Another look at collective neutrino oscillations

Evgeny Akhmedov
Max-Planck Institute für Kernphysik, Heidelberg

In collaboration with A. Mirizzi
ν oscillations in dense neutrino backgrounds

(Early Universe, supernovae) – interesting collective oscillation effects, absent in the case of the usual MSW oscillations:

Synchronized neutrino oscillations, bi-polar oscillations, spectral splits & swaps, multiple spectral splits

Our goal: to study late-time decoherence effects on collective neutrino oscillations in the simplest possible system: uniform and isotropic neutrino gas

Decoherence: in momentum space – due to de-phasing of different momentum modes at late times

In coordinate space: due to spatial separation of wave packets corresponding to different propagation eigenstates

The two descriptions are equivalent
Decoherence by wave packet separation

For SN neutrinos: $\sigma_{x_P} \sim 10^{-11}$ cm (Kersten 2012) ⇒ estimated $L_{\text{coh}} \sim 10$ km. By the time $\nu$’s reach the region of coll. oscillations ($r \sim 100$ km) they should have already lost their coherence.
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E.g. for 2-flavor $\nu_e \rightarrow \nu_x$ oscillations in vacuum and Gaussian WPs:

$$P_{ee} = c^4 + s^4 + 2c^2s^2e^{-t^2/L_{coh}^2}\cos\phi,$$

$$P_{ex} = 2c^2s^2(1 - e^{-t^2/L_{coh}^2}\cos\phi)$$

$$\phi \equiv (\Delta m^2/2E)t.$$
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\begin{align*}
P_{ee} &= c^4 + s^4 + 2c^2s^2e^{-t^2/L_{\text{coh}}^2}\cos\phi, \\
P_{ex} &= 2c^2s^2(1 - e^{-t^2/L_{\text{coh}}^2}\cos\phi)
\end{align*}
\]

\[\phi \equiv (\Delta m^2/2E)t.\]

\(\blacklozenge\) At the same time: synchronized oscillations in a dense neutrino gas (i.e. with sufficiently large \( \mu \)) show no trace of averaging out!
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Why does this happen? Why is WP separation inoperative in this case?
Coherent neutrino - neutrino scattering

2-flavour evolution equation:

\[ \left( \begin{array}{c} \nu_e \\ \nu_\mu \end{array} \right) = \left( \begin{array}{cc} -\frac{\Delta m^2}{4E} \cos 2\theta_0 + V_e(t) + V_{ee} & \frac{\Delta m^2}{4E} \sin 2\theta_0 + V_{e\mu} \\ \frac{\Delta m^2}{4E} \sin 2\theta_0 + V_{\mu e} & \frac{\Delta m^2}{4E} \cos 2\theta_0 + V_{\mu \mu} \end{array} \right) \left( \begin{array}{c} \nu_e \\ \nu_\mu \end{array} \right) \]

For isotropic backgrounds:

- \( V_e(t) = \sqrt{2} G_F n_e(t) \)  \( \nu_e + e \rightarrow \nu_e + e \)  (Wolfenstein, 1978)
- \( V_{ee} = \sqrt{2} G_F (2n_{\nu_e} + n_{\nu_\mu}) \)  \( \nu_e + \nu_\mu \rightarrow \nu_e + \nu_\mu \)  (Fuller et al., 1987; Nötzold & Raffelt, 1988)
- \( V_{\mu \mu} = \sqrt{2} G_F (n_{\nu_e} + 2n_{\nu_\mu}) \)  

- \( V_{\alpha\beta} = \sqrt{2} G_F \int \frac{d^3 q}{(2\pi)^3} \left[ \rho_{\alpha\beta}(\vec{q}) - \bar{\rho}_{\alpha\beta}(\vec{q}) \right] \)  \( \nu_\alpha(\vec{p}) + \nu_\beta(\vec{q}) \rightarrow \nu_\alpha(\vec{q}) + \nu_\beta(\vec{p}) \)  (Pantaleone, 1992; Sigl & Raffelt, 1993)

In general backgrs.: \( V_{\alpha\beta} = \sqrt{2} G_F \int \frac{d^3 q}{(2\pi)^3} \left[ \rho_{\alpha\beta}(\vec{q}) - \bar{\rho}_{\alpha\beta}(\vec{q}) \right] (1 - \vec{v}_p \cdot \vec{v}_q) \)
The Schrödinger evolution equation

\[ i \frac{d}{dt} \nu = H \nu \]

Even in the case of pure neutrino states it is convenient to introduce neutrino density matrix in the flavour space:

\[ \rho_{\alpha\beta} = \langle \nu_\alpha \nu^*_\beta \rangle \quad (\text{Dolgov 1981; Sigl & Raffelt 1993, ...}) \]

\[ \langle \ldots \rangle : \text{summation over all neutrinos in the system and averaging over} \]
\[ \text{“microscopically large but macroscopically small” spatial volumes} \]

Equation of motion for the density matrix:

\[ \dot{\rho} = -i[H, \rho] \]
Flavour spin

A convenient method of analysis and visualization of flavour transitions (especially useful for oscillations in dense neutrino backgrounds!)

For 2-flavour oscillations:

$$\rho = \frac{n_\nu}{2} (P^0_\omega + \vec{P}_\omega \cdot \vec{\sigma}), \quad H = \frac{1}{2} (H_0 + \vec{H}_\omega \cdot \vec{\sigma}), \quad \omega \equiv \frac{\Delta m^2}{2p}.$$
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From \( \dot{\rho} = -i[H, \rho] \Rightarrow \dot{P}_0^0 = 0 \quad (n_{\nu} P_0^0 = n_{\nu \omega} = \text{const}) , \)

\[ \bigdiamond \quad \dot{\vec{P}}_\omega = \vec{H}_\omega \times \vec{P}_\omega \]

Precession of flavour “spin” around the “magnetic field” \( \vec{H}_\omega \).
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Precession of flavour “spin” around the “magnetic field” \( \vec{H}_\omega \).

\[ \vec{H}_\omega = \omega \vec{B} + \lambda \vec{L} + \mu \vec{D} , \]
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From $\dot{\rho} = -i[H, \rho] \Rightarrow \dot{P}^0_\omega = 0$ ($n_\nu P^0_\omega = n_\nu \omega = \text{const}$),

$$\dot{\vec{P}}_\omega = \vec{H}_\omega \times \vec{P}_\omega$$

Precession of flavour “spin” around the “magnetic field” $\vec{H}_\omega$.

$$\vec{H}_\omega = \omega \vec{B} + \lambda \vec{L} + \mu \vec{D},$$

$$\vec{B} = (\sin 2\theta_0, 0, -\cos 2\theta_0), \quad \vec{L} = (0, 0, 1),$$

$$\lambda = \sqrt{2}G_F n_e, \quad \mu = \sqrt{2}G_F n_\nu, \quad \vec{D} = \int_{-\infty}^{\infty} \vec{P}_\omega d\omega$$ (4)
The convention for antineutrinos: \( \bar{\rho}_{\alpha\beta} = \langle \bar{\nu}_\beta \bar{\nu}_\alpha^* \rangle \) and

\[
\bar{\rho} = \frac{1}{2} n_\nu \left( P^0_{-\omega} - \vec{P}_{-\omega} \vec{\sigma} \right).
\]

\( \vec{P}_{-\omega} \) describes the antineutrino flavour. \( P^0_{-\omega} \): \( n_\nu P^0_{-\omega} = n\bar{\nu}_\omega \).
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For \( \nu_e \leftrightarrow \nu_x \) oscillations (\( \nu_x = \nu_\mu, \nu_\tau \)):
\[
\rho_{ee} = \frac{n_\nu}{2} [P_{0} + P_{3}],
\]
\[
\rho_{xx} = \frac{n_\nu}{2} [P_{0} - P_{3}].
\]
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For \( \nu_e \leftrightarrow \nu_x \) oscillations \( (\nu_x = \nu_\mu, \nu_\tau) \):

\[
\rho_{ee} = \frac{n_\nu}{2} \left[ P^0_\omega + P^0_{-\omega} \right],
\]

\[
\rho_{xx} = \frac{n_\nu}{2} \left[ P^0_\omega - P^0_{-\omega} \right].
\]

(Figs. from Duan, Fuller & Qian, arXiv:1001.2799)
Rotating away ordinary matter effects

If the density of ordinary matter is constant or nearly constant \( \Rightarrow \) effects of ordinary matter (\( \lambda \vec{L} \) term) can be removed by going into a frame rotating around \( \vec{L} \) and replacing \( \theta_0 \rightarrow \theta \) (\( \theta \) small).

\[
\Downarrow
\]

\[
\vec{H}_\omega = \omega \vec{B} + \lambda \vec{L} + \mu \vec{D} \Rightarrow \vec{H}_\omega = \omega \vec{B} + \mu \vec{D}
\]

with

\[
\vec{B} = (\sin 2 \theta_0, 0, -\cos 2 \theta_0),
\]

\( \theta \ll 1 \). Used in many papers (also by us).
Flavour spin formalism

EoMs for individual $\omega$-modes:

$$
\dot{\vec{P}}_\omega = \vec{H}_\omega \times \vec{P}_\omega
$$

Precession of “flavour spin” around the “magn. field” $\vec{H}_\omega$.

$$
\vec{H}_\omega = \omega \vec{B} + \mu \vec{P}
$$

$$
\vec{B} = (s_{20}, 0, -c_{20}) , \quad s_{20} \equiv \sin 2\theta_0 , \quad c_{20} \equiv \cos 2\theta_0 ,
$$

$$
\mu = \sqrt{2} G_F n_\nu , \quad \vec{P} = \int \vec{P}_\omega d\omega , \quad \omega = \frac{\Delta m^2}{2p} .
$$

$$
|\vec{P}_\omega| \equiv P_0 g_\omega = \text{const.} , \quad g_\omega \ - \ \text{normalized neutrino spectrum in } \omega .
$$

Initial conditions:

$$
\vec{P}_\omega (0) = P_0 g_\omega \vec{n}_z
$$
For WPs: $g_\omega$ – spectrum of an individual WP. Assumed to have a peak at $\omega \simeq \omega_0$ and effective width $\sigma_\omega$. E.g. for the Gaussian spectrum

$$g_\omega = \frac{1}{\sqrt{2\pi}\sigma_\omega} e^{-\frac{(\omega-\omega_0)^2}{2\sigma_\omega^2}}$$
EoM for flavour spin

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$$g_\omega = \frac{1}{\sqrt{2\pi} \sigma_\omega} e^{-\frac{(\omega-\omega_0)^2}{2\sigma_\omega^2}}$$

Integrating over $\omega$:

$$\dot{\vec{P}} = \int d\omega [\omega \vec{B} + \mu \vec{P}] \times \vec{P}_\omega = \vec{B} \times \vec{S},$$

$$\vec{S} \equiv \int d\omega \omega \vec{P}_\omega.$$ Initial condition: $\vec{P}(0) = P_0 \hat{n}_z.$
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\]

Integrating over \( \omega \):

\[
\dot{\vec{P}} = \int d\omega [(\omega \vec{B} + \mu \vec{P}) \times \vec{P}_\omega] = \vec{B} \times \vec{S},
\]

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Initial condition:

\[
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\]

From EoM:

\[
\vec{B} \cdot \dot{\vec{P}} = 0 \quad \Rightarrow \quad \vec{P} \cdot \vec{B} = \text{const.} = -c_2 P_0.
\]

But: \(|\vec{P}| \neq \text{const.}!\)
EoM for flavour spin

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From EoM:

$$\vec{B} \cdot \dot{\vec{P}} = 0 \quad \Rightarrow \quad \vec{P} \cdot \dot{\vec{B}} = \text{const.} = -c_{20}P_0.$$

But: $|\vec{P}| \neq \text{const.}$! From conservation of $|\vec{P}_\omega|$ and $\vec{P} \cdot \vec{B}$:

$$\diamond c_{20}P_0 \leq |\vec{P}(t)| \leq P_0$$
EoM for flavour spin

From the EoMs:

\[ E_{\text{tot}} \equiv \frac{\mu P^2}{2} + \vec{B} \cdot \vec{S} = \text{const.} \]

Can be interpreted as conservation of the ‘total energy’ of the system of flavour spins (Duan, Fuller & Qian, 2006).
Oscillations in uniform isotropic neutrino gas

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Three main regimes:
Oscillations in uniform isotropic neutrino gas


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- Large \( \mu \) regime – “perfect synchronization”. All \( \vec{P}_\omega \) evolve in a synchronized way despite differences in \( \omega \). \( \vec{P} \) satisfies

\[
\dot{\vec{P}} = \omega_0 \vec{B} \times \vec{P}
\]

– precesses around \( \vec{B} \) with frequency \( \omega_0 \); \( P = P_0 \).
Oscillations in uniform isotropic neutrino gas


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- Small $\mu$ regime – complete de-synchronization at late times. Oscillations average out, no evolution at asymptotically large times. $|\vec{P}|$ shrinks to its minimal value $P_{min} = c_{20}P_0$. 
Oscillations in uniform isotropic neutrino gas

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- Small $\mu$ regime – complete de-synchronization at late times. Oscillations average out, no evolution at asymptotically large times. $|\vec{P}|$ shrinks to its minimal value $P_{\text{min}} = c_{20}P_0$.

- Intermediate $\mu$ regime – partial de-synchronization at late times. Precession around $\vec{B}$ with some frequency $\omega_s$; $P_{\text{min}} < P < P_0$. 


Order parameter \( R_A = \frac{P_\perp}{\sin 2\theta_0} \) for Gaussian (solid line) and box-type (dashed line) neutrino spectra.

Approximations used in the analytic approach:

- "Sudden approximation" – \( \vec{P} \) replaced by its asymptotic (late time) value starting immediately at \( t = 0 \).
- The angles between \( \vec{P}_\omega \) and \( \vec{B} \) at the onset of asymptotic regime are taken to be those corresponding to \( t = 0 \) (i.e. all equal \( 2\theta_0 \)).
- In the co-rotating frame \( \vec{P}_\omega \) are replaced by their asymptotic averages assumed to be given by their projections on \( \vec{H}'_\omega \): \( \vec{P}_\omega \rightarrow \frac{\vec{P}_\omega \cdot \vec{H}'_\omega}{\vec{H}'_\omega^2} \vec{H}'_\omega \).

Very good agreement with numerical results.
Spectral moments formalism

I. Exact relations.

\[ \dot{\vec{P}} = \int d\omega [(\omega \vec{B} + \mu \vec{P}) \times \vec{P}_\omega] = \vec{B} \times \vec{S}, \]

Contains 2 global flavour spin vectors (spectral moments):

\[ \vec{P} \equiv \int d\omega \vec{P}_\omega, \quad \vec{S} \equiv \int d\omega \omega \vec{P}_\omega. \]

Introduce

\[ \vec{K}_n(t) = \int d\omega \omega^n \vec{P}_\omega(t), \quad n \geq 0. \]

Well defined if the neutrino spectrum \( g_\omega \) goes to zero fast enough for \( |\omega| \to \infty \)
(satisfied e.g. for Gaussian and box-type neutrino spectra).

\[ \vec{P}(t) = \vec{K}_0(t), \quad \vec{S}(t) = \vec{K}_1(t). \]

Multiplying \( \dot{\vec{P}}_\omega = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_\omega \) by \( \omega^n \) and integrating over \( \omega \): \( \Rightarrow \)
**EoMs for** $\vec{K}_n(t)$

\[
\dot{\vec{K}}_n = \vec{B} \times \vec{K}_{n+1} + \mu \vec{P} \times \vec{K}_n
\]

Combining equations for $\vec{K}_n$ and $\vec{K}_{n+1}$:

\[
\vec{B} \cdot \dot{\vec{K}}_{n+1} + \mu \vec{P} \cdot \dot{\vec{K}}_n = 0
\]

(Alternatively, can be derived using $\vec{H} \omega \cdot \dot{\vec{P}}_\omega = 0$ and taking $\int d\omega \omega^n$).

For $n = 0$  \Rightarrow  $\vec{B} \cdot \dot{\vec{S}} + \mu \vec{P} \cdot \dot{\vec{P}}$. Can be integrated: $\frac{\mu P^2}{2} + \vec{B} \cdot \vec{S} = \text{const.} = E_{\text{tot}}$

– already known.

For $n = 1$:

\[
\vec{B} \cdot \dot{\vec{K}}_2 + \mu \vec{P} \cdot \dot{\vec{S}} = 0.
\]

Can also be integrated! From $\dot{\vec{P}} = \vec{B} \times \dot{\vec{S}}$: $\dot{\vec{P}} \cdot \vec{S} = 0$. \Rightarrow

\[
\vec{P} \cdot \dot{\vec{S}} = \vec{P} \cdot \dot{\vec{S}} + \dot{\vec{P}} \cdot \vec{S} = (d/dt)(\vec{P} \cdot \vec{S}).
\]
EoMs for $\vec{K}_n(t)$

\[ \vec{B} \cdot \vec{K}_2 + \mu \vec{P} \cdot \vec{S} = \text{const.} \]

− **New conservation law!** Valid also for systems of $\nu$’s and $\bar{\nu}$’s. (Can also be derived from the results of Pehlivan et al., 2011)

Above formulas: exact and satisfied for all $t$. Now consider the regime of asymptotically large times, at which synchronized oscillations set in.
A simplified analytic approach

Assume that at late $t$ evolution of the system conserves the length of $\vec{P}$:

$$\vec{P} \cdot \vec{\dot{P}} = \vec{P} \cdot (\vec{B} \times \vec{S}) = 0.$$
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Can be satisfied if one (or more) of the following conditions is fulfilled:
(a) $\vec{P} = 0$;  (b) $\vec{P} \parallel \vec{B}$;  (c) $\vec{S} = 0$;  (d) $\vec{S} \parallel \vec{B}$;  (e) $\vec{S} \parallel \vec{P}$. Non-trivial realization – case (e), $\vec{S} \parallel \vec{P}$. 
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For now we assume:

$$\vec{S}(t) = \omega_s \vec{P}(t).$$

(Additional assumption for the longitudinal components. Satisfied well if $\mu$ is not too close to the threshold $\mu_0$).
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From $\dot{\vec{P}} = \vec{B} \times \vec{S}$ \quad $\Rightarrow$

$$\vec{P} = \omega_s \vec{B} \times \vec{P}$$

$\vec{P}$ precesses around $\vec{B}$ with the constant frequency $\omega_s$. 
In principle, \( \omega_s \) could be time dependent (but we shall see that it is constant).

\[
\vec{S} = \omega_s \vec{P} \quad \Rightarrow \quad \omega_s = \frac{\int d\omega \omega \vec{P}_\omega}{\int d\omega \vec{P}_\omega}
\]

(in \( N \) and \( D \) components of \( \vec{P}_\omega \) along a fixed direction should be taken).

Substitute \( \vec{S} = \omega_s \vec{P} \) into \( E_{tot} \):

\[
\frac{\mu}{2} \left[ P_0^2 - P^2 \right] = c_{20} P_0 (\omega_0 - \omega_s)
\]

- All quantities except possibly \( \omega_s \) are constant \( \Rightarrow \) \( \omega_s \) must be constant.
- Since l.h.s. is non-negative, so must be r.h.s. \( \Rightarrow \) \( \omega_s \leq \omega_0 \).
- Because \( \omega_0 \) and \( \omega_s \) are certain averages of \( \omega \) over the spectrum \( g_\omega \) of effective width \( \sigma_\omega \), \( \omega_0 - \omega_s \lesssim \sigma_\omega \). \( \Rightarrow \)
- In the limit \( \mu \to \infty \): asympt.-lly \( P \to P_0 \) (no shrinkage of \( \vec{P} \) for large \( \mu \)).
Rewrite equation for $E_{tot}$:

$$\frac{P_0^2 - P}{P_0^2} = 2c_{20} \frac{\omega_0 - \omega_s}{\mu P_0} \lesssim 2c_{20} \frac{\sigma_\omega}{\mu P_0},$$

‘Perfect synchronization’ ($P = P_0$) achieved for $\mu \gg$ some $\mu_0$; $\Rightarrow$

$$\mu_0 \sim \sigma_\omega,$$

while it is irrelevant whether or not $\mu$ is large compared to $\omega_0, \omega_s$. 
A simplified analytic approach – contd.

From EoM $\dot{\vec{P}} = \omega_s \vec{B} \times \vec{P}$ and $\dot{\vec{S}} = \omega_s \vec{P}$:

$$\dot{\vec{S}} = \omega_s \vec{B} \times \vec{S}$$

– at asymptotic times $\vec{S}$ satisfies EoM similar to that of $\vec{P}$.

$\vec{S} \cdot \dot{\vec{S}} = 0$; on the other hand, from EoM for $\vec{K}_n$ with $n = 1$:

$$\dot{\vec{S}} = \vec{B} \times \vec{K}_2 + \mu \vec{P} \times \vec{S}.$$  

$\vec{S} \cdot \dot{\vec{S}} = 0 \Rightarrow \vec{S} \cdot (\vec{B} \times \vec{K}_2) = 0$.

Non-trivially realized only if at asymptotically large times $\vec{K}_2$ is parallel or antiparallel to $\vec{S}$ (and therefore also to $\vec{P}$), that is $\vec{K}_2(t) = \omega_1 \vec{S}(t)$.
Comparing EoMs $\dot{\mathbf{S}} = \omega_s \mathbf{B} \times \mathbf{S}$ and $\dot{\mathbf{S}} = \mathbf{B} \times \mathbf{K}_2 + \mu \mathbf{P} \times \mathbf{S}$:

$\omega_1 = \omega_s$, that is, asymptotically

$$\mathbf{K}_2(t) = \omega_s \mathbf{S}(t) = \omega_s^2 \mathbf{P}(t).$$

Substituting this into the conservation law $\mathbf{B} \cdot \mathbf{K}_2 + \mu \mathbf{P} \cdot \mathbf{S} = \text{const.}$:

$$-\omega_s^2 c_20 P_0 + \omega_s \mu P^2 = P_0 [\mu P_0 \omega_0 - c_20 (\omega_0^2 + \sigma_\omega^2)].$$

Combining this with $E_{tot} = \text{const.}$, $\Rightarrow$ quadratic equation for $\omega_0 - \omega_s$:

$$(\omega_0 - \omega_s)^2 - \frac{\mu P_0}{c_20} (\omega_0 - \omega_s) + \sigma_\omega^2 = 0.$$
The solution:

\[ \omega_0 - \omega_s = \frac{1}{2c_{20}} \left[ \mu P_0 - \sqrt{\mu^2 P_0^2 - 4c_{20}^2 \sigma^2_\omega} \right]. \]

Exhibits a threshold behaviour: real solution only exists (and synchronized oscillations are only take possible) for \( \mu > \mu_0 \) where

\[ \mu_0 P_0 = 2c_{20} \sigma_\omega. \]

For \( \mu P_0 \gg 2c_{20} \sigma_\omega \) (far above the threshold):

\[ \omega_0 - \omega_s \simeq c_{20} \frac{\sigma_\omega}{\mu P_0} \sigma_\omega \ll \sigma_\omega \quad (\omega_s \to \omega_0). \]

\( \omega_0 - \omega_s \) reaches its maximum at the threshold \( \mu = \mu_0 \): \( \omega_0 - \omega_s(\mu_0) = \sigma_\omega \).

The asymptotic value of \( P \) vs \( \mu \):

\[ P = P_0 \left[ 1 - \frac{\mu_0^2}{\mu^2} \right]^{1/4}. \]
Results for the asymptotic regime are quite reasonable in a number of respects:

- demonstrate the existence of the threshold \( \mu_0 \)
- give a good estimate for its value (typically within a factor of 2 of the true value)
- lead to a reasonable behaviour of \( P \) far above the threshold.

But: they lead to a wrong result for the value of \( P \) at the threshold: \( P = 0 \).
(Due to conservation of \( \vec{P} \cdot \vec{B} \) the length of \( \vec{P} \) cannot be smaller than \( c_{20} P_0 \)).

This problem is related to the additional assumption about the relation between the longitudinal components of \( \vec{S} \) and \( \vec{P} \) that is not valid close to the threshold.
II. More accurate consideration

The goal: to get a better description close to the threshold $\mu_0$.

$\Rightarrow$ Remove an additional assumption regarding the relation between the longitudinal components of $\vec{P}$ and $\vec{S}$, $\vec{P}_\parallel \equiv (\vec{P} \cdot \vec{B})\vec{B}$ and $\vec{S}_\parallel \equiv (\vec{S} \cdot \vec{B})\vec{B}$.

Non-trivial realization of the condition $\vec{P} \cdot (\vec{B} \times \vec{S}) = 0$:

$$\vec{S}_\perp (t) = \omega_s \vec{P}_\perp (t)$$

In previous analysis: $\omega_s = const.$ (followed from $E_{tot} = const.$ and late-time relation $P = const$).

Now: $\omega_s$ relates only the transverse components of $\vec{P}$ and $\vec{S}$, whereas $E_{tot} = const.$ constrains only the longitudinal component of $\vec{S}$. $\Rightarrow$ cannot prove $\omega_s = const.$.

In general, no longer possible to directly use the conservation laws to find $\mu_0$ and the asymptotic value of $P$. $\Rightarrow$ Follow a different strategy.
Different strategy

Assume that the asymptotic steady-state evolution of $\vec{P}$ is a simple precession with constant angular velocity, that is $\omega_s = const.$

Differentiating $\vec{S}_\perp = \omega_s \vec{P}$ and using EoM for $\vec{P}$:

$$\dot{\vec{S}}_\perp = \omega_s \vec{B} \times \vec{S}_\perp.$$ 

From conservation of $E_{tot}$ and $P^2 = const.$: $S_\parallel = \vec{B} \cdot \vec{S}$ is also constant. \Rightarrow

$$\dot{\vec{S}} = \omega_s \vec{B} \times \vec{S}.$$ 

From $\vec{S} \cdot \dot{\vec{S}} = 0$ and $\dot{\vec{S}} = \vec{B} \times \vec{K}_2 + \mu \vec{P} \times \vec{S}$:

$$\vec{S} \cdot (\vec{B} \times \vec{K}_2) = 0.$$ 

Non-trivial realization:

$$\vec{K}_{2\perp}(t) = \omega_1 \vec{S}_\perp(t)$$

with constant $\omega_1$ (steady-state evolution assumption). But: $\omega_1 \neq \omega_s$! (cannot be immediately found).
Different strategy – contd.

Just as for $\vec{S}$: $\vec{B} \cdot \vec{K}_2$ is conserved, $\dot{\vec{K}}_2 = \omega_s \vec{B} \times \vec{K}_2$, $\vec{K}_2 \perp = \omega_1 \omega_s \vec{P} \perp$.

Can be shown by induction: if for some $n$

$$
\dot{\vec{K}}_n = \omega_s \vec{B} \times \vec{K}_n, \quad \vec{K}_n \perp = \alpha_n \vec{P} \perp
$$

with constant $\alpha_n$ (steady-state evolution assumption), the the same relations hold for $\vec{K}_{n+1}$. $\Rightarrow$ In particular

$$
\vec{B} \cdot \dot{\vec{K}}_n(t) \equiv \int d\omega \omega^n \vec{B} \cdot \dot{\vec{P}}_\omega(t) = 0.
$$

Has to be satisfied for all $n$, which is only possible if

$$
\vec{B} \cdot \dot{\vec{P}}_\omega = \vec{B} \cdot (\mu \vec{P} \times \vec{P}_\omega) = 0,
$$

Non-trivial realization:

$$
\vec{P}_\omega \perp(t) = a_\omega \vec{P} \perp(t), \quad a_\omega = \text{const.}
$$
Different strategy – contd.

\[
P_{\omega \perp} = P_\omega \sin \theta_\omega = P_0 g_\omega \frac{\mu P_{\perp}}{\sqrt{(\omega - \omega_r)^2 + (\mu P_{\perp})^2}},
\]

\[
P_{\omega \parallel} = P_\omega \cos \theta_\omega = P_0 g_\omega \frac{(\omega - \omega_r)}{\sqrt{(\omega - \omega_r)^2 + (\mu P_{\perp})^2}},
\]

Longitudinal and transverse components of all \( \vec{K}_n \) can be reconstructed as integrals of \( P_{\omega \parallel} \) and \( P_{\omega \perp} \) with proper \( \omega \)-dependent factors.

E.g., integrating \( P_{\omega \perp} \) and \( P_{\omega \parallel} \) over \( \omega \) gives \( P_{\perp} \) and \( P_{\parallel} \) \( \Rightarrow \)

\[
\diamond \quad 1 = \mu P_0 \int d\omega g_\omega \frac{1}{\sqrt{(\omega - \omega_r)^2 + (\mu P_{\perp})^2}},
\]

\[
\diamond \quad - c_{20} = \int d\omega g_\omega \frac{(\omega - \omega_r)}{\sqrt{(\omega - \omega_r)^2 + (\mu P_{\perp})^2}}.
\]

Coincide with results of Raffelt and Smirnov (arXiv:0705.1830, 0709.4641) and also of Duan, Fuller & Qian (arXiv:0706.4293) obtained under completely different assumptions!
Different strategy – contd.

\[ \omega_s P_\perp = S_\perp = \int d\omega \omega P_\perp \quad \Rightarrow \]

\[ \omega_s(\mu) = \mu P_0 \int d\omega g_\omega \frac{\omega}{\sqrt{(\omega - \omega_r)^2 + (\mu P_\perp)^2}}. \]

Defines \( \omega_s(\mu) \) implicitly (r.h.s. depends on \( \omega_s \) through \( \omega_r \)).

Qualitative results:

- In the limit \( \mu \gg \mu_0 \):

\[ 1 = \frac{P_0}{P} \int d\omega g_\omega \]

Gives \( P = P_0 \). Then

\[ \omega_s = \int d\omega g_\omega \omega = \omega_0. \]
Assume $\mu P_0 \lesssim \sigma_\omega$. Because of the factor $g_\omega, (\omega - \omega_r)^2$ in the integrands is $\sim \sigma_\omega^2$.  

\[
1 = \frac{\mu P_0}{\sigma_\omega} C_1, \quad C_1 \lesssim 1.
\]

Can only be satisfied for $\mu P_0 \gtrsim \sigma_\omega$  \Rightarrow there should exist a minimum value $\mu_0$ such that $\mu_0 P_0 \sim \sigma_\omega$.

For $\mu < \mu_0$ only trivial solution possible $P_{\perp} = 0$ \Rightarrow complete decoherence.
Assume $\mu P_0 \lesssim \sigma_\omega$. Because of the factor $g_\omega, (\omega - \omega_r)^2$ in the integrands is $\sim \sigma_\omega^2$. \[1 = \frac{\mu P_0}{\sigma_\omega} C_1, \quad C_1 \lesssim 1.\]

Can only be satisfied for $\mu P_0 \gtrsim \sigma_\omega \implies$ there should exist a minimum value $\mu_0$ such that $\mu_0 P_0 \sim \sigma_\omega$.

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Solutions for box-type spectrum using $\int d\omega P_{\omega||} = -c_{20} P_0$ and $E_{\text{tot}} = \text{const.}$:

\[\mu_0 P_0 = \sigma \quad \text{(N.B.: } \sigma = \sqrt{3}\sigma_\omega)\]

\[P_\perp = s_{20} P_0 \sqrt{1 - \frac{\mu_0^2}{\mu^2}}.\]
Different strategy – contd.

Consistency check: substitute into the second conservation law:

\[ \vec{B} \cdot \vec{K}_2 + \mu \vec{P} \cdot \vec{S} = P_0 \left\{ \mu P_0 \omega_0 - c_{20} \left( \omega_0^2 + \frac{1}{3} \sigma^2 \right) - \frac{1}{6} s_{20}^2 c_{20} \sigma^2 \right\}. \]

Last term (\( \sim s_{20}^2 \sigma^2 / (\mu P_0 \omega_0) \), \( s_{20}^2 \sigma^2 / \omega_0^2 \)) violates the cons. law. Violation can be sizeable near threshold (\( \mu \sim \mu_0 \)).

\[ P_{\perp} / s_{20} \text{ vs. } \mu / \mu_0 \text{ for box-type spectrum of width } 2\sigma. \quad \theta_0 = 0.5. \]

Red curve – numerical, green curve – analytical with \( \mu_0 P_0 = \sigma \), blue curve – analytical with \( \mu_0 P_0 = 0.5 \sigma \).
Gaussian spectrum, $\theta_0 = 0.5$, $\sigma_\omega = 0.2$, $\mu = 1.1\mu_{0}^{\text{theo}}$.

Red: $P_{\omega\|}^{\text{num}}$, green: $P_{\omega\|}^{\text{theo}}$.

Gaussian spectrum, $\theta_0 = 0.5$, $\omega_0 = 1$, $\sigma_\omega = 0.2$, $\mu = \mu_{0}^{\text{theo}}$.

Red: $|\vec{P}|$, green: $P_\perp$.

Note: $t_{\text{max}} = 10000 \gg L_{\text{coh}}^{\text{naive}} \sim \sigma_\omega^{-1}!$
Synchronization/de-synchronization and decoherence
by WP separation
Decoh. by WP separation and flavour spin

Propagation eigenstate basis: the basis which diagonalizes $\mathcal{H}$.

(Evolution in prop. eigenstate basis considered by Galais, Kneller & Volpe, 2011)
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WP separation: In the propagation eigenstate basis off-diagonal elements of the density matrix $\rho_{ik} = \langle \nu_i \nu_k^* \rangle$ get suppressed
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Each $\vec{P}_\omega$ aligns (or anti-aligns) with its $\vec{H}_\omega = \omega \vec{B} + \mu \vec{P}$. 
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This can happen for all $\omega$ only when $\vec{P} = \int d\omega \vec{P}_\omega$ becomes collinear with $\vec{B}$.
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◊ Decoherence means alignment of the flavour spin vector $\vec{P}$ with $\vec{B}$ ($P_\perp = 0$). Also: all $P_{\omega \perp} = 0$. 
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(Taken as the definition of complete decoherence in Raffelt & Tamborra, arXiv:1006.0002; integration over spectrum rather than WP separation has been considered).
Propagation eigenstates and adiabaticity

Propagation eigenstates: Physically meaningful when the adiabaticity condition is satisfied – the rate of change of the Hamiltonian is small compared to the characteristic frequency of neutrino oscillations.

(Adiabaticity violation parameter \( \lambda \ll 1 \))

In ordinary matter: adiabaticity \( \iff \) matter density varies slowly along the neutrino path compared to in-matter “oscillation frequency” \( 2\pi/|E_1 - E_2| \).

In dense neutrino environments: adiabaticity may be violated even when \( n_\nu = \text{const.} \). The Hamiltonian \( H \) depends not only on the density of neutrino gas, but also on its flavour composition which changes during evolution.

Adiabaticity condition in terms of flavour spin: the rate of evolution of \( \vec{H}_\omega \) (angular velocity \( \Omega_H \) of its rotation and the rate of its length decrease) is small compared to the frequency \( |\vec{H}_\omega| \) of precession of individual \( \vec{P}_\omega \) around their \( \vec{H}_\omega \) – “tracking” of moving \( \vec{H}_\omega \) by \( \vec{P}_\omega \).
Strong or moderate non-adiabaticity

What happens when adiabaticity is violated?

- Propagation eigenstates are not physically meaningful; WP separation does not occur (or is suppressed).
- In adiabatic regime: prop. eigenstate evolve independently without going into each other.
- If adiabaticity is strongly violated: prop. eigenstates strongly mix – fully (or almost fully) interchange on a time scale $\tau$ that is short compared to $L_{\text{coh}}$. The slow state becomes the fast one and vice versa – small WP separation ($\ll \sigma_x$) over the period $\tau$ is compensated during the next period $\tau$.
- If adiabaticity is moderately violated, the shuffling of prop. eigenstates occurs with amplitude $< 1$ – only partial compensation of WP separation that occurred during the period $\tau$. Over long times ($> L_{\text{coh}}$) the overlap of different prop. eigenstates tends to a finite value.
Adiabaticity – contd.

I. Large $\mu$ limit ($\mu P_0 \gg \omega_0$): $\lambda \simeq s_{20} \frac{\omega_0}{\mu P_0} \ll 1$ – good adiabaticity.

II. Subcritical $\mu$ ($\mu < \mu_0$): $P_\perp \to 0 \Rightarrow \lambda = 0$ (perfect adiabaticity).

III. $\mu P_0 \sim \omega_0$: $\lambda \sim 1$ (or $\gg 1$) : strong violation of adiabaticity.
Adiabaticity – contd.

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Synchronized oscillations:

- Case $\mu P_0 \gg \omega_0$. Good adiabaticity. Propagation eigenstates physically meaningful. At neutrino production $\theta(0) \approx \pi/2$ – produced flavour state practically coincides with one of the propagation eigenstates (mixing strongly suppressed). $\Rightarrow$ No WP separation $\Rightarrow$ no decoherence.
Adiabaticity — contd.

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- **Case $\sigma_\omega \ll \mu P_0 \lesssim \omega_0$.** Moderate or strong adiabaticity violation. Go to the co-rotating frame: $\omega \to \omega' = (\omega - \omega_s)$. $\Rightarrow$ $\mu P_0 \gg |\omega'|$. Good adiabaticity, suppressed mixing. Reduces to the previous case.
Adiabaticity – contd.

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- Case $\mu P_0 < \mu_0 P_0 \sim \sigma_\omega$. Perfect adiabaticity; $\theta \simeq \theta_0$ (mixing not suppressed); for $L > L_{coh}$ WPs separate $\implies$ complete decoherence. No synchronization.
Adiabaticity & adiabaticity violation

The produced state: $\nu(t_0) = \nu_e$. In prop. eig. basis: $\nu(t_0) = \nu_e = (c_0 \ s_0)^T$.

In the large $t$ limit ($\Omega t \gg 1$) or equivalently averaging over the $\omega$-spectrum:

$$\rho_{12} \simeq \tilde{s}_2 s_0 c_0 - \frac{1}{2} \tilde{s}_2 \tilde{c}_2 (c_0^2 - s_0^2).$$

$$\tilde{s}_2 = \sin 2\tilde{\theta}, \quad \tilde{c}_2 = \cos 2\tilde{\theta}, \quad \tilde{\theta} - \text{mixing of prop. eigenstates},$$

$$\tilde{s}_2 = \frac{\lambda}{\sqrt{1 + \lambda^2}}, \quad \tilde{c}_2 = \frac{1}{\sqrt{1 + \lambda^2}}.$$
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Mixing angle in ‘matter’ (neutrino gas) at $t = t_0$: $\theta(t_0)$ (≠ vac. mix. angle $\theta_0$!)

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(1) Maximal adiabaticity violation: $\tilde{s}_2 = 1$, $\tilde{c}_2 = 0 \Rightarrow \rho_{12} = s_0 c_0$ – coincides with that at $t = t_0$. No WP separation.
Adiabaticity & adiabaticity violation

The produced state: $\nu(t_0) = \nu_e$. In prop. eig. basis: $\nu(t_0) = \nu_e = (c_0 \ s_0)^T$. In the large $t$ limit ($\Omega t \gg 1$) or equivalently averaging over the $\omega$-spectrum:

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(1) Maximal adiabaticity violation: $\tilde{s}_2 = 1$, $\tilde{c}_2 = 0 \Rightarrow \rho_{12} = s_0 c_0$ – coincides with that at $t = t_0$. No WP separation.

(2) Perfect adiabaticity: $\tilde{s}_2 = 0 \Rightarrow \rho_{12} = 0$ – complete decoherence at late $t$. 
The produced state: $\nu(t_0) = \nu_e$. In prop. eig. basis: $\nu(t_0) = \nu_e = (c_0 \ s_0)^T$. In the large $t$ limit ($\Omega t \gg 1$) or equivalently averaging over the $\omega$-spectrum:

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(2) Perfect adiabaticity: $\tilde{s}_2 = 0 \Rightarrow \rho_{12} = 0$ – complete decoherence at late $t$.

(3) Moderate non-adiabaticity: $\tilde{s}_2 = \mathcal{O}(1) \Rightarrow$ partial decoherence.
Summary

- We have studied decoherence effects on synchronized neutrino oscillations in a dense (uniform and homogeneous) neutrino gas. This system is the simplest model for collective neutrino oscillations that can occur in the early Universe or core-collapse supernovae.

- We developed an exact formalism of spectral moments $\vec{K}_n$ of neutrino flavour spin and found a new conservation law for the uniform system of self-interacting neutrinos (or neutrinos and antineutrinos).

- We developed two simple analytic approaches to decoherence dense neutrino gases. They demonstrate in a straightforward way the existence of threshold $\mu_0$ delineating the coherent and decoherent regimes and allow to estimate $\mu_0$ within a factor of 2.
The accuracy of description of the numerical results not very good close to the threshold but is excellent away from it.

This is probably related to the fact that the condition $|\vec{P}| = \text{const}.$ is never satisfied at the threshold because the relaxation time $\to \infty$ as $\mu \to \mu_0$.

- We gave an interpretation of synchronization and de-synchronization (i.e. coherence and decoherence) in terms of the WP separation, both in the adiabatic and non-adiabatic cases.

- Effects of decoherence by WP separation in realistic settings still remain to be studied.
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In vacuum:

\[
\frac{\Delta v_g}{v_g} \approx \frac{\Delta m^2}{2p^2} \quad \Rightarrow \quad L_{coh} \approx \frac{2p^2}{\Delta m^2 \sigma_x}
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In ordinary matter:

$$\frac{\Delta v_g}{v_g} \simeq \frac{\Delta m^2}{2p^2} \cdot \frac{\Delta m^2}{2p^2} - V_e c_2 \sqrt{\left(\frac{\Delta m^2}{2p} c_2 - V_e \right)^2 + \left(\frac{\Delta m^2}{2p} s_2 \right)^2}$$

Typically of the same order as in vacuum (exception: close to the MSW res.).
In dense neutrino backgrounds: the Hamiltonian $\mathcal{H}$ depends on the state of the neutrino system $\Rightarrow$ $\Delta v_g = \frac{\partial}{\partial p} \Delta E$ depends on the solution of the evolution equation.

$$\Delta E = |\vec{H}_\omega| = \sqrt{(\omega - c_{20} \mu P_0)^2 + \mu^2 P_\perp^2}, \quad \omega \equiv \frac{\Delta m^2}{2p}$$

$\downarrow$

$$\Delta v_g = \frac{\partial \omega}{\partial p} \left( \frac{\partial}{\partial \omega} |\vec{H}_\omega| \right) = \frac{\Delta m^2}{2p^2} \cdot \frac{\omega - c_{20} \mu P_0}{\sqrt{(\omega - c_{20} \mu P_0)^2 + \mu^2 P_\perp^2}}$$

The second factor is $O(1)$ (except in vicinity of $\omega = c_{20} \mu P_0$) $\Rightarrow$ $\Delta v_g \simeq \Delta m^2 / (2p^2)$. 
Different strategy – contd.

⇒ $\vec{P}_\omega(t)$ satisfy the same evolution equations as all $\vec{K}_n(t)$:

$$\dot{\vec{P}}_\omega = \omega_s \vec{B} \times \vec{P}_\omega.$$

⇒ Asymptotic evolution of the system at late times:

All $\vec{P}_\omega$ are always in the same plane which rotates around $\vec{B}$ with a constant (but in $\mu$-dependent) angular velocity $\omega_s = \omega_s(\mu)$. Same holds for all $\vec{K}_n$ which are various linear superpositions of $\vec{P}_\omega$.

At late times: $\dot{\vec{P}}_\omega = \omega_s \vec{B} \times \vec{P}_\omega$. At all times: $\dot{\vec{P}}_\omega = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_\omega$.

Consistency condition:

$$\omega_s(\mu) = \omega - c_{20} \mu P_0 - \frac{1}{a_\omega} \mu P_\omega \parallel$$

⇒ $\omega - \mu P_\omega \parallel / a_\omega$ must be $\omega$-independent.

N.B: $P_\omega \parallel \equiv \vec{P}_\omega \cdot \vec{B}$ and $a_\omega$ can be of either sign.