Primordial vorticity, gradient expansion and large-scale magnetic fields

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- ## Magnetized Univers

A variety of observations imply that stars, planets, galaxies, clusters of galaxies are all magnetized.

- ## Magnetic field strengths

Galaxies and galaxy clusters → μG
Planets → G
Neutron stars → 10^12 G

Primordial vorticities?

- According to big bang model, prior to photon decoupling, Universe is a strongly coupled electromagnetic plasma where the angular momentum transfer between ions, electrons and photons in an expanding space-time geometry leads to the formation of large-scale vortices.

$$\vec{\omega}_{\mathrm{e}}(\vec{x},\tau) = \vec{\nabla} \times \vec{v}_{\mathrm{e}}$$

- The primordial vorticities lead to the formation of large-scale magnetic fields.

Fully inhomogeneous geometry

$$g_{00} = N^2 - N_k N^k, \qquad g_{ij} = -\gamma_{ij}, \qquad g_{0i} = -N_i,$$
$$g^{00} = \frac{1}{N^2}, \qquad g^{ij} = \frac{N^i N^j}{N^2} - \gamma^{ij}, \qquad g^{0i} = -\frac{N^i}{N^2}$$

it is sometimes referred to as the ADM ecomposition.

$$R_\mu^\nu = \ell_{\mathrm{P}}^2 \left[\left(T_\mu^\nu - \frac{T}{2} \delta_\mu^\nu \right) + \mathcal{T}_\mu^\nu \right]$$

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p g_{\mu\nu}, \qquad \mathcal{T}_\mu^\nu = \frac{1}{4\pi} \left(-F_{\mu\alpha} F^{\nu\alpha} + \frac{\delta_\mu^\nu}{4} F_{\alpha\beta} F^{\alpha\beta} \right)$$

Charged fluids and gradient expansion

- In terms of the ADM decomposition the field strengths and their duals are

$$F^{0i} = -\frac{\mathcal{E}^i}{N^2}, \qquad F^{ij} = -\gamma_{mk}\,\eta^{ijk}\,\frac{\mathcal{B}^m}{N}$$

$$\tilde{F}^{0i} = -\frac{\mathcal{B}^i}{N^2}, \qquad \tilde{F}^{ij} = \gamma_{mk}\,\eta^{ijk}\,\frac{\mathcal{E}^m}{N}.$$

$$\eta^{ijk} = \frac{\epsilon^{ijk}}{\sqrt{\gamma}}, \qquad \eta_{ijk} = \sqrt{\gamma}\,\epsilon_{ijk} \qquad E^i = \frac{\sqrt{\gamma}}{N}\mathcal{E}^i, \qquad B^i = \frac{\sqrt{\gamma}}{N}\mathcal{B}^i.$$

- Maxwell equations in a four-dimensional curved space-time can be written as

$$\nabla_\mu F^{\mu\nu} = 4\pi j^\nu, \qquad \nabla_\mu \tilde{F}^{\mu\nu} = 0$$

$$j^{\nu} = j^{\nu}_{(e)} + j^{\nu}_{(i)}\,, \qquad j^{\nu}_{(e)} = -e\tilde{n}_{e}\, u^{\nu}_{e}\,, \qquad j^{\nu}_{(i)} = e\tilde{n}_{i}\, u^{\nu}_{i}$$

where $\tilde{n}_e$ and $\tilde{n}_i$ denote, respectively, the concentrations of electrons and ions.

u^{μ} is four velosity

The covariant conservation of the energy momentum tensor of the charged species implies

$$\nabla_{\mu} T^{\mu\nu}_{(e)} = j^{(e)}_{\alpha} F^{\nu\alpha}, \qquad \nabla_{\mu} T^{\mu\nu}_{(i)} = j^{(i)}_{\alpha} F^{\nu\alpha}$$

where $\quad T^{\mu\nu}_{(e)} = \rho_{e} u^{\mu}_{e} u^{\nu}_{e}, \qquad T^{\mu\nu}_{(i)} = \rho_{i} u^{\mu}_{i} u^{\nu}_{i}$

By choosing the free index to be time-like and space-like, the evolution of the electron and ion energy densities and the evolution equation for the electron and ion velocity can be derived

$$\partial_{\tau}\rho_{e} - NK\rho_{e} = -en_{e}\frac{\vec{v}_{e}\cdot\vec{E}}{\gamma}, \qquad \partial_{\tau}\rho_{i} - NK\rho_{i} = en_{i}\frac{\vec{v}_{i}\cdot\vec{E}}{\gamma}$$

$$\partial_{\tau}\vec{v}_{e} + \mathcal{H}\,\vec{v}_{e} = -\frac{en_{e}}{\rho_{e}\,a^{4}}[\vec{E} + \vec{v}_{e}\times\vec{B}] - \vec{\nabla}\phi + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{e}} a\Gamma_{\gamma\,e}(\vec{v}_{\gamma} - \vec{v}_{e}) + a\Gamma_{e\,i}(\vec{v}_{i} - \vec{v}_{e})$$

$$\partial_{\tau}\vec{v}_{i} + \mathcal{H}\,\vec{v}_{i} = \frac{en_{i}}{\rho_{i}\,a^{4}}[\vec{E} + \vec{v}_{i}\times\vec{B}] - \vec{\nabla}\phi + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{i}} a\Gamma_{\gamma\,i}(\vec{v}_{\gamma} - \vec{v}_{i}) + a\Gamma_{e\,i}\frac{\rho_{e}}{\rho_{i}}(\vec{v}_{e} - \vec{v}_{i}),$$

- For typical temperatures T < Teγ the electrons and the ions are more strongly coupled than the electrons and the photon. Then, effective evolution can be described in terms of the one-fluid magnetohydrodynamical equations where J is the total current and the center of mass vorticity of the electron-ion system is introduced

$$\vec{\omega}_{\rm b} = \frac{m_{\rm i}\vec{\omega}_{\rm i} + m_{\rm e}\vec{\omega}_{\rm e}}{m_{\rm e} + m_{\rm i}}.$$

- Using evolution eqs. of velocities and Moxwell's eqs with together

$$\vec{E} + \vec{v}_{\rm b} \times \vec{B} = \frac{\vec{J}}{\sigma} + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm b}}\frac{m_{\rm i}}{e} a^2 \Gamma_{\gamma{\rm e}}(\vec{v}_\gamma - \vec{v}_{\rm b})$$

- The effective set of evolution equations can then be written

$$\partial_\tau \vec{\omega}_{\rm b} + \mathcal{H}\vec{\omega}_{\rm b} = \frac{\vec{\nabla} \times (\vec{J} \times \vec{B})}{a^4\, \rho_{\rm b}} + \frac{\epsilon'}{R_{\rm b}}(\vec{\omega}_\gamma - \vec{\omega}_{\rm b}),$$

$$\partial_\tau \vec{B} = \vec{\nabla} \times (\vec{v}_{\rm b} \times \vec{B}) + \frac{\nabla^2 \vec{B}}{4\pi\sigma} + \frac{m_{\rm i} a}{e\, R_{\rm b}} \epsilon' (\vec{\omega}_{\rm b} - \vec{\omega}_\gamma)$$

$$\partial_\tau \vec{\omega}_\gamma = \epsilon'(\vec{\omega}_{\rm b} - \vec{\omega}_\gamma),$$

- By eliminating the baryon-photon rate between these Eqs. and by neglecting the spatial gradients in, following pair of approximate conservation laws can be obtained

$$\partial_\tau\left(\vec{B} + \frac{m_{\rm i}}{e}\, a\, \vec{\omega}_{\rm b}\right) = 0$$

$$\partial_\tau\left(\vec{B} - \frac{a}{R_{\rm b}}\frac{m_{\rm i}}{e}\vec{\omega}_\gamma\right) = 0.$$

By combining these relations, the vorticity of the photons can be directly related to the vorticity of the ions

$$\vec{\omega}_{\rm i}(\vec{x},\tau) = -\frac{e}{m_{\rm i}}\frac{\vec{B}(\vec{x},\tau)}{a(\tau)} + \frac{a_{\rm r}}{a(\tau)}\vec{\omega}_{\rm r},$$

$$\vec{\omega}_\gamma(\vec{x},\tau) = \frac{R_{\rm b}(\tau)}{a(\tau)}[\vec{\omega}_{\rm r} - a(\tau)\vec{\omega}_{\rm i}(\vec{x},\tau)]$$

Maximal vorticity induced by the geometry

- From the momentum constraint, the total velocity field can be written, formally

$$
\begin{aligned}
v^i &= -\frac{N\,S^i}{2S^2}\left[1-\sqrt{1+4S^2}\right] \simeq NS^i\left[1-S^2+\mathcal{O}(\epsilon^3)\right]+\mathcal{O}(\epsilon^4),\\
S^i &= \frac{1}{\ell_{\rm P}^2(p+\rho)}\nabla_k\left(K^{ki}-K\gamma^{ki}\right),
\end{aligned}
$$

- the total vorticity can be written as

$$
\omega^i_{\rm tot} = \partial_j\left\{N\Lambda^{ij}_m S^m\left[1-S^2+\mathcal{O}(\epsilon^3)\right]\right\}
$$

- To implement the gradient expansion we parametrize the geometry as

$$\gamma_{ij}(\vec{x},\tau) = a^2(\tau)[\alpha_{ij}(\vec{x}) + \beta_{ij}(\vec{x},\tau)], \qquad \gamma^{ij}(\vec{x},\tau) = \frac{1}{a^2(\tau)}[\alpha^{ij}(\vec{x}) - \beta^{ij}(\vec{x},\tau)]$$

- After long calculations we will find

$$\omega^i_{\rm tot} = \partial_j \mathcal{A}^{ij}, \qquad \mathcal{A}^{ij} = \frac{N^2\,\gamma^{kj}\,\gamma^{in}\,\eta_{kmn}}{\ell_{\rm P}^2(p+\rho)} \nabla_a \Big(K^{am} - \gamma^{am} K \Big)$$

Vorticity to first-order in the gradient expansion

- The simplest parametrization of αij(~x) which does not contain spatial gradients can be written as

$$\alpha_{ij}(\vec{x}) = e^{-2\Psi(\vec{x})}\delta_{ij}, \qquad \alpha = \det \alpha_{ij} = e^{-6\Psi(\vec{x})}$$

- In this case, it is easy to show that total vorticity is zero, and the first-order in the gradient expansion vanishes. To have non zero value for vorticity, the contribution of the tensor modes must be included and αij(~x) will then given by:

$$\alpha_{ij}(\vec{x}) = \left[\delta_{ij} + h_{ij}(\vec{x})\right]$$

- where hij is divergenceless and traceless.

- Finally, the total vorticity can be derived directly

$$\omega^i_{\rm tot} = -\mathcal{L}(\tau,w)\,\epsilon^{mij}\partial_j\left[h^{a\ell}h^b_\ell\partial_b h_{am} + h_{mq}h^{ba}\partial_b h_{qa}\right] + \mathcal{O}(\epsilon^3)$$
$$\mathcal{L}(\tau,w) = \frac{\mathcal{H}}{3\mathcal{H}_1^2(w+1)}\left(\frac{a}{a_1}\right)^{3w+1}$$

Using this Eq., the maximal obtainable magnetic field will be

$$B^i_{\rm max}(\vec{x},\tau) = -\frac{\rho_{\rm i}\sqrt{\gamma}}{e\,N^2\tilde{n}_{\rm i}}\omega^i_{\rm tot}(\vec{x},\tau)$$

which can also be written as

$$B^i_{\rm max}(\vec{x},\tau) = \left\{\mathcal{L}(\tau,w)\,\epsilon^{mij}\partial_j\left[h^{a\ell}h^b_\ell\partial_b h_{am} + h_{mq}h^{ba}\partial_b h^q_a\right] + \mathcal{O}(\epsilon^3)\right\}a(\tau)\left[1+\mathcal{O}(\epsilon^2)\right]$$

- By calculating the correlation between vorticities and then average it, we estimate the amount of magnetic field

$$\langle \omega^i(\vec{x},\tau)\,\omega^i(\vec{y},\tau)\rangle = \mathcal{L}^2(\tau,w)\epsilon^{jmi}\,\epsilon^{j'm'i}\frac{\partial^2}{\partial y^{j'}\,\partial y^{b'}}\frac{\partial^2}{\partial x^j\,\partial x^b}\mathcal{T}_{bmb'm'}$$

$$\begin{aligned}\langle B^2(r)\rangle &= 6.348\times 10^{-76}\left(\frac{r_{\rm T}}{0.32}\right)^3\left(\frac{\mathcal{A}_{\mathcal{R}}}{2.43\times 10^{-9}}\right)^3\left(\frac{z_{\rm dec}+1}{1089.2}\right)^2\\ &\times \left(\frac{h_0^2\Omega_{\rm M0}}{0.134}\right)^6\left(\frac{h_0^2\Omega_{\rm R0}}{4.15\times 10^{-5}}\right)^{-6}\mathcal{C}(n_{\rm T},r)\ {\rm G}^2,\\ \mathcal{C}(n_{\rm T},r) &= c(n_{\rm T})\left(\frac{r}{r_{\rm p}}\right)^{8-3n_{\rm T}}+d(n_{\rm T}),\end{aligned}$$

$$\begin{aligned}B_{\rm max} &= 2.519\times 10^{-38}\left(\frac{r_{\rm T}}{0.32}\right)^{3/2}\left(\frac{\mathcal{A}_{\mathcal{R}}}{2.43\times 10^{-9}}\right)^{3/2}\left(\frac{z_{\rm dec}+1}{1089.2}\right)\\ &\times \left(\frac{h_0^2\Omega_{\rm M0}}{0.134}\right)^3\left(\frac{h_0^2\Omega_{\rm R0}}{4.15\times 10^{-5}}\right)^{-3}{\rm G}.\end{aligned}$$

Conclusion

- It has been computed the vorticity for the treatment of fully inhomogeneous plasmas which are also gravitating.
- A set of general conservation laws has been derived.
- The total vorticity has been estimated to lowest order in the gradient expansion.
- The maximal comoving magnetic field that is induced is epends upon the tensor to scalar ratio and it is, at most, of the order of $10{-}37$ G over the typical comoving scales ranging between 1 and 10 Mpc.


Thanks for your attention

Some Relations

$$
\begin{aligned}
R_0^0 &= \frac{\partial_\tau K}{N} - \mathrm{Tr} K^2 + \frac{\nabla^2 N}{N} - \frac{N^m}{N}\nabla_m K + \frac{N^q}{N}\mathcal{L}_q, \\
R_i^0 &= \frac{1}{N}\mathcal{L}_i, \\
R_i^j &= \frac{1}{N}\partial_\tau K_i^j - K K_i^j - r_i^j + \frac{1}{N}\nabla_i \nabla^j N - \frac{N^m}{N}\nabla_m K_i^j \\
&+ \frac{1}{N}\nabla_m N^j K_i^m - \frac{1}{N}\nabla_i N^m K_m^j - \frac{N^j}{N}\mathcal{L}_i,
\end{aligned}
$$

$$
{}^{(3)}\nabla_i = \nabla_i, \qquad \mathcal{L}_i = \left(\nabla_i K - \nabla_k K_i^k\right).
$$

The second relation give Momentom constrain.

- To give an explicit estimate of the primordial vorticity the relevant cosmological parameters have be taken to be the ones determined on the basis of the WMAP 7yr data alone

Using relation between vorticity and velocity for each species

$$\partial_\tau \vec{\omega}_{\rm e} + \mathcal{H}\,\vec{\omega}_{\rm e} = \frac{e n_{\rm e}}{\rho_{\rm e}\, a^4}\Big[\partial_\tau \vec{B} + (\vec{v}_{\rm e}\cdot\vec{\nabla})\vec{B} + \theta_{\rm e}\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{v}_{\rm e}\Big] + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm e}} a\Gamma_{\gamma\,\rm e}(\vec{\omega}_\gamma - \vec{\omega}_{\rm e}) + a\Gamma_{\rm e\,i}(\vec{\omega}_{\rm i} - \vec{\omega}_{\rm e}),$$

$$\partial_\tau \vec{\omega}_{\rm i} + \mathcal{H}\,\vec{\omega}_{\rm i} = -\frac{e n_{\rm i}}{\rho_{\rm i}\, a^4}\Big[\partial_\tau \vec{B} + (\vec{v}_{\rm i}\cdot\vec{\nabla})\vec{B} + \theta_{\rm i}\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{v}_{\rm i}\Big] + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm i}} a\Gamma_{\gamma\,\rm i}(\vec{\omega}_\gamma - \vec{\omega}_{\rm i}) + a\Gamma_{\rm e\,i}\frac{\rho_{\rm e}}{\rho_{\rm i}}(\vec{\omega}_{\rm e} - \vec{\omega}_{\rm i}),$$

$$\partial_\tau \vec{\omega}_\gamma = a\Gamma_{\gamma\rm i}(\vec{\omega}_{\rm i} - \vec{\omega}_\gamma) + a\Gamma_{\gamma\rm e}(\vec{\omega}_{\rm e} - \vec{\omega}_\gamma)$$