Revisiting Black Hole Temperature in Horndeski Gravity

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Standard Picture of Entropy and Temperature

- Consider (a black hole with) a Killing horizon "H" that is generated by ξ_{H}.

  Surface Gravity:

  $$\xi_{\mathsf{H}} \cdot \nabla \xi_{\mathsf{H}}^{\mu} = -\kappa \xi_{\mathsf{H}}^{\nu}$$

  (Hawking-)Unruh Temperature:

  $$T_0 = \frac{\kappa}{2\pi}$$

- If this is a solution to theory with Lagrangian $\mathcal{L}$ the entropy is given by

  $$S = -2\pi \int_{\mathsf{H}} \frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\mu\nu}} \hat{\epsilon}_{\alpha\beta} \hat{\epsilon}_{\mu\nu}$$

- They satisfy the 1st Law of black hole thermodynamics e.g.

  $$\delta M = T_0\, \delta S - \Omega_{\mathsf{H}}.\, \delta J$$

- It works in generic situations.

A counterexample

Black hole in Horndeski: Wald entropy + Hawking temperature fail!

- Consider following theory that belongs to Horndeski class

$$\mathcal{L} = \frac{1}{16\pi G_{\mathrm{N}}}\left(R - F_{\mu\nu}F^{\mu\nu} + 2\gamma G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

- It admits a black hole solution

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$h = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{q^4}{12r^4}, \qquad f = \frac{4r^4 h}{(2r^2 - q^2)^2} \qquad A = \left(\frac{q}{r} - \frac{q^3}{6r^3}\right)dt, \qquad \phi'(r) = \sqrt{\frac{-q^2}{2\gamma r^2 f}}.$$

- Standard argument:

$$M = \frac{m}{G_N}, \qquad Q = \frac{q}{G_N}, \qquad S_{\mathrm{BH}} = \frac{\pi r_H^2}{G_N}$$
$$T_0 = \frac{2r_H^2 - q^2}{8\pi r_H^3}, \qquad \Phi_{\mathrm{H}} = \frac{q}{r_{\mathrm{H}}} - \frac{q^3}{6r_{\mathrm{H}}^3}$$

- But the 1st law does Not hold $\delta M \neq T_0\,\delta S + \Phi_{\mathrm{H}}\delta Q$! [Feng-Liu-Lu-Pope '16]

Resolutions:

- Adding extra terms to first law! [Feng-Liu-Lu-Pope '16]
  It is a bit ad-hoc!

- Revising the Wald's entropy formula!
  But you may check entropy using other methods!

  Our observation/suggestion: Revising the Temperature!

  - In Horndeski theories gravitons speed of graviton c_g and photon $c = 1$ are different.
  - So, gravitons experience an effective metric and surface gravity $c_g \kappa$.
  - By translating κc_g in units of original metric one may obtain the true black hole temperature

$$T_{\text{BH}} = \alpha c_g^2 T_0$$

  - αc_g^2 depends on solution as well as theory!

Outline

- Wald entropy and ambiguities
- Entropy in solution phase space method
- Review of Horndeski Gravity
- Effective metric and new proposal for temperature
- Examples

Lagrangians, Noether currents and charges

- Consider a diff. inv. theory with Lagrangian $\mathbf{L} = \star\mathcal{L}$ and dynamical fields $\Phi = (g_{\mu\nu}, \phi, ...)$ in n-dim.
  The first variation: $\delta\mathbf{L} = \mathbf{E}\delta\Phi + d\,\boldsymbol{\Theta}(\Phi, \delta\Phi)$

- Vector ξ^μ generates a diff. $\delta_\xi\Phi = \mathcal{L}_\xi\Phi$. It implies a Noether current $(n-1)-$form $\mathbf{J}$

$$\mathbf{J}_\xi = \boldsymbol{\Theta}(\delta_\xi\Phi) - \xi\cdot\mathbf{L} \quad \Longrightarrow \quad d\mathbf{J}_\xi = -\mathbf{E}\;\delta_\xi\Phi$$

- $\mathbf{J}$ is conserved on-shell: $\mathbf{E} = 0 \quad \Longrightarrow \quad d\mathbf{J} = 0$.

- So at least locally $\mathbf{J} = d\mathbf{Q}$ one may defined a global Noether charge $(n-2)$-form $\mathbf{Q}$ in terms of local field Φ and ξ [Wald '90,Iyer-Wald '94]

$$\mathbf{Q} = \mathbf{W}_\mu(\Phi)\xi^\mu + \mathbf{X}^{\mu\nu}(\Phi)\nabla_{[\mu}\xi_{\nu]} + \mathbf{Y}(\Phi, \delta_\xi\Phi) + d\mathbf{Z}(\Phi, \xi)$$

$$(\mathbf{X}^{\mu\nu})_{\mu_3\ldots\mu_n} = -\frac{\delta\mathcal{L}}{\delta R_{\alpha\beta\mu\nu}}\boldsymbol{\epsilon}_{\alpha\beta\mu_3\ldots\mu_n}$$

- But definition of $\mathbf{J}$ and $\mathbf{Q}$ are ambiguous.

Noether-Wald charge and ambiguities

- There are 3 sources of ambiguity in $\mathbf{Q}$ by adding exact forms to $\mathbf{Q}, \boldsymbol{\Theta}$ or $\mathbf{L}$:
  - ★ $\mathbf{Q} \to \mathbf{Q} + d\mathbf{Z}$
  - ★ $\begin{cases} \boldsymbol{\Theta} \to \boldsymbol{\Theta} + d\mathbf{Y}(\phi, \delta\phi) \\ \mathbf{L} \to \mathbf{L} + d\boldsymbol{\mu} \end{cases} \quad \Longrightarrow \quad \mathbf{J} \to \mathbf{J} + d(\xi \cdot \boldsymbol{\mu}) + d\mathbf{Y}$
- $\mathbf{J} = d\mathbf{Q}$ implies the most general form of ambiguous Noether-Wald charge [Iyer-Wald '94]

$$\mathbf{Q} = \xi \cdot (\mathbf{W}(\Phi) + \boldsymbol{\mu}) + \mathbf{X}^{\mu\nu}(\Phi)\nabla_{[\mu}\xi_{\nu]} + \mathbf{Y}(\Phi, \delta_\xi \Phi) + d\mathbf{Z}(\Phi, \xi)$$

- $\mathbf{Y}$ and $\mathbf{Z}$ are linear in $\delta_\xi \Phi$ and ξ.
- Note that $\mathbf{L} \to \mathbf{L} + d\boldsymbol{\mu}$ also changes $\mathbf{X}^{\mu\nu}$ which is given the Wald entropy.

Wald entropy formula

- Let's consider a stationary black hole with a bifurcate Killing horizon. Where the Killing field is

$$\xi_{\mathsf{H}} = \partial_t + \Omega_{\mathsf{H}} \cdot \partial_\varphi$$

- Based on covariant phase space formalism, Wald argues for a charge associated to ξ_{H}

$$\delta \int_{\mathsf{H}} \mathbf{Q}[\xi_{\mathsf{H}}] = \delta M - \Omega_{\mathsf{H}} \cdot \delta J \equiv T_{\mathsf{BH}} \delta S$$

  M (mass) and J (angular momenta) are charges associated to ∂_t and ∂_φ and defined at infinity.

- Since $\xi_{\mathsf{H}}\big|_{\mathsf{H}} = 0$ and $\mathcal{L}_{\xi_{\mathsf{H}}}\Phi = 0$, he shows $\mathbf{W}$, $\mathbf{Y}$ and $\mathbf{Z}$ do not contribute to $\mathbf{Q}[\xi_{\mathsf{H}}]$. Using $\nabla_\mu \xi_{\mathsf{H}\nu}\big|_{\mathsf{H}} = \kappa \boldsymbol{\epsilon}_{\mu\nu}$:

$$\delta \int_{\mathsf{H}} \mathbf{Q}[\xi_{\mathsf{H}}] = \delta \int_{\mathsf{H}} \mathbf{X}^{\mu\nu}(\Phi) \nabla_{[\mu}\xi_{\nu]} = \kappa\, \delta \int_{\mathsf{H}} \mathbf{X}^{\mu\nu}(\Phi) \boldsymbol{\epsilon}_{\mu\nu} = \frac{\kappa}{2\pi} \delta S$$

$$S \equiv 2\pi \int_{\mathsf{H}} \mathbf{X}^{\mu\nu} \boldsymbol{\epsilon}_{\mu\nu}, \quad T_{\mathsf{BH}} \equiv \frac{\kappa}{2\pi}$$

  $\boldsymbol{\epsilon}_{\mu\nu}$ is the binormal to H, $\boldsymbol{\epsilon}_{\mu\nu}\boldsymbol{\epsilon}^{\mu\nu} = -2$

Wald entropy and its ambiguities

- There is a loophole: what happens when Φ or $d\Phi$ is singular on the horizon?!
- Is it clear namely $\xi_{\text{H}} \cdot \mathbf{W}(\Phi) = 0$ or $\boldsymbol{\mu}$ does not contribute to the entropy or $\mathbf{Q}[\xi_{\text{H}}]$?
- In this case, one may doubt about expressions for entropy S or black hole temperature T_{BH}

$$S \overset{?}{\equiv} 2\pi \int_{\text{H}} \mathbf{X}^{\mu\nu} \boldsymbol{\epsilon}_{\mu\nu} \qquad \text{or} \qquad T_{\text{BH}} \overset{?}{\equiv} \frac{\kappa}{2\pi}$$

- Naturally, one may take $\frac{\kappa}{2\pi}$ as the black hole temperature (based on Hawking's argument) and so looking for an alternative definition for entropy.
- Note that the Hawking's calculation is almost depends on the geometry and not the details of theory. Hence, looking for an alternative definition for entropy seems reasonable natural.
- However, we think this is not the whole story!
- As we will argue in certain cases one ought to revise definition of black hole temperature.

Ambiguity of Wald entropy in Horndeski: An explicit example

- Consider following theory that belongs to Horndeski class

$$\mathcal{L} = \frac{1}{16\pi G_{\mathsf{N}}}\Big(R - F_{\mu\nu}F^{\mu\nu} + 2\gamma \underbrace{G^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi}_{R^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - \frac{1}{2}R(\partial\phi)^2}\Big)$$

$$R^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi = -(\nabla_\rho\nabla_\sigma\phi)^2 + (\Box\phi)^2 + \nabla_\mu\mathcal{W}^\mu, \qquad \mathcal{W}^\mu = (\nabla_\nu\phi\nabla^\nu\nabla^\mu\phi - \Box\phi\nabla^\mu\phi)$$

- The explicit dependence of Lagrangian to $R_{\mu\nu}$ can be removed in favor of derivatives of ϕ.

- Note that $\nabla_\mu\mathcal{W}^\mu$ is same as $d\boldsymbol{\mu}$.

- It changes Wald entropy

$$(\mathbf{X}^{\mu\nu})_{\mu_3\dots\mu_n} \to (\mathbf{X}^{\mu\nu})_{\mu_3\dots\mu_n} + \lambda\frac{\delta}{\delta R_{\alpha\beta\mu\nu}}R^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\boldsymbol{\epsilon}_{\alpha\beta\mu_3\dots\mu_n}.$$

where λ is an arbitrary number and last term is not zero!

Using $\nabla_{[\mu}\xi_{\nu]} = 2\kappa\boldsymbol{\epsilon}_{\mu\nu}$ and isometry condition $\delta_\xi\phi = \xi_\mu\nabla^\mu\phi = 0$

$$
\begin{aligned}
\frac{\delta\left(R^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right)}{\delta R_{\alpha\beta\mu\nu}}\boldsymbol{\epsilon}_{\mu\nu} &= \frac{\delta\left(R^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right)}{\delta R_{\alpha\beta\mu\nu}}\,\frac{\nabla_{[\mu}\xi_{\nu]}}{\kappa} \\
&= \frac{-1}{\kappa}\nabla^\alpha(\nabla^\nu\phi)\nabla^\beta\phi\xi_\nu
\end{aligned} \tag{1}
$$

For the over example the pull-back of the result to the bifurcation surface of horizon is non-zero

$$
\oint_{\mathsf{H}} \frac{-\sqrt{-g}\,d\theta\,d\varphi}{\kappa}\nabla^t(\nabla^\nu\phi)\nabla^r\phi\,\xi_{\mathsf{H}\,\nu} = \frac{-\pi\,q^2}{\gamma}. \tag{2}
$$

Therefore, there is a non-vanishing ambiguity in the Wald entropy.

Solution phase space method (SPSM):

Entropy without ambiguity (I)

- There are a couple of alternative to obtain black hole entropy without using the Wald formula e.g. Euclidean on-shell action, SPSM, a certain limit of holographic entanglement entropy, etc.

- In the case of stationary black hole, one can show the Killing field ξ_{H} generates the variation of entropy [Ashtekar-Bombellim-Koul-Reula '87,Lee-Wald '90,'Iyer-Wald '94, Hajian-Sheikh Jabbari'16]

$$\boxed{\delta S_{\text{BH}} := \frac{1}{T_{\text{BH}}} \int_{\text{H}} \delta \mathbf{Q}_{\xi_{\text{H}}} - \xi_{\text{H}} \cdot \mathbf{\Theta}(\delta\Phi, \bar{\Phi})}$$

- Note, here $1/T_{\text{BH}}$ plays role of integrating factor and $S_{\text{BH}} = \int \delta S_{\text{BH}}$

- It is similar to standard thermodynamics where temperature is integrating factor for heat (Clausius eq.).

Solution phase space method (SPSM): Entropy without ambiguity (II)

- So an appropriate $T_{_{\rm BH}}$ should lead to an integrable expression for $\delta S_{\rm BH}$.
- This method provides an expression for $S_{\rm BH}$ without ambiguity.
- Now the main question is how to fix $T_{_{\rm BH}}$.
- In usual situations, the standard Hawking temperature $T_{_{\rm BH}} = \frac{\kappa}{2\pi} = T_0$ is fine.
- One may apply this formula for the mentioned example with T_0 but $\delta S_{\rm BH}$ is not integrable and so the Hawking Temperature fails! as $T_{_{\rm BH}} \neq T_0$
- In particular, when the speed of gravitons and photons are different as happens in Horndeski Gravity.

Review of Horndeski Gravity

- A class of scalar-tensor theories with second order field equation (a generalization for Lovelock's theorem)

$$S_{\text{Horn.}} = \frac{1}{16\pi G_{\text{N}}}\int d^n x\sqrt{-g}\,\mathcal{L}_{\text{Horn.}}$$

$$\mathcal{L}_{\text{Horn.}} = \mathcal{G}_2(\phi,\mathcal{X}) - \mathcal{G}_3(\phi,\mathcal{X})\Box\phi + \mathcal{G}_4(\phi,\mathcal{X})R + \mathcal{G}_4'(\phi,\mathcal{X})\left((\Box\phi)^2 - (\partial_{\mu\nu}\phi)^2\right)$$
$$- \mathcal{G}_5(\phi,\mathcal{X})G^{\mu\nu}\partial_{\mu\nu}\phi - \frac{\mathcal{G}_5'(\phi,\mathcal{X})}{6}\left((\Box\phi)^3 + 2(\partial_{\mu\nu}\phi)^3 - 3\Box\phi(\partial_{\mu\nu}\phi)^2\right)$$

- R is Ricci scalar and $G_{\mu\nu}$ is the Einstein tensor, $\mathcal{G}_i' = d\mathcal{G}_i/d\mathcal{X}$

$$\partial_{\mu\nu}\phi \equiv \nabla_\mu\nabla_\nu\phi \qquad \Box\phi \equiv g^{\mu\nu}\partial_{\mu\nu}\phi \qquad \mathcal{X} \equiv -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

We restrict ourselves to a class with $\mathcal{G}_4(\phi)$ and $\mathcal{G}_5(\phi)$

$$\boxed{\mathcal{L}_{\text{Horn.}} = \mathcal{G}_2 + (\mathcal{G} - \mathcal{G}'\mathcal{X})R + \mathcal{G}'G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}$$

where $\mathcal{G} = \mathcal{G}_4 + \mathcal{X}\,d\mathcal{G}_5/d\phi$, $\tilde{\mathcal{G}}_2 = \mathcal{G}_2 + 2d^3\mathcal{G}_5/d\phi^3, \tilde{\mathcal{G}}_3 = \mathcal{G}_3 + 3d^2\mathcal{G}_5/d\phi^2$.

Speed of graviton in Horndeski gravity

- To find speed of graviton c_g, one should study linearized eom around a given background.
- But here, we use "$\phi + 3$" decomposition as a shortcut to obtain c_g

$$g_{\mu\nu} = h_{\mu\nu} + \sigma\phi_\mu\phi_\nu, \qquad \phi_\mu := \frac{\partial_\mu \phi}{|\partial\phi|}, \qquad \sigma = \frac{\phi_\mu\phi^\mu}{|\phi_\mu\phi^\mu|} = \pm 1$$

- σ is -1 for cosmological backgrounds and $+1$ for black holes.
- and $h_{\mu\nu}$ is the metric along constant ϕ surface, $h_{\mu\nu}\phi^\nu = 0$.
- Similar to $1+3$ one may obtain

$$\mathcal{L} = \mathcal{G}_2 + \mathcal{G}\ {}^{(3)}R + (\mathcal{G} - 2\mathcal{X}\mathcal{G}')(K_{\mu\nu}K^{\mu\nu} - K^2)$$
$$+ 2\sqrt{-2\mathcal{X}}\mathcal{G}_{,\phi}K + \text{total derivative terms}$$

- For a typical black hole solution ϕ_μ is along the radial direction and normal to horizon. So the time is in 3 part $h_{\mu\nu}$. It allows to read speed of graviton as

$$c_g^2 = \begin{cases} \frac{\mathcal{G} - 2\mathcal{X}\mathcal{G}'}{\mathcal{G}} & \text{for gravitons moving along } \phi_\mu \\ \frac{\mathcal{G}}{\mathcal{G}} = 1 & \text{for gravitons moving normal to } \phi_\mu \end{cases}$$

Effective Metric for Gravitons (EMG) and effective surface gravity

- One may show that graviton experiences an effective metric $\tilde{g}_{\mu\nu}$. It is related to the original metric via a "disformal map"

$$\tilde{g}_{\mu\nu} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')g_{\mu\nu} - \mathcal{G}'\partial_\mu\phi\partial_\nu\phi$$

- This metric also admits a Killing horizon and the associate surface gravity (as seen by gravitons) is given by $c_g\kappa$.

$$d\xi_{\mathrm{H}} = 2\kappa c_g \tilde{\boldsymbol{\epsilon}}\Big|_{\mathrm{H}}, \quad \tilde{\boldsymbol{\epsilon}}_{\mu\nu}\tilde{\boldsymbol{\epsilon}}^{\mu\nu} = -2 \quad c_g^2 = \frac{\mathcal{G} - 2\mathcal{X}\mathcal{G}'}{\mathcal{G}}$$

- But κc_g is defined in terms of new metric units. To translate it we can use the relation between binomials

$$\tilde{\boldsymbol{\epsilon}} = \sqrt{\mathcal{G}(\mathcal{G} - 2\mathcal{X}\mathcal{G}')}\,\boldsymbol{\epsilon} \qquad \Longrightarrow \qquad d\xi_{\mathrm{H}} = 2\kappa(\mathcal{G} - 2\mathcal{X}\mathcal{G}')\boldsymbol{\epsilon},$$

- So we claim that the temperature of black hole is given by

$$\boxed{T_{\mathrm{BH}} = T_{\mathrm{graviton}} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')\,T_0,}$$

- It is the correct integrating factor for δS_{BH}.

Example 1: Solution

$$\mathcal{L} = \frac{1}{16\pi G_{\mathsf{N}}}\Big(R - F_{\mu\nu}F^{\mu\nu} + 2\gamma G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big)$$

$$\mathcal{G}_2 = 0, \mathcal{G}_4 = 1, \mathcal{G}_5 = 2\gamma\phi, \mathcal{G} = 1 + 2\gamma\mathcal{X}$$

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2),$$

$$h = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{q^4}{12r^4}, \qquad f = \frac{4r^4 h}{(2r^2 - q^2)^2}.$$

$$A = \left(\frac{q}{r} - \frac{q^3}{6r^3}\right)dt, \qquad \phi'(r) = \sqrt{\frac{-q^2}{2\gamma r^2 f}}.$$

Example 1: Thermodynamics

Standard argument fails:

$$M = \frac{m}{G_N}, \qquad Q = \frac{q}{G_N}, \qquad S_{\mathsf{BH}} = \frac{\pi r_H^2}{G_N}$$

$$T_0 = \frac{\kappa}{2\pi} = \frac{2r_H^2 - q^2}{8\pi r_H^3}, \qquad \Phi_{\mathsf{H}} = \frac{q}{r_{\mathsf{H}}} - \frac{q^3}{6r_{\mathsf{H}}^3}$$

and the first law does Not hold!

$$\delta S \neq \frac{1}{T_0}(\delta M - \Phi_{\mathsf{H}}\delta Q)$$

Our proposal works:

$$T_{\mathsf{BH}} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')\Big|_{\mathsf{H}} T_0 = \left(1 - \frac{q^2}{2r_{\mathsf{H}}^2}\right) T_0 = \frac{1}{4\pi r_{\mathsf{H}}}\left(1 - \frac{q^2}{2r_{\mathsf{H}}^2}\right)^2$$

$$\delta S_{\mathsf{BH}} = \frac{1}{T_{BH}}(\delta M - \Phi_{\mathsf{H}}\delta Q) \qquad \xrightarrow{S=\int \delta S} \qquad S_{BH} = \frac{\pi r_H^2}{G_N}$$

Example II

$$\mathcal{L} = \frac{1}{16\pi G_{\mathrm{N}}}\Big(R - 2\Lambda - 2(\alpha g^{\mu\nu} - \gamma G^{\mu\nu})\partial_\mu\phi\partial_\nu\phi\Big)$$

$$\mathcal{G} = 1 + 2\gamma\mathcal{X}, \qquad \mathcal{G}_2 = 4\alpha\mathcal{X} - 2\Lambda, \quad \gamma < 0$$

$$ds^2 = -h dt^2 + \frac{dr^2}{h} + r^2(d\varphi - \frac{j}{r^2}dt)^2,$$

$$h = -m + \frac{\alpha r^2}{\gamma} + \frac{j^2}{r^2}, \quad \phi'(r) = \sqrt{\frac{-(\alpha+\gamma\Lambda)}{2\alpha\gamma h}}$$

$$M = \frac{(\alpha - \Lambda\gamma)m}{16\alpha G_{\mathrm{N}}}, \qquad J = \frac{(\alpha-\Lambda\gamma)j}{8\alpha G_{\mathrm{N}}}, \qquad S_{\mathrm{BH}} = 2\pi r_{\mathrm{H}}/(4G_{\mathrm{N}})$$

$$\kappa_\pm = \frac{\alpha(r_+^2 - r_-^2)}{\gamma r_\pm}, \qquad \Omega_\pm = \frac{j}{r_\pm^2},$$

$$T_{\mathrm{BH}} = \left(\frac{\alpha - \Lambda\gamma}{2\alpha}\right) T_0,$$

$$\delta S_{\mathrm{BH}} = \frac{1}{T_{\mathrm{BH}}}(\delta M - \Omega_{\mathrm{H}}\delta J)$$

Example III

$$\mathcal{L} = \frac{1}{16\pi G_{\mathrm{N}}}\Big((1+\beta\sqrt{-\mathcal{X}})R - 2\Lambda + \eta\mathcal{X} - \frac{\beta}{2\sqrt{-\mathcal{X}}}((\Box\phi)^2 - (\partial_{\mu\nu}\phi)^2)\Big) \tag{3}$$

$$\mathcal{G} = 1 + 2\beta\sqrt{-\mathcal{X}}$$

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2),$$

$$h = f = 1 - \frac{2m}{r} - \frac{\beta^2}{2\eta r^2} - \frac{\Lambda r^2}{3}, \quad d\phi = \frac{\sqrt{2}\beta}{\eta r^2\sqrt{h}}dr.$$

For this theory $\mathcal{G} - 2\mathcal{G}'\mathcal{X} = 1$ hence $T_{BH} = T_0$ and first law $T_{\mathrm{BH}}\delta S_{\mathrm{BH}} = \delta M$ is satisfied for the following standard charges:

$$M = \frac{m}{G_{\mathrm{N}}}, \qquad \kappa = \frac{\beta^2 + 2\eta(r_{\mathrm{H}}^2 - \Lambda r_{\mathrm{H}}^4)}{4\eta r_{\mathrm{H}}^3}, \quad S_{\mathrm{BH}} = \pi r_{\mathrm{H}}^2/G_{\mathrm{N}} \tag{4}$$

Summary

- Wald's entropy formula suffers from some ambiguities in certain cases.
- One may obtain variation of entropy without any ambiguity using solution phase space method.
- As an example it does not lead to the correct first law in Horndeski theories.
- For this family of theories speeds of gravitons and photons are different.
- In this case, one should consider a modified temperature for black hole than can be obtained from the effective metric for graviton.
- This is true integrating factor for charge associated to the generator of Killing horizon.
- One can argue that our result is consistent with 2nd law e.g. by adding a box of radiation.
- We expect this formalism works in more general situation where $c \neq c_g$.
- It may help us to fix ambiguities in Wald entropy.

Thank you for your attention!