Charge Fluctuation Entropy of Hawking Radiation

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- Entanglement entropy (EE) is hard to compute for a subsystem. It requires the replica trick and replica wormholes in gravity.
- If a system has symmetries, EE of subregions are bounded below by simple observables related to charges, namely the fluctuation entropy

$$S_R \geq S_f$$

In this talk

The fluctuation entropy leads to the violation of global symmetry in the black hole background.

It also bounds the gauge coupling in certain toy models.

Outline

- Fluctuation entropy and symmetry resolved entropy
- Fluctuation entropy in JT gravity
- Bound on a gauge coupling
- Non-perturbative corrections
- Ensemble average resolution

Symmetry reolution of Entanglement

For any region R, the charge $Q_R = \int_R j_0$

$$\text{if } [Q_{tot}, \rho_{tot}] = 0 \;\Rightarrow\; [Q_R, \rho_R] = 0 \;\Rightarrow\; \rho_R = \sum_q p_R(q) \rho_R(q) \qquad (1)$$

- Fluctuation entropy: $S_f = -\sum_q p_R(q) \log p_R(q)$
- Symmetry resolved entropy: $S_R(q) = -\mathrm{Tr}\left[\rho_R(q) \log \rho_R(q)\right]$

$$S_R = \sum_q p_R(q) S_R(q) + S_f \qquad \Rightarrow \qquad S_R \geq S_f$$

- Fluctuation entropy: Correlation between charges of subsystems
- Symmetry resolved entropy: Correlations among configurations of given charged subsystems

Symmetry reolution of Entanglement

- S, S_f have UV divergences. The volume dependence is unaffected.
- To compute probabilities (for an example of $U(1)$):

$$p_R(q) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\alpha e^{-i\alpha q} \langle e^{i\alpha Q_R} \rangle,$$

which is a replica-free!

Example: Massless Dirac fermion in 1+1 D

- Bosonization technique

$$j_\mu \propto \epsilon_{\mu\nu}\partial^\nu \phi \qquad \Rightarrow \qquad Q_R \propto (\phi(x_2) - \phi(x_1))$$

$$\langle e^{i\alpha Q_R} \rangle = \langle e^{i\alpha\phi(x_2)} e^{-i\alpha\phi(x_1)} \rangle$$

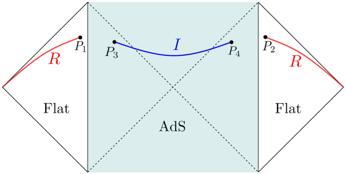
- For large intervals $l \gg \beta$ at finite temperature, probabilities are Gaussian,

$$S_f = \frac{1}{2} \log\left(\frac{2\pi l}{\beta}\right) + \mathcal{O}(l^0)$$

- EE for a single interval (Dirac fermion) in finite temperature $S_R = \frac{\pi}{3\beta} l + \mathcal{O}(l^0)$. For large intervals, $S_f \leq S_R$ is trivially satisfied.
- For $U(1)^c$, the fluctuation entropy is $S_f = \frac{c}{2} \log(\frac{2\pi l}{\beta}) + \mathcal{O}(l^0)$.
- Non-Abelian symmetries: $U(N)$ level k WZW models in a particular large l and large N limit. see also [Calabrese, et al'21]

$$S_f = \frac{N^2}{4} \log\left(k \frac{2\pi l}{\beta}\right) + \mathcal{O}(l^0)$$

Fluctuation entropy in JT gravity



East coast model [AMM'19, AHMST'19]

$$I = -S_0 \chi(M) - \int_M \phi(R+2) - 2\int_{\partial M} \phi_b K + I_{CFT}$$

I_{CFT} : CFT action with central charge c with $U(1)^c$ symmetry

- working in the regime

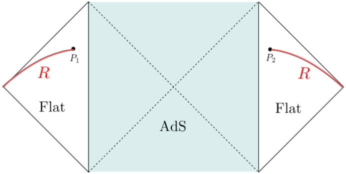
$$c \to \infty, \qquad S_0/c = \text{fixed and kind of large}, \qquad \frac{\phi_r}{\beta c} \geq 1$$

- EE of Hawking radiation in region R at late times:

$$S(R) \approx 2S_0 + \frac{4\pi\phi_r}{\beta}$$

Fluctuation entropy in JT gravity

$$Z_1(\alpha) \sim \langle e^{-i\alpha\phi(P_1)} e^{i\alpha\phi(P_2)} \rangle \underset{t \gg \beta}{\sim} \exp\left[-\frac{\alpha^2}{\pi\beta} t\right]$$



$$S_f = -\sum_q p_R(q) \log p_R(q) = \frac{c}{2} \log(2\pi t/\beta) + \mathcal{O}(1)$$

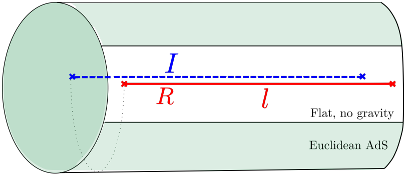
$$t^\star \sim \beta e^{4S_0/c + \frac{8\pi\phi_r}{\beta c}} \times \mathcal{O}(1)$$

- Around $t = t^\star$, the fluctuation entropy exceeds the EE of Hawking radiation and the black hole coarse-grained entropy which is a contradiction.

Taking large subsystems in the bra-ket wormholes gives a similar contradiction when $l \gtrsim \beta e^{4S_0/c}$[Chen, Gorbenko, Maldacena'20].

$$S_R^{\text{island}} \approx 2S_0$$

$$S_f(R) \approx \frac{c}{2}\log(\pi l/\beta)$$



Adding gauge fields

- What happens if the matter has a gauge symmetry?
- Presumably gauge symmetries are allowed.

Adding gauge fields

- We can either consider the gauge field everywhere or in the gravitational region only.
- If the gauge field is everywhere, EE is defined as $S_R = \sum_q p_R(q) S_R(q) + S_f$.

Adding gauge fields

The matter content: $1+1D$ massless Dirac fermions coupled to YM gauge fields with the coupling g_0

$$\mathcal{L} = \frac{1}{2g_0^2}F^2 + \bar{\psi}(\partial_z + A_z)\psi + (\text{anti-chiral})$$

Extra subtleties:

- Fermions are not free
- Propagating gauge d.o.f
- Extra holonomy d.o.f in the path integral

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Excitations are massive in massless Schwinger model with a lowest mass $m = \frac{g_0}{\sqrt{\pi}}$.

Holonomy issue does not arise in calculating the fluctuation entropy.

case 1) small g_0

- The Compton length associated to g_0 is larger than other scales. The result is the same as the global symmetry case.

$$g_0 \gtrsim \max_{\beta} \frac{1}{t^\star} = \frac{c}{\phi_r} \exp\left(-4S_0/c\right) \times \mathcal{O}(1),$$

- For $U(N)$ YM, we find

$$g_0\sqrt{N} \gtrsim \frac{N^2}{\phi_r} \exp\left(-8S_0/N^2\right) \times \mathcal{O}(1)$$

case 2) large g_0

- The Compton length is small. The fluctuation entropy does not grow with length (or time in the eternal BH) even if the gauge field only exists in the gravity region.

So far

- Massless fermions with different global symmetries coupled to gravity lead to a contradiction.
- If the theory is coupled to gauge fields, the issue is resolved if g_0 is not too small.

$4d$ magnetic black holes

- $4d$ near-extremal magnetic black holes mimic the two-dimensional setup [...,Maldacena, Milekhin, Popov'18].
- To get $U(1)^c$, we assume that a black hole is charged under separate $U(1)$ with unit charges.

Near-extremality conditions:

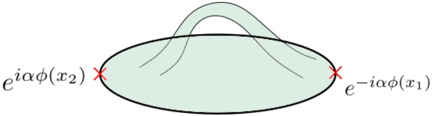
$$\phi_r \sim \frac{r_e^3}{l_p^2}, \qquad S_0 = \frac{\pi r_e^2}{l_p^2}, \qquad g_0 \propto \frac{g_{4d}}{r_e}$$

$$r_e \geq \frac{\sqrt{\pi c}\, l_p}{g_{4d}}, \qquad \Rightarrow \qquad \frac{1}{g_{4d}^3} \gtrsim \exp\left(-\frac{4\pi^2}{g_{4d}^2}\right)$$

This is always satisfied in weakly-coupled theories.
The bound satisfies in the complex SYK model $g_0 \sim c/\phi_r$.

Non-perturbative corrections

- The result for the fluctuation entropy relies on calculating $\langle e^{i\alpha\phi(x_2)} e^{-i\alpha\phi(x_1)} \rangle$.
- The exponential decay is inconsistent with unitarity when correlators become smaller than $e^{-\mathcal{O}(1)S_0}$[Maldacena '01].
- At late times, the contributions from higher topologies are presumably important [Saad'19].



Non-perturbative corrections

$$\langle e^{i\alpha Q_R}\rangle = \langle e^{i\alpha Q_R}\rangle_0 + e^{-2S_0} f(\alpha)$$
$$p(q) = p_0(q) + e^{-2S_0}\tilde{f}(q), \qquad |\tilde{f}(q)| \leq 1$$
$$p_0(q) = \sqrt{\frac{\pi}{2I}} e^{-\frac{\pi^2 q^2}{2I}}$$

- Non-perturbative corrections are small as long as[1] $I \lesssim e^{4S_0}$.

The calculation is valid unless there is a missing contribution around $t, I \sim e^{(\mathcal{O}(1)S_0/c)}$, for instance a saddle point of order $e^{-2S_0/c}$.

[1] In vacuum, I should be replaced with $\log I$.

Ensemble average resolution

If gravity is an ensemble average, usual unitarity arguments do not apply. Two options:

- Global symmetry is part of each member of ensemble.
- Global symmetry is emergent only after averaging.

Is the large fluctuation entropy compatible with unitarity in the ensemble average?

There can be different probabilities associated to charges q and ensemble parameter J:

- conditional probability distribution: $p(q|J)$
- averaged charge distribution $p(q) \equiv [p(q|J)]_J$
- $\int d\alpha e^{-i\alpha q}\langle e^{i\alpha Q_R}\rangle_{\text{gravity}} = p(q)$

$$\text{concavity:} \qquad S_f(p(q)) \geq [S_f\left(p(q|J)\right)]_J$$

$S_f(p(q))$ can be unbounded while $[S_f\left(p(q|J)\right)]_J$ is bounded.

However, this implies[2]

$$S_{\text{ens}}(p(J)) \geq S_f(p(q)) - [S_f\left(p(q|J)\right)]_J,$$

and the ensemble entropy has to be unbounded.

[2] For continuous distributions, this holds under an extra assumption.

Further directions

- What happens in 4d for massive charged particles? An easier exercise is to compute the fluctuation entropy when fermions are massive in 2d.
- Fluctuation entropy for higher form symmetries
- Fluctuation entropy in cosmology
- Understanding holonomies in the island prescription better.

Further directions

- What happens in 4d for massive charged particles? An easier exercise is to compute the fluctuation entropy when fermions are massive in 2d.
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Thank You!

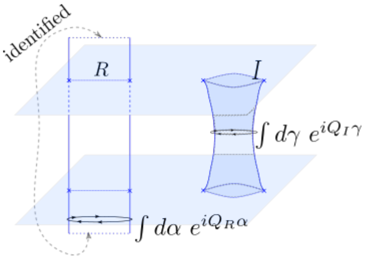
A puzzle about holonomies in island prescription

- There are non-trivial holonomies changing the phase of fermions in the replica manifold.

$$e^{iQ_R\alpha}\rho^{semi}(R\cup I)e^{-i\alpha Q_R}\neq \rho^{semi}(R\cup I)$$

- The proposal $\rho^{semi}_{gauge}(R\cup I):=\int_{-\pi}^{\pi}d\gamma e^{iQ_I\gamma}\rho^{semi}(R\cup I)e^{-iQ_I\gamma}$. It looks like coarse-graning!

- $\rho^{semi}_{gauge}(R\cup I)$ has a positive spectrum.
- $[\rho^{semi}_{gauge}(R\cup I), Q_R]=0$
- $\mathrm{Tr}_I\rho^{semi}_{gauge}(R\cup I)=\rho^{semi}(R)$



Calculating $S_{vN}(\rho^{semi}_{gauge}(R\cup I))$ is hard. However, it is bounded as

$$\max(S_f(R),S_f(I))\leq S(\rho^{semi}_{gauge}(R\cup I)\leq S(\rho^{semi}(R\cup I))+\min(S_f(R),S_f(I))$$