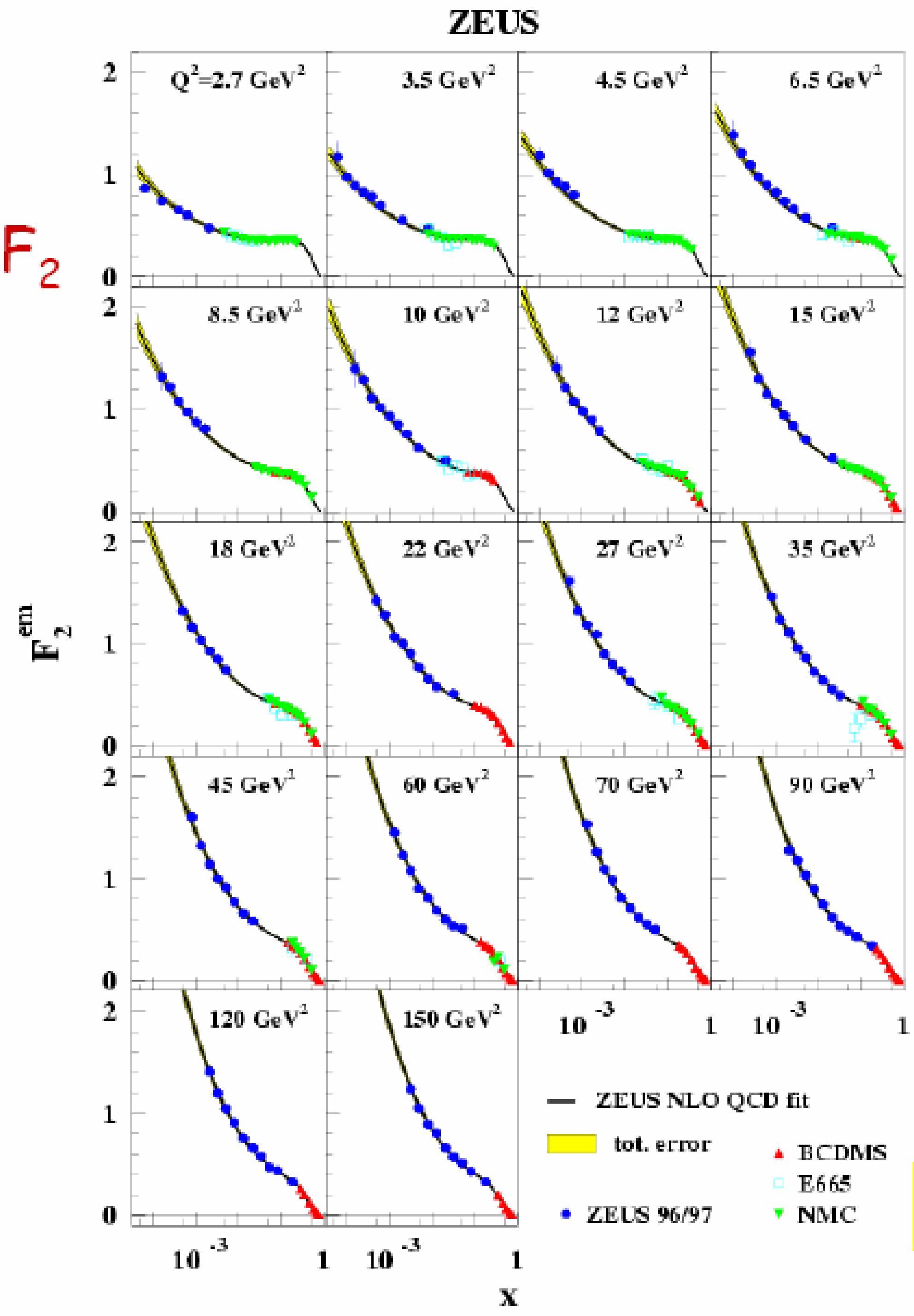

$Q^2 = 15\ GeV^2$
MRS D^0
MRS D^-
CTEQ1MS
GRV
DOLA

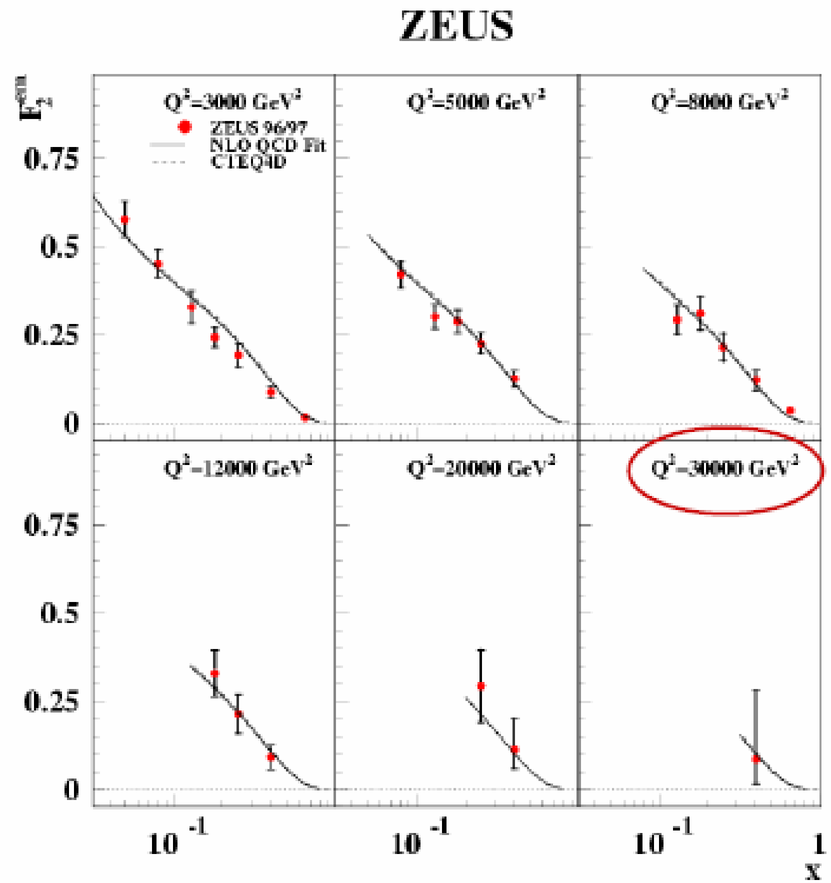

$F_2(x,Q^2)$
Q^2 = 4 - 8 (GeV)^2
Q^2 = 30 - 50 (GeV)^2
Q^2 = 2 - 4 (GeV)^2
Q^2 = 15 - 30 (GeV)^2
Q^2 = 1 - 2 (GeV)^2
Q^2 = 8 - 15 (GeV)^2
x
x


x


x

F$_2$



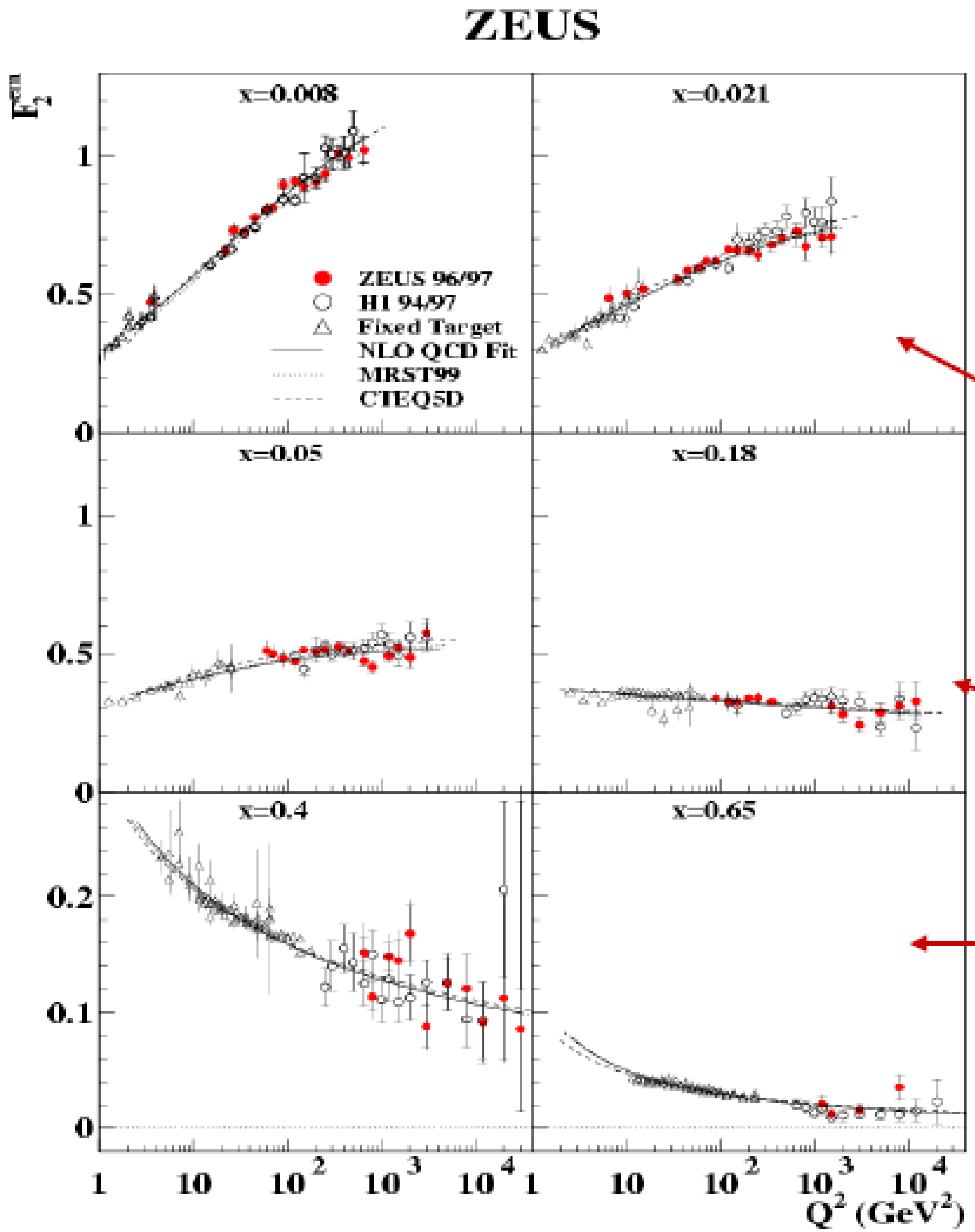


F_2 grows with Q^2 at low x
(*sea region*)

F_2 Decreases with Q^2 at
high x (valence region)

Scaling Violation!

Scaling Violation



$$F(x) \to F(x, Q^2)$$

Rising with increasing Q at small x

Flat behavior at medium x

Decreases with increasing Q at small x

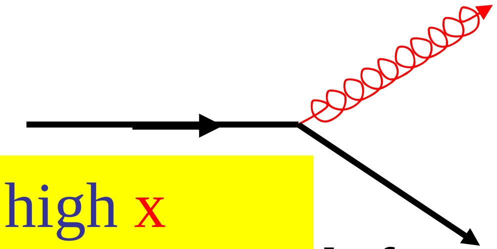
Scaling Violation

Good news, because we can apply perturbative QCD. Quark Parton Model is the 0^{th} order of perturbation expansion.

Q dependence is inherent in QCD. Renormalization group equation (RGE) governs the scale dependence of parton distributions and hard cross section

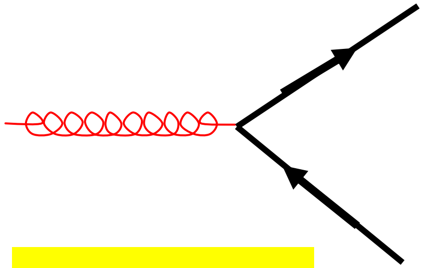
⇒ Dokshitzer- Gribov-Lipatov-Altarelli- parisi (DGLAP) equation

F_2 varies with Q^2



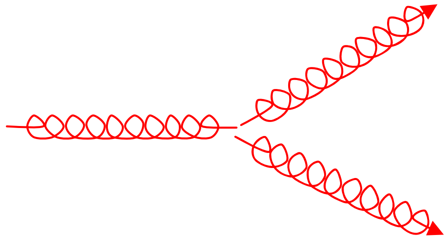
At high x

Gluon radiation shifts quark to lower x



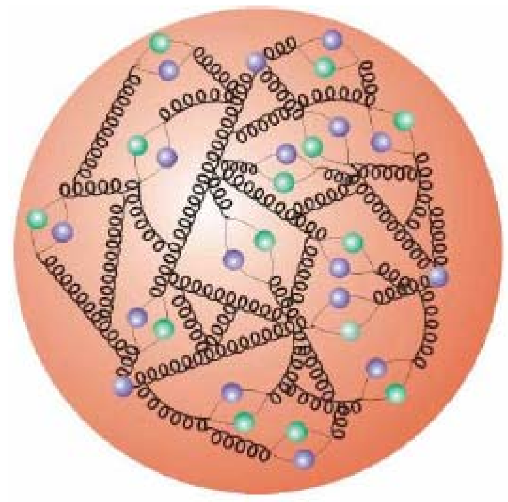
At low x

Gluon splitting enhances quark density ⇒ F_2 rises with Q^2



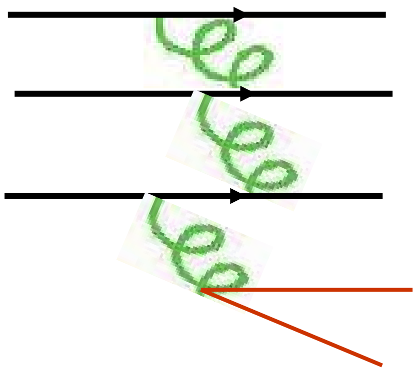
gluon radiates gluons

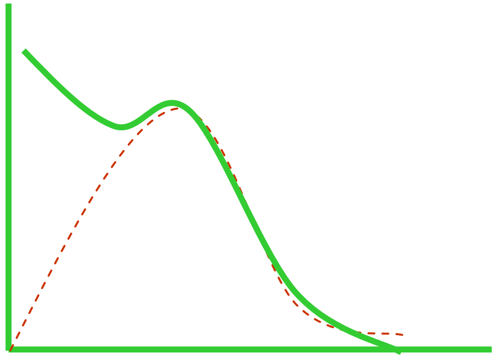
So, proton is really a complicated object



And what do we expect for F_2 as a function of x at a fixed Q^2 to look like?

All of the processes on the last slide says that branchings shift the densities to lower x





F_2 varies with Q^2

The variation is logarithmic $\frac{\partial F_2}{\partial \ell n Q^2} \sim \alpha_s x g$

Why logarithmic? It comes from the QCD diagrams

Why logarithmic?

Technical

Recall that at the parton level, F_2 is just $\hat{F}_2 = e_f^2 x$. At the proton level we must multiply it by the probability of finding parton inside the proton, $q(x)$ and integrate over x. we write this as

$$F_2(x) = e_f^2 x \int_0^1 d\xi\, q(\xi)\,\delta(x-\xi) = e_f^2 x \int_0^1 \frac{d\xi}{\xi} q(\xi)\,\delta\left(1-\frac{x}{\xi}\right)$$

At the $\mathcal{O}(\alpha_s)$ (i.e. one loop) many diagrams contribute. We are looking for scaling violations and therefore, we must investigate ultraviolet, infrared and mass singularities of any kind. It turns out that ultraviolet singularities are proportional to the free result and simply renormalize the charge. Infrared singularities cancel among the diagrams. Only the mass singularity present in diagram (d) when the momentum of the gluon parallel to the gluon survives. Keeping only the logrithmic terms, the calculation at $\mathcal{O}(\alpha_s)$ amounts to the replacement

$$\delta\left(1-\tfrac{x}{\xi}\right) \to \delta\left(1-\tfrac{x}{\xi}\right) + \tfrac{\alpha_s}{2\pi} P\left(\tfrac{x}{\xi}\right) \ell n \tfrac{Q^2}{\lambda^2}$$

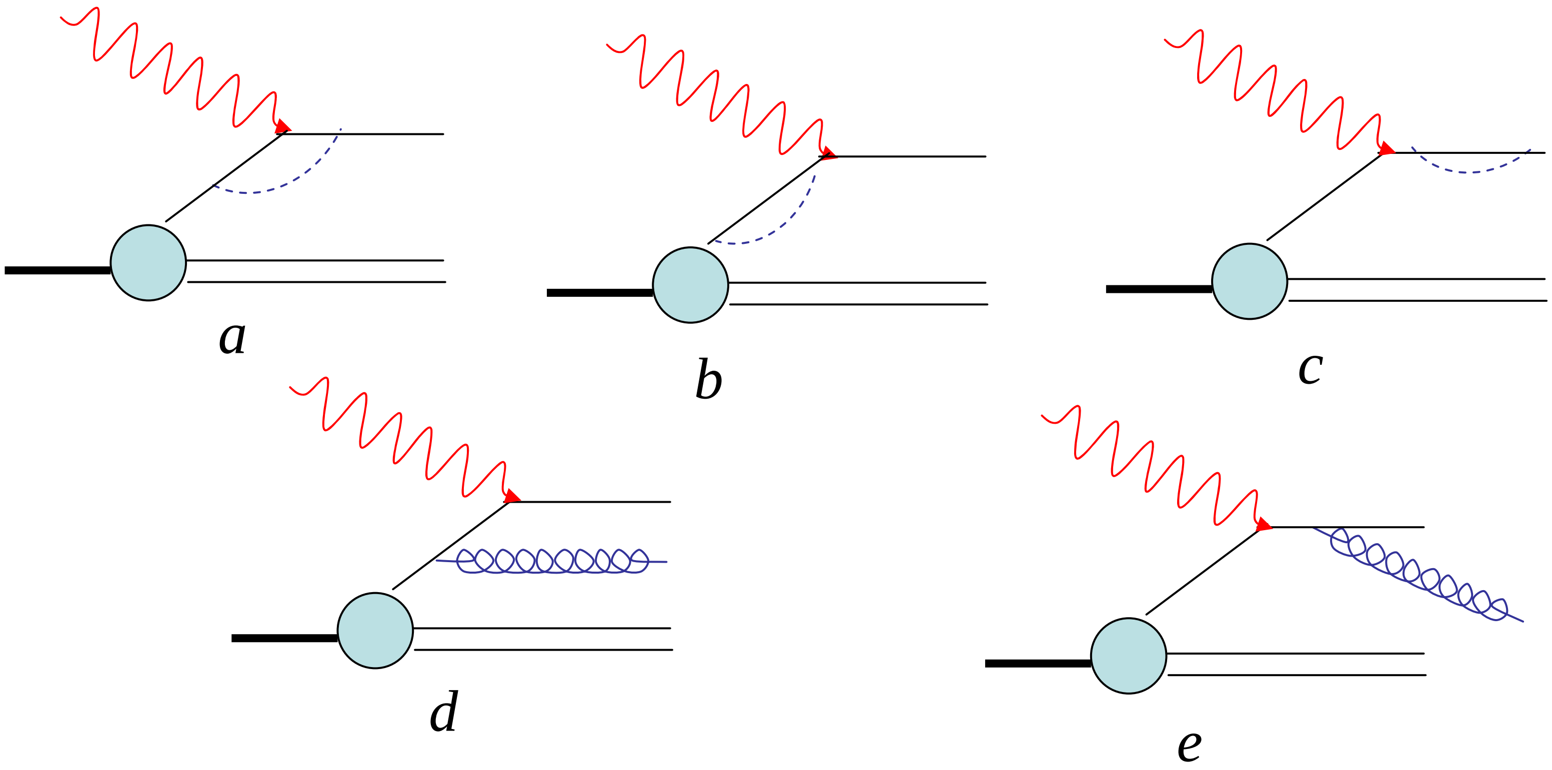
Technical

λ^2 is an infrared regulator and P is given by

$$P = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right]$$

$1/(1-z)_+$ defined by

$$\int_0^1 \tfrac{1}{(1-z)_+} f(z)\,dz = \int_0^1 \tfrac{f(z)-f(1)}{1-z} dz$$



Only the logarithmic term is retained for this lecture. Now we understand the reason for writing things in apparently such a complicated way. First of all, scaling violation appear through the contribution of real soft partons. ξ is the original momentum fraction of the nucleon that is carried by the parton, upon radiation it is reduced to $x \le \xi$ after the emission of the soft gluon. Photon interact with the parton which carrying fraction x. *Then*

$$F_2\left(x, Q^2\right) = e_i^2 x \left[\underbrace{q_0\left(x\right)}_{\text{Bare parton of QPM}} + \underbrace{\int_x^1 \frac{d\xi}{\xi} q_0\left(x\right) \frac{\alpha_s}{2\pi} P\left(\frac{x}{\xi}\right) \ell n \frac{Q^2}{\lambda^2}}_{\text{QCD Correction}} \right]$$

Technical

Define renormalized PDF

$$q\left(x, \mu^2\right) = q_0\left(x\right) + \int_x^1 \frac{d\xi}{\xi} q_0\left(x\right) \frac{\alpha_s}{2\pi} P\left(\frac{x}{\xi}\right) \ell n \frac{Q^2}{\lambda^2}$$

We get:

$$F_2\left(x, Q^2\right) = e_i^2 x \left[q\left(x, \mu^2\right) + \int_x^1 \frac{d\xi}{\xi} q\left(x, \mu^2\right) \frac{\alpha_s}{2\pi} P\left(\frac{x}{\xi}\right) \ell n \frac{Q^2}{\mu^2} \right]$$

Now the infrared regulator is eliminated (hidden in the bare PDF, note that this is an IR ill-defined object because of the soft gluon emission) at the expense of introducing a renormalization- scale dependence. These are the sought after scaling ciolations.

Technical

DGLAP Equation

Let us introduce the variable : $t = \frac{1}{2}\ell n \frac{\mu^2}{\Lambda_{QCD}^2}$ It follow from

$$q\left(x,\mu^2\right) = q_0\left(x\right) + \int_x^1 \frac{d\xi}{\xi} q_0\left(x\right) \frac{\alpha_s}{2\pi} P\left(\frac{x}{\xi}\right) \ell n \frac{Q^2}{\lambda^2}$$

$$\Longrightarrow \quad \frac{\partial}{\partial t} q\left(x,t\right) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} q\left(\xi,t\right) P\left(\frac{x}{\xi}\right)$$

DGLAP Equation

Which immediately translates into differential equation for F_2

DGLAP equations summarize the rate of change of parton distributions with t

It is simpler to work with the moments, rather than with the distribution functions itself. Define the moments by:

$$q(n,t) = \int_0^1 dx\, x^{n-1} q(x,t)$$

Technical

Introduce the anomalous dimension γ_n as

$$\gamma_n = \int_0^1 dx\, x^{n-1} P(x)$$

The convolution over the fractional momentum ξ transforms into a product. That is the simplification of working in moment space rather than working in x space: we get

$$\frac{\partial}{\partial t} q(n,t) = \frac{\alpha_s}{\pi} \gamma_n q(n,t)$$

This leads to following scaling behavior for the moments of the structure functions

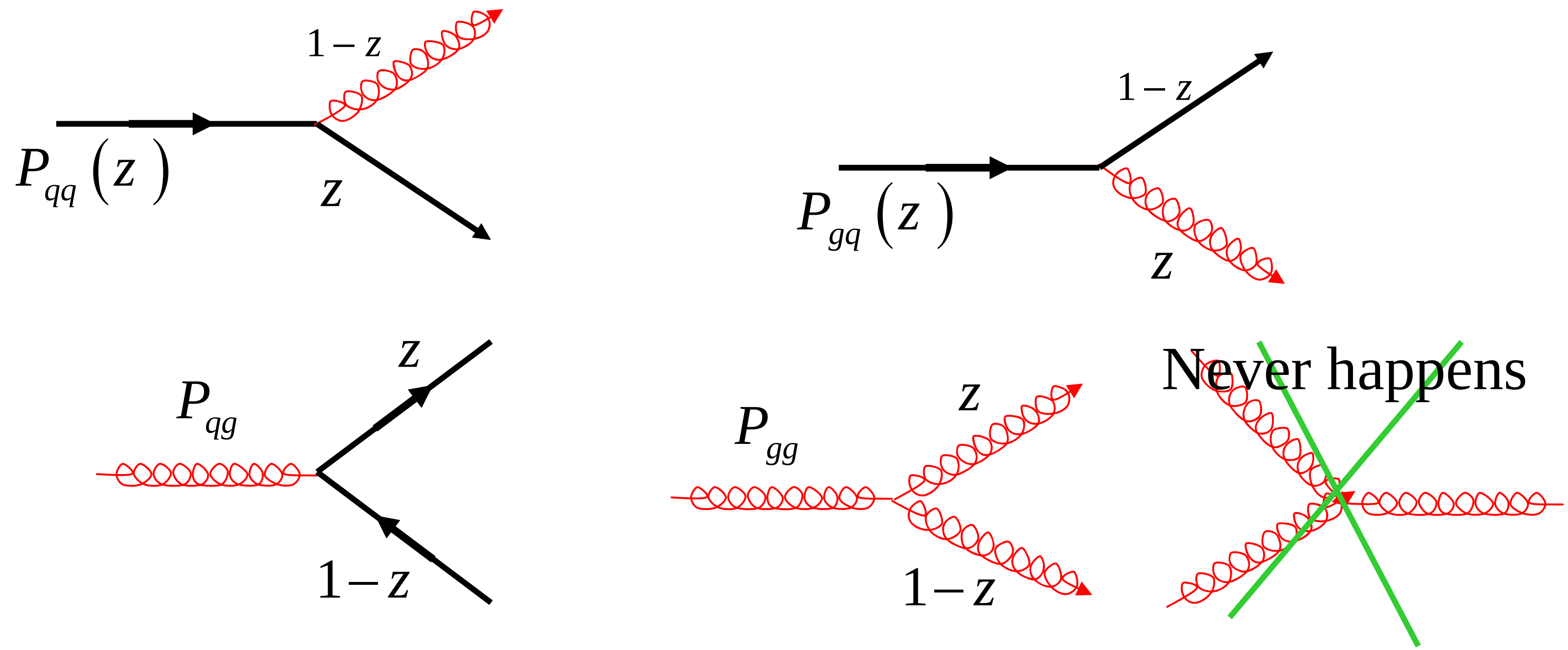
$$F_2(n,Q^2) = F_2(n,Q_0^2) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\frac{\gamma_n}{\beta_1}}$$

Let us put the technicality into perspective

DGLAP Equation is essentially a diff. equation of the form

$$\frac{\partial}{\partial \ell n Q^2} f_p \sim f_p \otimes P, \qquad f_p : q, g \text{ density} \qquad P : \text{Splitting function}$$

In the LO we need to consider 4 splitting functions, correponding to



P_{ba} : Probability that parton a will radiate a parton b with the fraction z of the original momentum carried by a

So, the DGLAP equation, $\frac{\partial}{\partial \ell n Q^2} f_p \sim f_p \otimes P$ For quarks and gluons are:

$$\frac{\partial}{\partial \ell n Q^2}\Sigma\left(x,Q^2\right)=\frac{\alpha_s\left(Q^2\right)}{2\pi}\left(\left[\Sigma\otimes P_{qq}\right]+\left[g\otimes 2n_f P_{qg}\right]\right)$$

Where $\Sigma\left(x,Q^2\right)=\sum_i\left(q_i(x,Q^2)+\bar{q}_i(x,Q^2)\right)$ is the quark density summed over all active flavors.

And for the gluon we have

$$\frac{\partial}{\partial \ell n Q^2} g\left(x,Q^2\right)=\frac{\alpha_s\left(Q^2\right)}{2\pi}\left(\left[\Sigma\otimes P_{gq}\right]+\left[g\otimes P_{gg}\right]\right)$$

More commonly: for non-singlet (valence), singlet,gluon

$$\frac{\partial}{\partial \ln Q^2} q_{NS}^{\pm} = P_{NS}^{\pm} \otimes q_{NS}^{\pm}$$

$$\frac{\partial}{\partial \ln Q^2}\begin{bmatrix}\Sigma\left(Q^2\right)\\ g\left(Q^2\right)\end{bmatrix}=\begin{bmatrix}P_{qq} & P_{qg}\\ P_{gq} & P_{gg}\end{bmatrix}\otimes\begin{bmatrix}\Sigma\left(\mu^2\right)\\ g\left(\mu^2\right)\end{bmatrix}=P\otimes q$$

$$\frac{\partial}{\partial \ln Q^2} q_{NS}^{\pm} = P_{NS}^{\pm} \otimes q_{NS}^{\pm}$$

$$\frac{\partial}{\partial \ln Q^2} \begin{bmatrix} \Sigma\left(Q^2\right) \\ g\left(Q^2\right) \end{bmatrix} = \begin{bmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{bmatrix} \otimes \begin{bmatrix} \Sigma\left(\mu^2\right) \\ g\left(\mu^2\right) \end{bmatrix} = P \otimes q$$

Where in the leading order

$$P(x) = a_S P^{(0)}(x) + a_S^2 \left[P^{(1)}(x) - \beta_0 \ln \frac{\mu_F^2}{\mu_R^2} P^{(0)}(x) \right] \qquad \text{with} \qquad a_S = \frac{\alpha_S(\mu_R^2)}{4\pi}$$

$$\frac{da_S}{d \ln \mu_R^2} = \beta(a_S) = \sum_{l=0}^{\infty} a_S^{l+2} \beta_l \cong a_S^2 \beta_0 + a_S^3 \beta_1 \qquad \text{where} \quad \beta_0 = 11 - \frac{2}{3} N_F$$

$$\text{and} \quad \beta_1 = 102 - \frac{38}{3} N_F$$

Parton distributions

In general we do not know how to compute the parton distributions, even for $Q^2 \to \infty$. Only their evolution can be reliably computed either through OPE or using DGLAP equation. We can, however use the experimental data to find parton distributions.

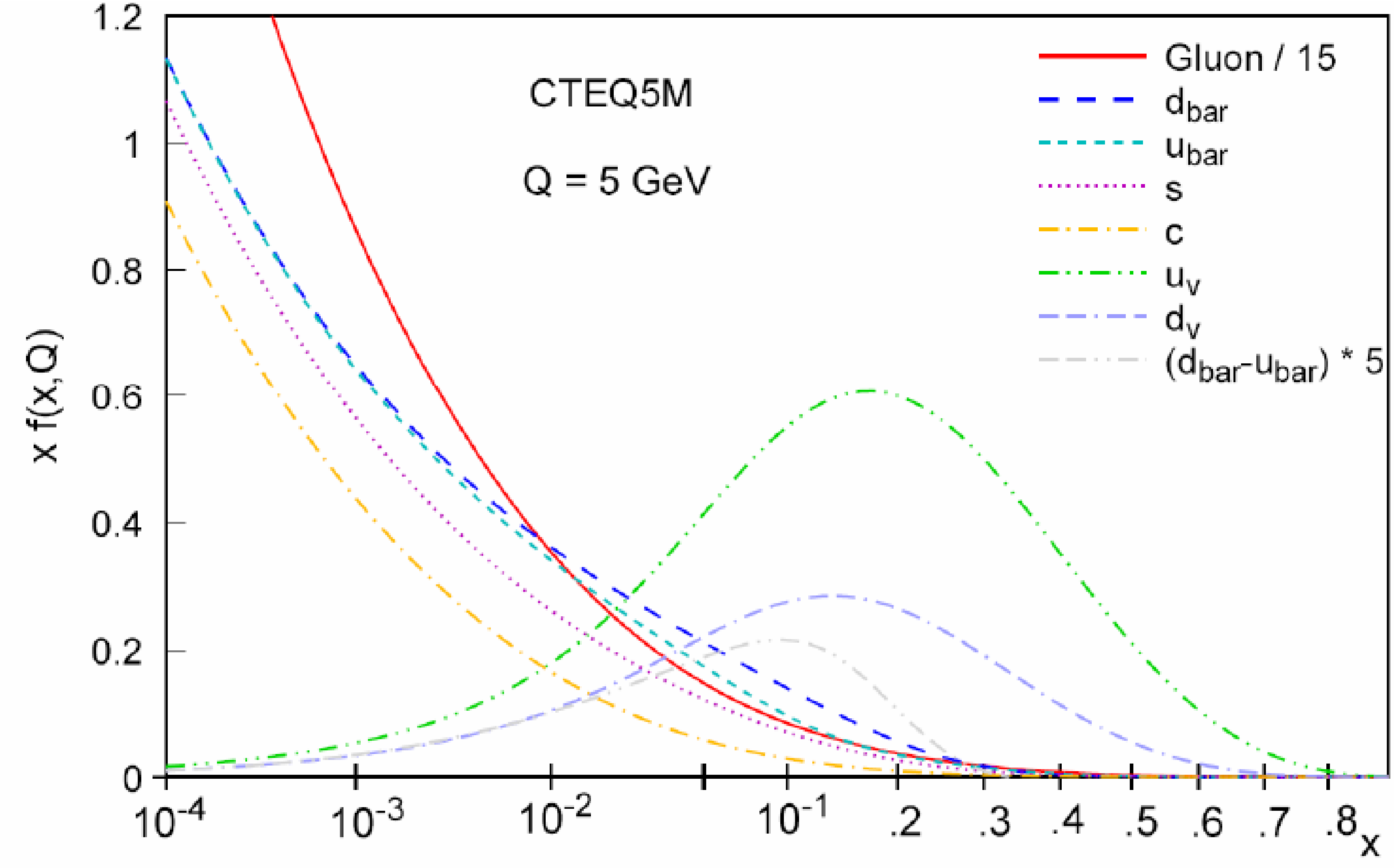
The recipe is:

Step 1: parameterize the parton densities, $q(x), g(x)$ at some Q^2 as:

$$q(x) = p_1 x^{p_2} (1-x)^{p_3} \left(1 + p_4 \sqrt{x} + p_5 x\right)$$

Step 2: find $p_1 \cdots p_5$ by fitting to DIS (and other) data, using DGLAP evolution equation and then evolve $q(x)$ in Q^2

Overview of parton distributions



A final note

The scaling behavior is governed by the anomalous dimensions.
At the Leading Order they are

$$\gamma_{qq}(j) = C_F\left[-\frac{1}{2} + \frac{1}{j(j+1)} = 2\sum_{k=2}^{j}\frac{1}{k}\right]$$

$$\gamma_{qg}(j) = T_R\left[\frac{2+j+j^2}{j(j+1)(j+2)}\right]$$

$$\gamma_{gq}(j) = C_F\left[\frac{2+j+j^2}{j\left(j^2-1\right)}\right]$$

$$\gamma_{gg}(j) = 2C_A\left[-\frac{1}{2} + \frac{1}{j-1} + \frac{1}{(j+1)(j+2)} - \sum_{k=2}^{j}\frac{1}{k}\right] - \frac{2N_f}{3}T_R$$

Sea also, Arash, Khorramian, PRC 67, 045201 (2003) for the NLO calculations

C_F, C_A, and T_R are group factors

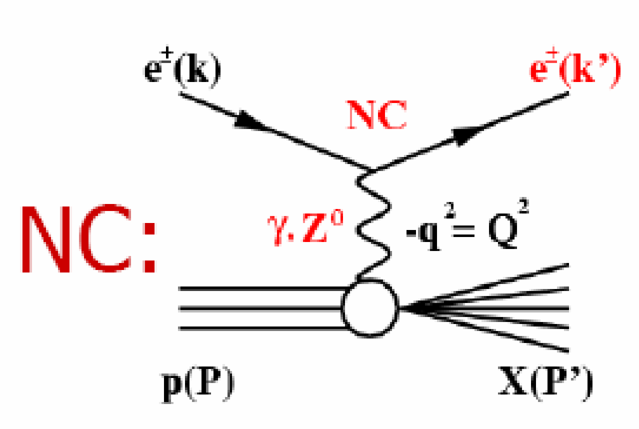
A final note-2

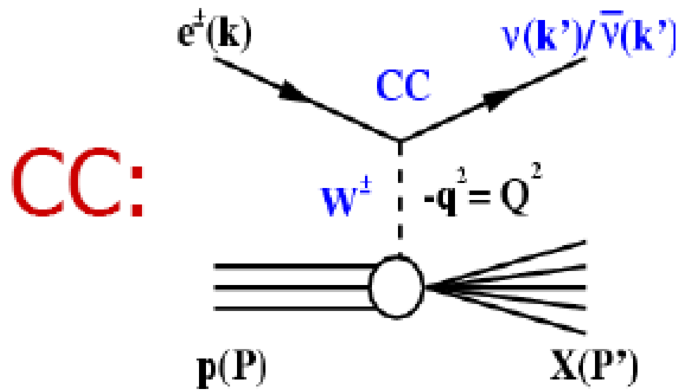
We worked with F_2, but it is the same with any other structure function.

If we used neutrino, instead of electron, or considered Z probe (instead of γ^*) then, we would have an additional structure function, F_3. The reason: γ_5 coupling

$$\mathrm{NC}: \quad e^{\pm} + p \to e^{\pm} + X, \qquad \mathrm{CC}: \; e^{\pm} + p \to \bar{\nu}_e(\nu_e) + X$$

$W^{\pm}$ Probe gives CC, and Z (like photon) corresponds to NC





ep Neutral Current (γ*,Z) Cross Section

$$\frac{d^2\sigma_{ep}}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}\left(Y_+F_2 - y^2F_L \mp Y_-xF_3\right)$$

$Y_\pm \equiv \left(1 \pm (1-y)^2\right)$, the inelasticity parameter

$F_2(x,Q^2) = x \sum_q e_q^2 \left(q(x,Q^2) + \bar{q}(x,Q^2)\right)$

- the sum of the quark and anti-quark densities

$xF_3(x,Q^2) = x \sum_q e_q^2 \left(q(x,Q^2) - \bar{q}(x,Q^2)\right)$

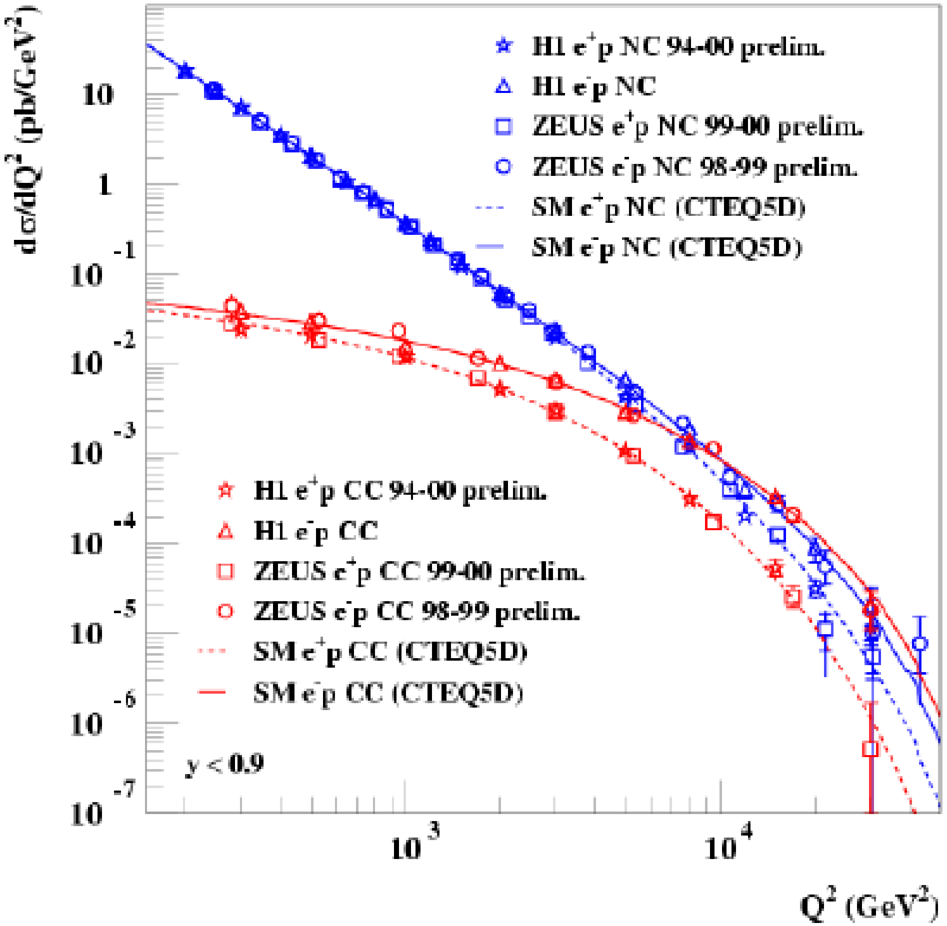
- the difference of the quark and anti-quark densities, small for $Q^2 \ll M_Z^2$

$F_L(x,Q^2) \sim F_2 - xg(x,Q^2)$

- the longitudinal structure function which vanishes at LO in QCD and is damped by y^2 in the cross section

NC helps to separate sea and valence quark distributions

Comparing NC and CC cross-sections at HERA: EW Unification



NC cross section sharply decreases with increasing Q^2(dominantly γ exchange)

$$\sim 1/Q^4$$

CC cross section approaches a constant at low Q^2

$$\sim \left[M_W^2 / \left(Q^2 + M_W^2 \right) \right]^2$$

Dramatic confirmation of the unification of electromagnetic and weak interactions of SM in Deep Inelastic Scattering!

Polarized

Now, let me go back to hadron tensor $W_{\mu\nu}$ for the polarized case

Now in addition to variables p, q we also have the hadron's spin s_h

$$\Rightarrow W_{\mu\nu} = W_{\mu\nu}\left(p, q, s_h\right)$$

As in the case of $\ell^{\mu\nu}$ We will have additional tensor quantities

$$i\,\varepsilon_{\mu\nu\lambda\sigma} q^{\lambda} s^{\sigma},\; i\,\varepsilon_{\mu\nu\lambda\sigma} q^{\lambda}\, p\cdot q\, ^{\sigma} s, \text{and } i\,\varepsilon_{\mu\nu\lambda\sigma} q^{\lambda}\left(s\cdot q\right) p^{\sigma}$$

They enter into our earlier decomposition as additional terms

$$W_{\mu\nu} = F_1\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right), + \frac{F_2}{p.q}\left(p_\mu - \frac{\left(p\cdot q\right) q_\mu}{q^2}\right)\left(p_\nu - \frac{\left(p\cdot q\right) q_\nu}{q^2}\right) \qquad \text{Unpol.}$$

$$+ \frac{i g_1}{p.q}\varepsilon_{\mu\nu\rho\sigma} q^{\lambda} s^{\sigma} + \frac{i g_2}{\left(p.q\right)^2}\varepsilon_{\mu\nu\rho\sigma} q^{\lambda}\left(p.q\, s^{\sigma} - \left(s.q\right) p^{\sigma}\right) \qquad \text{Polarized}$$

The leptonic tensor $\ell^{\mu\nu}$ is conserved $q^{\mu}\ell^{\mu\nu} = q^{\nu}\ell^{\mu\nu} = 0$

So, we better simplify $W_{\mu\nu}$ befor contracting with $\ell^{\mu\nu}$

$$W_{\mu\nu} = -F_1 g_{\mu\nu} + \frac{F_2}{p.q} p_\mu p_\nu + \frac{ig_1}{\nu} \varepsilon_{\mu\nu\lambda\sigma} q^\lambda s_h^\sigma + \frac{ig_2}{\nu^2} \varepsilon_{\mu\nu\lambda\sigma} q^\lambda \left(p.q\, s_h^\sigma - (s_h \cdot q) p \right)$$

symmetric under $\mu\nu$

Spin part is antisymmetric under $\mu\nu$

$$mW_1 = F_1, \quad \nu W_2 = F_2 \qquad m\nu^2 G_1 = g_1, \quad m\nu^2 G_2 = g_2$$

We have already seen that the symmetric part of $\ell_{\mu\nu}$ is independent of the lepton spin, and the spin dependent part of $\ell_{\mu\nu}$ is antisymmetric in $\mu\nu$. Thus the conbination $W^{\mu\nu}\ell_{\mu\nu}$ has no terms which have only the hadron spin, or only the lepton spin; all terms contain either both or none.

The structure functions F_1 and F_2 can be measured using unpolarized beam and target, but to measure g_1 and g_2 requires both a polarized beam and polarized target. There is no advantage to do experiment with only a polarized beam or only a polarized target. This is not true for spin greater than ½

Cross section for spin ½ target

Combining $\ell^{\mu\nu}$:

$$\ell^{\mu\nu} \approx 2\left(k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - g^{\mu\nu}k\cdot k' - i\,\varepsilon^{\mu\nu\alpha\beta}q_{\alpha}s_{\ell\beta}\right)$$

And $W_{\mu\nu}$:

$$W_{\mu\nu} = F_1\left(-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2}\right), + \frac{F_2}{p.q}\left(p_{\mu} - \frac{(p\cdot q)q_{\mu}}{q^2}\right)\left(p_{\nu} - \frac{(p\cdot q)q_{\nu}}{q^2}\right) \qquad \text{Unpol.}$$

$$+\frac{ig_1}{p.q}\varepsilon_{\mu\nu\rho\sigma}q^{\lambda}s^{\sigma} + \frac{ig_2}{(p.q)^2}\varepsilon_{\mu\nu\rho\sigma}q^{\lambda}\left(p.q\,s^{\sigma} - (s.q)p^{\sigma}\right) \qquad \text{Polarized}$$

In the cross section

$$\frac{d^2\sigma}{dx\,dy\,d\phi} = \frac{e^4}{16\pi^2 Q^4}\,y\,\ell_{\mu\nu}W^{\mu\nu}(p,q)_{\lambda\lambda}$$

And using the identity

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\lambda\sigma} = -2\left(g^{\alpha}_{\lambda}g^{\beta}_{\sigma} - g^{\alpha}_{\sigma}g^{\beta}_{\lambda}\right)$$

$$\frac{d^2\sigma}{dx\,dy\,d\phi} = \frac{e^4 ME}{4\pi^2 Q^4}\left[xy^2 F_1 + (1-y) F_2 + y^2 g_1 \left(2x \frac{s_\ell \cdot s_h}{p\cdot q} + \frac{q\cdot s_\ell}{p\cdot q}\frac{q\cdot s_h}{p\cdot q} \right) + 2xy^2 g_2 \left(\frac{s_\ell \cdot s_h}{p\cdot q} - \frac{p\cdot s_\ell}{p\cdot q}\frac{q\cdot s_h}{p\cdot q} \right) \right]$$

I want to concentrate on two cases which are more intersting for the present days experiments: Longitudinal and Transversal polarizations target and longitudinally polarized lepton.

For longitudinally polarized lepton beam incident on the target which is either longitudinally or transversely polarized. The polarization of the lepton is $s_\ell = \mathcal{H}_\ell k$, where $\mathcal{H}_\ell = \pm$ is the lepton helisity . The lepton polarized term in the cross section, above can be written as

$$\frac{q\cdot s_\ell}{p\cdot q} = \mathcal{H}_\ell \frac{q\cdot k}{p\cdot q} = -\mathcal{H}_\ell x, \quad \frac{p\cdot s_\ell}{p\cdot q} = \mathcal{H}_\ell \frac{p\cdot k}{p\cdot q} = \frac{\mathcal{H}_\ell}{y}, \quad \frac{s_h\cdot s_\ell}{p\cdot q} = \mathcal{H}_\ell \frac{s_h\cdot k}{p\cdot q}$$

Longitudinally polarized target:

A target polarized along the incident beam has $\vec{s}_h = M\,\mathcal{H}_h \hat{\mathbf{z}}$, where

$\mathcal{H}_h = \pm$ for a target polarized parallel (+) or antiparallel (-) to the beam

In the cross section expression we only have $q\cdot s_h$ and $k\cdot s_h$

$$\frac{d^2\sigma}{dx\,dy\,d\phi} = \frac{e^4 ME}{4\pi^2 Q^4}\left[xy^2F_1 + (1-y)F_2 + y^2 g_1\left(2x\frac{s_\ell \cdot s_h}{p\cdot q} + \frac{q\cdot s_\ell}{p\cdot q}\frac{q\cdot s_h}{p\cdot q}\right) + 2xy^2 g_2\left(\frac{s_\ell\cdot s_h}{p\cdot q} - \frac{p\cdot s_\ell}{p\cdot q}\frac{q\cdot s_h}{p\cdot q}\right)\right]$$

$$\frac{s_h\cdot s_\ell}{p\cdot q} = \mathcal{H}_\ell \frac{s_h\cdot k}{p\cdot q}$$

Also recall

$$q = k - k' = \left(E-E', E'\sin\theta\cos\phi, E'\sin\theta\sin\phi, E-E'\cos\theta\right)$$

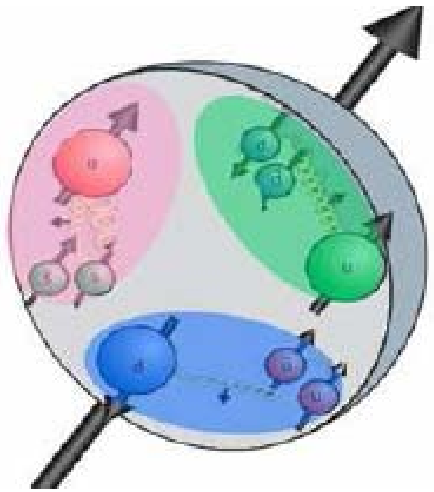
⇨ k and q both has 0 and 3 components which are almost equal

Thus, s_h can be replaced by $-\mathcal{H}_h p$ when evaluating cross section in the DIS limit. (note the additional minus sign, which comes from the relative minus sign between the space and time components in the dot product, such as $k.s$). The result is

$$\frac{d^2\sigma}{dxdy} = \frac{e^4 ME}{2\pi Q^4}\left[xy^2 F_1 + (1-y)F_2 - \mathcal{H}_\ell \mathcal{H}_h y(2-y)x g_1 + \mathcal{O}\left(M^2/Q^2\right)\right]$$

The effect of g_2 term is suppressed relative to the leading term

Polarization asymmetry in the cross section can be used to measure the structure function g_1



Arash, F. and Taghavi, F.

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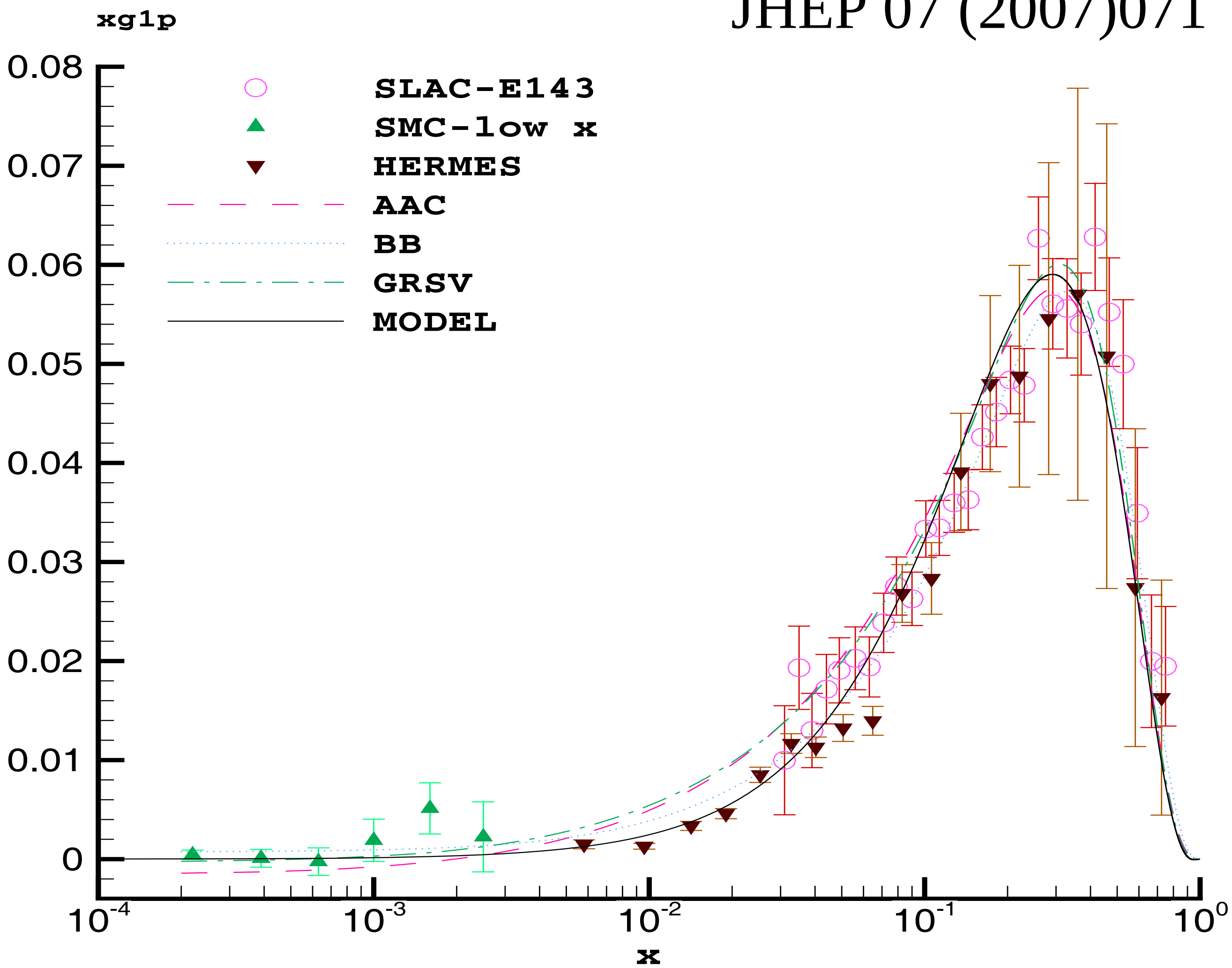

DGLAP in Moment space

$$\delta M_{NS\pm}(n,Q^2) = \left\{1+\frac{\alpha_s(Q^2)-\alpha_s(Q_0^2)}{2\pi}\left(\frac{-2}{\beta_0}\right)\left(\delta \mathbf{P}_{NS\pm}^{(1)n}-\frac{\beta_1}{2\beta_0}\delta \mathbf{P}_{qq}^{(0)n}\right)\right\}\mathbf{L}^{-(\frac{2}{\beta_0})\delta P_{qq}^{(0)n}}$$

$$\begin{pmatrix}\delta M_S(n,Q^2)\\ \delta M_G(n,Q^2)\end{pmatrix} = \left\{\mathbf{L}^{-(\frac{2}{\beta_0})\delta\hat{P}^{(0)n}}+\frac{\alpha_s(Q^2)}{2\pi}\hat{\mathbf{U}}\mathbf{L}^{-(\frac{2}{\beta_0})\delta\hat{P}^{(0)n}}-\frac{\alpha_s(Q_0^2)}{2\pi}L^{-(\frac{2}{\beta_0})\delta\hat{P}^{(0)n}}\hat{\mathbf{U}}\right\}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$\mathbf{L} \equiv \alpha_s(Q^2)/\alpha_s(Q_0^2)$, and $\delta\hat{P}^{(0)n}$ is 2×2 singlet matrix of splitting functions.

$$\delta\hat{P}^{(0)n} = \begin{pmatrix}\delta P_{qq}^{(0)n} & 2f\delta P_{qg}^{(0)n}\\ \delta P_{gq}^{(0)n} & \delta P_{gg}^{(0)n}\end{pmatrix},$$

$\delta P_{lm}^{(0)n}$ are the n^{th} moments of the polarized splitting functions

U accounts for the 2-loop contribution as an extension to LO

Make an inverse Mellin transform back to momentum space

$$\delta q_{valon}^{NS,S,G}(z,Q^2) = \frac{1}{\pi}\int_0^\infty Im[e^{i\phi}z^{-c-we^{i\phi}}\Delta M^{NS,S,G}(n=c+we^{i\phi},Q^2)]dw,$$

The polarized structure function

$$g_1\left(x,q^2\right)=\tfrac{1}{2}\sum_q e_q^2\left[\delta q\left(x,Q^2\right)+\delta\overline{q}\left(x,Q^2\right)\right]$$

where

$$\delta q\left(x,Q^2\right)=q^{\rightarrow}\left(x,Q^2\right)-q^{\rightarrow}\left(x,Q^2\right)$$

$$\delta\overline{q}\left(x,Q^2\right)=\overline{q}^{\rightarrow}\left(x,Q^2\right)-\overline{q}^{\rightarrow}\left(x,Q^2\right)$$

$\delta q\left(x,Q^2\right)$ Measures how much the parton of flavor q remembers the parent proton polarization.

The First moment of polarized parton distribution is defined as

$$\Delta q\left(Q^2\right)=\int_0^1 dx\ \delta q\left(x,Q^2\right)$$

They are related to the total z- component of quark (and gluon) spins by:

$$\langle S_z \rangle_q = \tfrac{1}{2}\Delta q, \qquad \langle S_z \rangle_g = \Delta g$$