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# Hopf algebras and Quantum groups with their treatments in particle physics

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Abstract. In the recent years, Hopf algebras have been introduced to describe certain combinatorial properties of quantum field theories.I have a short review of Hopf algebras and Quantum groups in this lecture. I will give a basic introduction to these algebras and objects and review some occurrences in particle physics and explain our conclude and ideas in this matter with some examples.

## Introduction

The mathematics of non-commutative geometry was pioneered by several great mathematicians, including the legendary Russian mathematician Israil Gelfand, who with his collaborators proved the first key theorems about $C^*-$algebras, and in modern times the Fields medallist Alain Connes. Connes recognised that non-commutative geometry could be immensely useful for theoretical physics. One of his ideas here, loosely speaking, is to reformulate the basic pattern of elementary particle physics by appending non-commutative "extra dimensions" to the usual classical notion of spacetime. This model is not about quantum gravity in the first instance; the space-time coordinates $x, y, z, t$(three for space and one for time) from an ordinary commutative algebraic system for spacetime as usual, but Connes then "takes on" non-commutative matrix coordinates that nearly encode all the different types of particles. His theory is not only exceedingly elegant, but makes a number of predictions that had previously eluded physicists, including predictions about the masses of fundamental particles like the mysterious Higgs boson.
Shahn Majid is Professor of Mathematics at Queen Mary, University of London. Educated at Cambridge and Harvard, he was one of the pioneers of the theory of

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quantum groups in the late 1980s and early 1990s. He is the author of numerous research articles and two textbooks on quantum algebra. He is currently on sabbatical at the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge. Majid has taken non-commutative geometry down a slightly different road. In his work he forgets about the classical picture of spacetime altogether. He still has the four objects that take the place of the four co-ordinates -$x, y, z$ for space and $t$ for time- but now these coordinates do not commute: xt is not the same as tx. Together the coordinates form a non-commutative algebraic system and spacetime would then be the mysterious geometrical space that we're hoping comes attached to this system.
But simply saying that the co-ordinates don't commute is not quite enough. If xt is not the same as tx, then in order to construct a specific system we must say by how much they differ. Majid suggests that a number called the Planck scale measures this amount of non-commutativity.
Heinz Hopf, one of the pioneers of Algebraic topology, first introduced these algebras in connection with the homology of Lie groups in 1939. Later, in the 1960s Milnor introduced the Steenrod algebra, the algebra of cohomology operations, which was another example of a Hopf algebra. More recently the study of these algbras has gained pace because of their applications in Physics as quantum groups, renormalisation and non-comutative geometry. In the late 1970s Rota introduced Hopf algebras into combinatorics and there is now a well established research field know as combinatorial Hopf algebras. An article by Woronowicz in 1987, which provided explicit examples of non-trivial Hopf algebras, triggered the interest of the physics community.
In turn, Hopf algebras have been used for integrable systems and quantum groups. In 1998 Kreimer and Connes re- examined renormalization of quantum field theories and showed that it can be described by a Hopf algebra structure.
Quantum groups, introduced in 1986 by Drinfeld, form a certain class of Hopf algebras. Up to date there is no rigorous, universally accepted definition, but it is generally agreed that this term includes certain deformations in one or more parameters of classical objects associated to algebraic groups, such as enveloping algebras of semisimple Lie algebras or algebras of regular functions on the corresponding algebraic groups. As one can relate algebraic groups with commutative Hopf algebras via group schemes, it is also agreed that the category of quantum groups should correspond to the opposite category of the category of Hopf algebras. This is why some authors define quantum groups as non-commutative and non-cocommutative Hopf algebras. We can say that a quantum group is a deformation of a Hopf algebra; this Hopf algebra is typically the algebra of $\mathbb{C}^\infty$ functions on a Lie group, or the universal enveloping algebra of its Lie algebra. Quantum groups arise in diverse areas of physics and mathematics, for example, quantum inverse scattering, knot theory and 3 manifold invariants and representation theory.

**Definition 0.0.1.** $(a)$ A lie algebra L is a vector space with a bilinear map $[,]: L \times L \to L$,called the Lie bracket, satisfying the following two conditions for all $x, y, z \in L$:
(i)(antisymmetry)

$$[x, y] = -[y, x],$$

(ii)(Jacobi identity)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

$(b)$ A morphism of Lie algebras f from the Lie algebra L into the Lie algebra $L^{'}$, is a linear map $f : L \to L^{'}$ such that $f([x,y]) = [f(x), f(y)]$ for all $x, y \in L$.

The Lie algebra $gl(2) = L(M_2(\mathbb{C}))$of $2 \times 2-$matrices with complex entries is four-dimensional. The four matrices

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis of $gl(2)$. Their commutators are easily computed. We get

$$[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y,$$

and

$$[I, X] = [I, Y] = [I, H]$$

The above relations show that $sl(2)$ is an ideal of $gl(2)$ and that there is an isomorphism of Lie algebras

$$gl(2) \cong sl(2) \bigoplus \mathbb{C}I$$

which reduces the investigation of the Lie algebra $gl(2)$ to that of $sl(2)$.

To any Lie algebra L we assign an(associative) algebra U(L), called the enveloping algebra of L, and a morphism of Lie algebras $i_L : L \to L(U(L))$. We define the enveloping algebra as follows. Let $I(L)$ be the two-sided ideal of the tensor algebra $T(L)$ generated by all elements of the form $xy - yx - [x, y]$ where x,y are elements of L. We define

$$U(L) = \frac{T(L)}{I(L)}$$

**Proposition 0.0.2.** *The enveloping algebra U(L) as a cocommutative Hopf algebra for maps $\Delta$, $\epsilon$ and $S$, for $x_1, ..., x_n \in L$:*

$$\Delta(x_1...x_n) = 1 \otimes x_1...x_n + \sum_{p=1}^{n-1} \sum_{\sigma} x_{\sigma(1)}...x_{\sigma(p)} \otimes x_{\sigma(p+1)}...x_{\sigma(n)} + x_1...x_n \otimes 1.$$

*where $\sigma$ runs over all $(p, q)-$shuffles of the symmetric group $S_n$, and*

$$S(x_1...x_n) = (-1)^n x_n...x_1.$$

The enveloping algebra

$$U = U(sl(2))$$

of $sl(2)$ is isomorphic to the algebra generated by the three elements $X, Y, H$ with the three relations

$$[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y,$$

## 1. The algebra $U_q(sl(2))$

Let me fix an invertible element $q$ of $\mathbb{C}$ different from 1 and $-1$ so that, the fraction $\frac{1}{q-q^{-1}}$ is well-defined. I review some notation.
For any integer n, set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + ... + q^{-n+3} + q^{-n+1}$$
$$[-n] = -[n] and [m+n] = q^n[m] + q^{-m}[n].$$

We also have the following versions of factorials and binomial coefficients. For integers $0 \leq k \leq n$, set $[0]! = 1$,

$$[k]! = [1][2]...[k]$$

if $k > 0$, and

$$H = \left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]![n-k]!}$$

for

$$[n] = q^{-(n-1)}(n)_{q^2}, [n]! = q^{\frac{-n(n-1)}{2}}(n)!_{q^2}$$

and

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = q^{-k(n-k)} \left( \begin{array}{c} n \\ k \end{array} \right)_{q^2}$$

**Definition 1.0.3.** We define $U_q = U_q(sl(2))$ as the algebra generated by the four variables $E, F, K, K^{-1}$ with the relations

$$KK^{-1} = K^{-1}K = 1$$
$$KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

Example:(The coordinate Hopf algebras $O(G)$ of simple matrix Lie groups).
Let $G$ denote one of the matrix groups $SL(N, \mathbb{C}), SO(N, \mathbb{C}) or Sp(N, \mathbb{C})$. Each element g of $G$ is a complex $N \times N$ matrix $g = (g_{ij})$. Define the coordinate functions $u_i^j$ on $G$ by $u_i^j(g) := g_{ij}, g = (g_{ij}) \in G$. For $g, h \in G$ we have

$$\Delta(u_i^j)(g, h) = u_i^j(gh) = (gh)_{ij} = \sum_k g_{ik}h_{kj} = \sum_k u_k^i(g)u_j^k(h)$$

and $u_i^j(e) = \delta_{ij}$, so that

$$\Delta(u_i^j) = \sum_k u_k^i \bigotimes u_j^k and \epsilon(u_i^j) = \delta_{ij}.$$

Let $A = O(G)$ be the subalgebra of $Fun(G)$ generated by the $N^2$ functions $u_i^j, i, j = 1, 2, ..., N$. Since $\Delta : Fun(G) \to Fun(G \times G)$ is an algebra homomorphism, we have $\Delta(A) \subseteq A \otimes A$ by the first formula of the above sequence. Any element $g \in G$ has determinant 1. Hence the function which is a constant equal to 1 on $G$ belongs to $A$, so $A$ has a unit. Further, it follows that there are polynomials $P_{ij}$ in $N^2$ indeterminates such that $(g^{-1})_{ij} = p_{ij}(g_{11}, g_{12}, ..., g_{NN}$, so that

$$S(u_i^j)(g) = u_i^j(g^{-1})_{ij} = P_{ij}(u_1^1(g), u_2^1(g), ..., u_N^N(g)$$
$$= P_{ij}(u_1^1(g), u_2^1(g), ..., u_N^N(g).$$

That is, we have $S(u_i^j) \in A$ and hence $S(A) \subseteq A$. Therefore, $A = O(G)$ is a Hopf algebra.

## 2. The Compact Quantum Group $SU_q(2)$

Recall that the quantum group $SU_q(2)$ is described by the Hopf $*-$algebra $O(SU_q(2))$ which is just the Hopf algebra $O(SL_q(2))$ for real q equipped with the $*-$structure defined in following way

$$a^* = d, b^* = -qc, c^* = -q^{-1}b, d^* = a$$

The corresponding Hopf algebra is called the coordinate algebra of the real quantum group $SU_q(2)$ and is denoted by $O(SU_q(2))$. It is called the compact real form of $SL_q(2)$. Therefore, the theory developed in the previous section applies in particular to $SU_q(2)$. But it still remains to deal with all questions and properties related to the $*-$structure.
Throughout this section $q$ is a real number such that $q \neq 0, \pm 1$ and $A$ denotes the Hopf $*-$algebra $O(SU_q(2))$.
In previous example we have seen that the algebraic properties of the group $SL(2, \mathbb{C})$ are stored in its coordinate Hopf algebra $O(SL(2))$. Let me briefly recall the structure of this Hopf algebra from example. As an algebra, $O(SL(2))$ is the quotient of the commutative polynomial algebra $\mathbb{C}[u_1^1, u_2^1, u_1^2, u_2^2]$ in four indeterminates $u_1^1, u_2^1, u_1^2, u_2^2$ ( the coordinate functions on $SL(2, \mathbb{C})$ by the two sided ideal generated by the element $u_1^1 u_2^2 - u_2^1 u_1^2 - 1$. On the generators the comultiplication $\Delta$, the counit $\epsilon$ and the antipode $S$ are given by

$$\Delta(u_j^i) = \sum_k u_k^i \otimes u_j^k$$

$$\epsilon(u_j^i) = \delta_{ij}$$

$$S(u_1^1) = u_2^2, S(u_2^1) = -u_2^1, S(u_1^2) = -u_1^2, S(u_2^2) = u_1^1$$

The Hopf algebras $O(SL_q(2))$ introduced are a one parameter deformation of this Hopf algebra $O(SL(2))$.

## 3. The Bialgebra $O(M_q(2))$

Let $O(M_q(2))$ be the complex (associative) algebra with generators $a, b, c, d$ satisfying the following relations:

$$ab = qba, ac = qca, bd = qdb, cd = qdc, bc = cb$$

$$ad - da = (q - q^{-1})bc$$

In order to shorten formulas we also write $u_1^1 = a, u_2^1 = b, u_1^2 = c, u_2^2 = d$.

**Proposition 3.0.4.** *There exists a unique bialgebra structure on the algebra $O(M_q(2))$ with comultiplication $\Delta$ and counit $\epsilon$ such that*

$$\Delta(a) = a \otimes a + b \otimes c, \Delta(b) = a \otimes b + b \otimes d$$

$$\Delta(c) = c \otimes a + d \otimes c, \Delta(d) = c \otimes b + d \otimes d$$

$$\epsilon(a) = \epsilon(d) = 1, \epsilon(b) = \epsilon(c) = 0$$

*In other words we can be written as $\Delta(u_i^j) = \sum_k u_k^i \otimes u_j^k$ and $\epsilon(u_i^j) = \delta_{ij}, i, j = 1, 2.$*

**Definition 3.0.5.** The bialgebra $O(M_q(2))$ is called the coordinate algebra of the quantum matrix space $M_q(2)$.

## 4. The Hopf algebra $O(SL_q(2))$

According to the above relations we have

$$ad - qbc = da - q^{-1}bc$$

This element of $O(M_q(2))$ is denoted $D_p$ and called the quantum determinant. This is a group like element (that is, $\Delta(D_p) = D_p \otimes D_p$ and $\epsilon(D_p) = 1$) belonging to the center of the algebra $O(M_q(2))$ Since $D_p$ is group like, the two sided ideal $< D_p - 1 >$ generated by the element $D_p - 1$ is a biideal of $O(M_q(2))$. Hence the quotient $O(SL_q(2)) := \frac{O(M_q(2))}{<D_p-1>}$ is again a bialgebra. and a Hopf algebra with the antipode map is determined by

$$S(a) = d, S(b) = -q^{-1}b, S(c) = -qc, S(d) = a$$

A direct computation shows that for the algebra $O(SL_q(2))$ the matrices

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$S(u) = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}$$

are inverse to each other. This fact is actually equivalent to the validity of the antipod condition for the generators $a, b, c, d$.
The Hopf algebra $O(SL_q(2))$ is called the coordinate algebra of the quantum group $SL_q(2)$.

## 5. Rooted trees

A rooted tree is a tree in which a special ("labeled") node is singled out. This node is called the "root" of the tree. Rooted trees are equivalent to oriented trees (Knuth 1997, pp. 385-399). A tree which is not rooted is sometimes called a free tree, although the unqualified term "tree" generally refers to a free tree. A rooted tree in which the root node has vertex degree 1 is known as a planted tree. The numbers of rooted trees on $n$ nodes for $n = 1, 2, ...$ are $1, 1, 2, 4, 9, 20, 48, 115, 286, 719, ....$ Denote the number of rooted trees with $n$ nodes by $T_n$ then the generating function is

$$T(x) = \sum_{n=0}^{\infty} T_n x^n = x + x^2 + 2x^3 + 4x^4 + 9x^5 + ...$$

Consider a set of rooted trees. An admissible cut of a rooted tree is any assignment of cuts such that any path from any vertex of the tree to the root has at most one cut. An admissible cut maps a tree t to a monomial in trees $t_1 \times ... \times t_{n+1}$. Precisely one of these subtrees $t_j$ will contain the root of $t$. We denote this distinguished tree by $R^c(t)$, and the monomial delivered by the n other factors by $P^c(t)$. The counit $\epsilon$ is given by:

$$\epsilon(e) = 1, \epsilon(t) = 0 fort \neq e$$

The coproduct $\Delta$ is given by:

$$\Delta(e) = e \otimes e$$

$$\Delta(t) = t \otimes e + e \otimes t + \sum_{adm.cutsC oft} P^c(t) \otimes R^c(t)$$

The antipode $S$ is given by:

$$S(e) = e$$
$$S(t) = -t - \sum_{adm.cutsCoft} S(P^c(t)) \otimes R^c(t)$$

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