System under study: heterotic string theory compactified on $T^5 \times S^1$.

We consider a BPS string state carrying w units of winding and n units of momentum along S^1.

The degeneracy of this states $\sim e^{4\pi\sqrt{nw}}$ for large n, w.

$\rightarrow S_{stat} = \ln(\text{degeneracy}) \simeq 4\pi\sqrt{nw}$

The goal is to see if we get the same expression for the Bekenstein-Hawking entropy by considering a BPS black hole of same mass and charge.

The (3+1) dimensional fields relevant for describing the black hole solution are:

$S,\ T,\ g_{\mu\nu},\ B_{\mu\nu},\ A_\mu^{(1)},\ A_\mu^{(2)}$

$G_{\mu\nu}$: string metric

$g_{\mu\nu}$: canonical metric

$G_{\mu\nu} = S^{-1}\, g_{\mu\nu}$

Define:

$$F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu\,, \qquad a = 1, 2\,,$$

$$H_{\mu\nu\rho} = \left[\partial_\mu B_{\nu\rho} + 2\left(A^{(1)}_\mu F^{(2)}_{\nu\rho} + A^{(2)}_\mu F^{(1)}_{\nu\rho}\right)\right]$$
$$+\text{cyclic permutations of } \mu,\ \nu,\ \rho\,.$$

The low energy effective action is given by:

$$
\begin{aligned}
\mathcal{S} \;=\; & \frac{1}{32\pi}\int d^4x\,\sqrt{-\det g}\,\Big[R_g - \frac{1}{2S^2}\,g^{\mu\nu}\,\partial_\mu S\partial_\nu S \\
& -\frac{1}{T^2}\,g^{\mu\nu}\,\partial_\mu T\partial_\nu T \\
& -\frac{1}{12}S^2 g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'} \\
& -ST^2\,g^{\mu\nu}\,g^{\mu'\nu'}\,F^{(1)}_{\mu\mu'}F^{(1)}_{\nu\nu'} \\
& -ST^{-2}\,g^{\mu\nu}\,g^{\mu'\nu'}\,F^{(2)}_{\mu\mu'}F^{(2)}_{\nu\nu'}\Big]\,,
\end{aligned}
$$

Note: The action is completely universal without any parameter.

Expectation values of S and T determine the string coupling constant and the radius of S^1.

$$
\langle S\rangle = \frac{1}{g^2}, \qquad \langle T\rangle = R\sqrt{\alpha'} = \frac{R}{4}
$$

We now want to construct an extremal black hole solution satisfying the following properties:

- It should have the same mass and charge as the elementary string state carrying quantum numbers (n, w).

- It should be a solution of the classical field equations.

- It should be invariant under half of the space-time supersymmetry transformations.

- Asymptotically $S \to \frac{1}{g^2}$, $T \to \frac{R}{4}$.

The extremal black hole solution:

$$
\begin{aligned}
ds_c^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu \\
&= -(F(\rho))^{-1/2} \rho dt^2 \\
&\quad + (F(\rho))^{1/2} \rho^{-1} \left(d\rho^2 + \rho^2 d\Omega_2^2 \right) , \\
F(\rho) &= (\rho + gwR/2)(\rho + 8gnR^{-1}) , \\
d\Omega_2^2 &\equiv d\theta^2 + \sin^2\theta \, d\phi^2 , \\
S &= g^{-2} (F(\rho))^{1/2} \rho^{-1} , \\
T &= \frac{1}{4} R \sqrt{(\rho + 8gnR^{-1})/(\rho + gwR/2)} , \\
F_{\rho t}^{(1)} &= \frac{16 g^2 R^{-2} n}{(\rho + 8gnR^{-1})^2} , \\
F_{\rho t}^{(2)} &= \frac{1}{16} \frac{g^2 w R^2}{(\rho + gwR/2)^2} , \\
H_{\mu\nu\rho} &= 0 , \\
ds^2_{string} &\equiv G_{\mu\nu} dx^\mu dx^\nu = S^{-1} ds_c^2 \\
&= -g^2 \rho^2 (F(\rho))^{-1} dt^2 + g^2 \, d\vec{x}^2 .
\end{aligned}
$$

'Horizon' is at $\rho = 0$.

Area of the horizon $= 0$

Naively this would imply that the Bekenstein-Hawking entropy of the black hole is zero!

However let us not give up immediately and study the solution in some detail.

We shall be interested in the limit of large n and w.

In this limit

$$\rho << gwR/2, \quad 8gnR^{-1}$$

Define:

$$r = g\,\rho, \qquad \tau = g^{-1}\,t/\sqrt{nw}$$

In this coordinate system the solution for large n, w takes the form:

$$\begin{aligned} ds^2_{string} &= -\frac{r^2}{4}\,d\tau^2 + dr^2 + r^2\,d\Omega_2^2 \\ S &= \frac{2\sqrt{nw}}{r}, \\ T &= \sqrt{\frac{n}{w}}, \\ F^{(1)}_{r\tau} &= \frac{1}{4}\sqrt{\frac{w}{n}} \\ , \\ F^{(2)}_{r\tau} &= \frac{1}{4}\sqrt{\frac{n}{w}}. \end{aligned}$$

Curvatures in string metric are small for $r >> 1$ but of order 1 for $r \sim 1$.

$\rightarrow$ higher derivative terms become important for $r \sim 1$.

We see that $S \sim \sqrt{nw}$ for $r \sim 1$

$S =$ Inverse string coupling^2

$\rightarrow$ string coupling $\sim (nw)^{-1/4}$ for $r \sim 1$.

Thus for large nw we can ignore string loop corrections to the effective action.

The relevant corrections come from higher derivative terms in the effective action appearing at string tree level.

In order to study the effect of these higher derivative corrections, we shall now try to simplify the solution further by using some exact symmetries of the string tree level effective action.

The effective action at string tree level has an exact symmetry:

$$G^{(10)}_{44} \to e^{2\beta} G^{(10)}_{44}, \qquad G^{(10)}_{4\mu} \to e^{\beta} G^{(10)}_{4\mu},$$
$$B^{(10)}_{4\mu} \to e^{\beta} B^{(10)}_{4\mu}$$

This corresponds to changing the radius of S^1.

In terms of four dimensional fields this becomes:

$$T \to e^{\beta} T, \qquad A^{(1)}_{\mu} \to e^{-\beta} A^{(1)}_{\mu},$$
$$A^{(2)}_{\mu} \to e^{\beta} A^{(2)}_{\mu}$$

Two solutions related by this transformation has the same entropy.

Choosing $e^{\beta} = \sqrt{w/n}$ we can map the original solution to the 'checked solution':

$$
\begin{aligned}
\check{ds}^2_{string} &= -\frac{r^2}{4}d\tau^2 + dr^2 + r^2 d\Omega_2^2 \\
\check{S} &= \frac{2\sqrt{nw}}{r}\,, \\
\check{T} &= 1\,, \\
\check{F}^{(1)}_{r\tau} &= \frac{1}{4}\,, \\
\check{F}^{(2)}_{r\tau} &= \frac{1}{4}\,.
\end{aligned}
$$

As before, the form of the solution is expected to change near $r \sim 1$ by higher derivative corrections.

Under another transformation

$$S \to K\,S, \quad T \to T, \quad G_{\mu\nu} \to G_{\mu\nu}, \quad F^{(a)}_{\mu\nu} \to F^{(a)}_{\mu\nu}$$

the tree level effective action gets multiplied by K.

This leaves the equations of motion unchanged.

This transformation also multiplies the entropy associated with a solution by a factor of K.

Choosing $K = 1/\sqrt{nw}$ we can map the checked solution to the 'hatted solution':

$$\begin{aligned}
\widehat{ds}^2_{string} &= -\frac{r^2}{4}d\tau^2 + dr^2 + r^2 d\Omega_2^2 \\
\widehat{S} &= \frac{2}{r}, \\
\widehat{T} &= 1, \\
\widehat{F}^{(1)}_{r\tau} &= \frac{1}{4}, \\
\widehat{F}^{(2)}_{r\tau} &= \frac{1}{4},
\end{aligned}$$

Note that this solution is completely universal, independent of any external parameter.

Thus its modification near $r \sim 1$ by higher derivative corrections will also be completely universal.

Modified form of the hatted solution by higher derivative terms:

$$\begin{aligned}
\widehat{ds}^2_{string} &= -\frac{f_1(r)}{f_3(r)}\,d\tau^2 + \frac{f_2(r)}{f_3(r)}\,(dr^2 + r^2 d\Omega_2^2) \\
\widehat{S} &= f_3(r)\,, \\
\widehat{T} &= f_4(r)\,, \\
\widehat{F}^{(1)}_{r\tau} &= f_5(r)\,, \\
\widehat{F}^{(2)}_{r\tau} &= f_6(r)\,, \\
\widehat{ds}^2_c &= -f_1(r)\,d\tau^2 + f_2(r)\,(dr^2 + r^2 d\Omega_2^2)
\end{aligned}$$

$f_i(r)$: universal functions

The entropy computed from this modified solution is also going to be a purely numerical constant a.

Naively,

$$a = \frac{A_{horizon}}{4G_N} = \frac{1}{8}\,4\pi\,\lim_{r\to 0}(r^2 f_2(r))\,.$$

We can now make inverse transformations to go back to the checked and then to the original solution.

The original solution:

$$
\begin{aligned}
ds^2_{string} &= -\frac{f_1(r)}{f_3(r)}\, d\tau^2 + \frac{f_2(r)}{f_3(r)}\, (dr^2 + r^2 d\Omega_2^2) \\
S &= \sqrt{nw}\, f_3(r)\,, \\
T &= \sqrt{\frac{n}{w}}\, f_4(r)\,, \\
F^{(1)}_{r\tau} &= \sqrt{\frac{w}{n}}\, f_5(r)\,, \\
F^{(2)}_{r\tau} &= \sqrt{\frac{n}{w}}\, f_6(r)\,.
\end{aligned}
$$

It is also easy to calculate the entropies associated with the checked and the original solutions in terms of a.

The entropy associated with the checked solution

$= \sqrt{nw} \times$ the entropy associated with the hatted solution

$= a\sqrt{nw}$

The entropy S_{BH} associated with the original solution

$=$ entropy associated with the checked solution

$= a\sqrt{nw}$

$$S_{BH} = a\sqrt{nw}$$

On the other hand

$$S_{stat} \equiv \ln d_{nw} \simeq 4\pi\sqrt{nw}$$

for large nw.

Thus we see that S_{BH} and S_{stat} has same dependence on n, w, g and R. (AS)

Q. Can we compute a?

A brief history of subsequent developments

1. Strominger and Vafa computed the statistical entropy S_{stat} of BPS black holes in five dimensions carrying three different types of charges by describing them as configurations of D-branes.

The corresponding black hole solutions have finite area event horizon and hence finite Bekenstein-Hawking entropy S_{BH}.

In the limit of large charge

$$S_{BH} = S_{stat}$$

2. This result was generalized to many other examples including black holes in four dimensional heterotic string theories carrying both electric and magnetic charges, in the limit where all charges are large.

3. For a special class of these four dimensional black holes, Maldacena, Strominger, Witten computed the subleading (in 1/charges) corrections to S_{stat}.

4. For these black holes, subleading corrections to S_{BH} were computed by Cardoso, de Wit, Mohaupt + Kapelli by taking into account a special class of higher derivative terms in the action.

It was found that including these subleading corrections we get

$$S_{BH} = S_{stat}$$

In computing the subleading corrections to S_{BH} we had to take into account modification of the Bekenstein-Hawking formula due to Wald.

5. Given the expression for the entropy of the black hole as a function of electric and magnetic charges, we can now set the magnetic charges to zero to compute entropy of purely electrically charged black holes.

In the leading approximation the answer vanishes.

However the full expression including the subleading corrections do not vanish.

Result for heterotic string wound on S^1:

$$S_{BH} = 4\pi\sqrt{nw} \qquad \to \qquad a = 4\pi$$

$\to$ Exact agreement with S_{stat}. Dabholkar

Instead of reviewing the detailed analysis we shall now give a brief outline of the steps which are involved in the computation of a.

(Lopes Cardoso, de Wit, Kappeli, Mohaupt; Dabholkar; AS; Hubeny, Maloney, Rangamani)

Tree level heterotic string effective action contains a term

$$\frac{1}{16\pi} \int d^4x \sqrt{-\det g}\, S\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

Supersymmetrization of this term gives many other terms.

These constitute a special class of higher derivative terms which are 'holomorphic'.

The analysis leading to the computation of a takes into account only these higher derivative corrections to the effective action.

Given this modified action we proceed as follows in order to determine the modified solution describing the heterotic string configuration carrying charges (n, w).

1. First we note that the modified solution describing the heterotic string configuration under study must satisfy the modified field equations derived from the new action.

2. We can also use the fact that we are trying to describe a BPS state that in invariant under a certain set of space-time supersymmetry transformations.

As a result the field configuration describing this state must also be invariant under these space-time supersymmetry transformations.

These give constraints on the field configurations.

3. Boundary condition: At large distance where higher derivative corrections are negligible, the solution must approach the leading order solution found earlier.

Now recall the general form of the 'hatted solution':

$$\begin{aligned} \widehat{ds}^2_{string} &= -\frac{f_1(r)}{f_3(r)}\, d\tau^2 + \frac{f_2(r)}{f_3(r)}\,(dr^2 + r^2 d\Omega_2^2) \\ \widehat{S} &= f_3(r)\,, \\ \widehat{T} &= f_4(r)\,, \\ \widehat{F}^{(1)}_{r\tau} &= f_5(r)\,, \\ \widehat{F}^{(2)}_{r\tau} &= f_6(r)\,. \end{aligned}$$

Substitute these into the field equations / BPS conditions.

This gives constraints on $f_1, \ldots f_6$.

Define $h(r) = \ln(f_1(r))$.

Then the constraints on $f_1, \ldots f_6$ may be expressed as:

$$
\begin{aligned}
f_1(r) &= e^{h(r)}\,, \\
f_2(r) &= e^{-h(r)}\,, \\
f_3(r) &= \frac{2}{r}\,\frac{1}{\sqrt{1+4\,(h'(r))^2}}\,, \\
f_4(r) &= \frac{1}{\sqrt{1+4\,(h'(r))^2}}\,, \\
f_5(r) &= \frac{1}{2}\,\partial_r\left(e^{h(r)}\,\sqrt{1+4\,(h'(r))^2}\right)\,, \\
f_6(r) &= \frac{1}{2}\,\partial_r\left(e^{h(r)}\,\sqrt{1+4\,(h'(r))^2}\right)\,.
\end{aligned}
$$

h satisfies the differential equation:

$$
\begin{aligned}
& h'\left(1+4\,(h')^2\right)+r\,h'' \\
= \; & \frac{r^2}{8}\,e^{-h}\,\left(1+4\,(h')^2\right)^{3/2}-\frac{r}{4}\,\left(1+4\,(h')^2\right)\,.
\end{aligned}
$$

At large r the equation for h admits a solution:

$$h = \ln \frac{r}{2}$$

$f_1, \ldots f_6$ calculated from this gives us back the supergravity results.

For small r the equation for h admits a solution:

$$h = 2 \ln \frac{r}{2}$$

Thus $f_2(r) = e^{-h} = 4/r^2$

This gives the naive entropy associated with the hatted solution:

$$a = \frac{\pi}{2} \lim_{r \to 0} (r^2 f_2(r)) = 2\pi$$

$\rightarrow$ finite area of the event horizon but wrong answer!

However due to higher derivative terms in the action the Bekenstein-Hawking formula itself gets modified (Wald)

After taking these corrections into account we get:

$$a = 4\pi$$

in exact agreement with the statistical entropy.

However our analysis is not complete yet.

$h(r)$ satisfies a second order differential equation.

It admits a solution $h = \ln(r/2)$ for large r that gives the correct asymptotic behaviour.

It admits a solution $h = 2\ln(r/2)$ at small r that gives the correct entropy.

However a second order differential equation has two integration constants.

Thus there is no guarantee *a priori* that a solution that has the small r behaviour $h = 2\ln(r/2)$ will approach the asymptotic form $h = \ln(r/2)$ at large r.

Study small fluctuations about the solution $h = \ln(r/2)$ at large r.

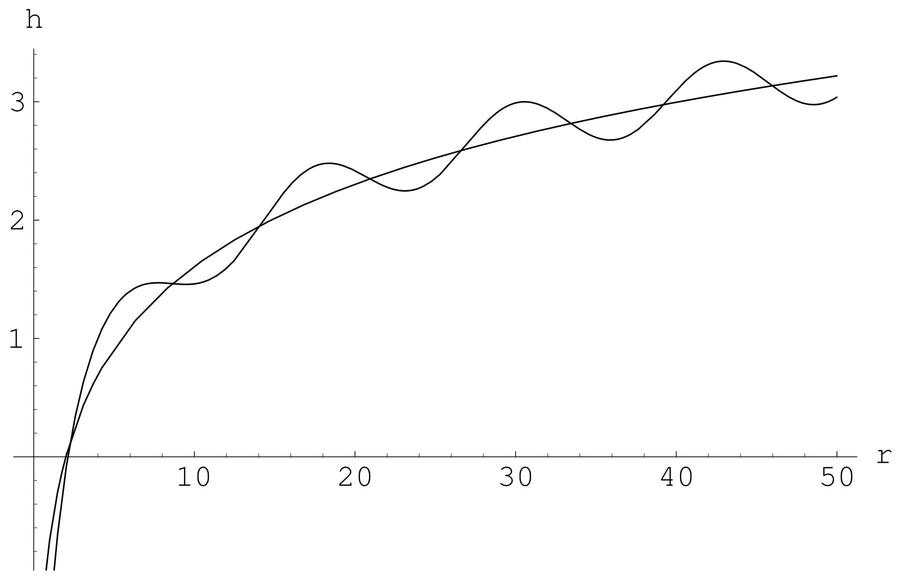
Result:

$$h \simeq \ln\frac{r}{2} + A \cos\left(\frac{r}{2} + B\right) + \mathcal{O}(A^2)$$

A, B: integration constants

Thus for a generic initial condition we expect the solution to oscillate about $h = \ln(r/2)$.

Numerical results show that this is exactly what happens.



Interpretation of these oscillations:

In the presence of higher derivative terms in the action, typically there are additional solutions of the equations of motion even at the linearized level.

Example: Take a scalar field ψ with action:

$$\frac{1}{2}\int d^4x\,\psi\,\partial_\mu\partial^\mu\left(1 - M^{-2}\,\partial^\mu\partial_\mu\right)\,\psi\,.$$

The equations of motion for ψ has solutions:

$$\psi = Ae^{ik.x}$$

with

$$k^2 = 0 \qquad \text{or} \qquad k^2 = -M^2$$

Similarly, in the presence of higher derivative terms, the equations of motion of the string effective action will also have these additional oscillatory solutions even at the linearized level.

$\rightarrow$ responsible for the oscillations seen in our analysis.

Quantization of these additional solutions will give rise to additional states in the spectrum which are not present in the string spectrum.

Solution (Zwiebach):

We must try to remove these higher derivative terms by field redefinition.

In the scalar field example we take:

$$\widetilde{\psi} = \left(1 - M^{-2}\partial_\mu\partial^\mu\right)^{1/2}\psi$$

This gives the standard kinetic term for $\widetilde{\psi}$ and maps $\psi = A\, e^{ik.x}$ with $k^2 = -M^2$ to 0.

The generalization of this construction to gauge field, metric etc. will remove the higher derivative terms from the action at the quadratic level and map the additional oscillatory solutions to zero.

These new fields are the correct ones to be used in describing string theory.

Thus we see that once we use these right field variables, our solution should approach the correct asymptotic form at large r.

Presumably when we use the correct field variables, the second order differential equation for h will be replaced by an ordinary equation with unique solution.

Can we explicitly carry out this field redefinition and verify this explicitly?

This requires reformulating the supergravity action in terms of a new set of variables.

Generalization to other heterotic string compactifications.

Heterotic on $K_5 \times S^1$.

K_5: any manifold / orbifold, possibly with background gauge fields etc., that preserves at least $N = 2$ supersymmetry.

($N = 2$ supersymmetry is needed to get the BPS states.)

Consider a heterotic string wrapped on S^1 with winding number w and carrying n units of momentum along S^1.

In the limit of large nw, the statistical entropy associated with this state is still given by:

$$S_{stat} = 4\pi\sqrt{nw}$$

(This is controlled by the central charge of the conformal field theory describing the heterotic string compactification.)

$\rightarrow$ does not depend on the choice of K_5.

The classical solution describing this heterotic string involves background fields along S^1 and the non-compact directions.

The tree level effective field theory involving these fields is independent of the choice of K_5 to all orders in α'.

As a result the classical solution describing the black hole solution does not depend on the choice of K_5.

$\rightarrow$ we get the same entropy of the black hole:

$$S_{BH} = 4\pi\sqrt{nw}$$

$\rightarrow$ The agreement between S_{stat} and S_{BH} continues to hold.

AS

The full ten dimensional space factorizes as:

$$K_5 \times \mathcal{M}_5$$

with dilaton and other fields depending on the coordinates of $\mathcal{M}_5$.

Thus the conformal field theory describing string propagation in this background is a direct sum of two conformal field theories:

1. The one associated with K_5, and

2. The one associated with $\mathcal{M}_5$.

Note: The CFT associated with $\mathcal{M}_5$ is a universal CFT without any parameters.

A detailed analysis of this CFT is likely to generate new insight into the black holes that they describe.

Finite charge corrections:

One of the advantages of working with the elementary string states is that we know their degeneracy very precisely.

The degeneracy d_{nw} of BPS states carrying charge quantum numbers (n, w) is determined from the formula

$$\sum_{N=0}^{\infty} d_{N-1} q^N = 16 \prod_{n=1}^{\infty} (1 - q^n)^{-24}$$

For large nw this gives:

$$d_{nw} \sim \exp(4\pi\sqrt{nw})$$

However we can calculate the corrections to this formula.

$$S_{stat} = \ln(d_{nw}) = 4\pi\sqrt{nw} - \frac{27}{2}\ln(\sqrt{nw}) + \mathcal{O}(1)$$

Question: Can we reproduce these corrections by keeping track of non-leading contribution to S_{BH}?

Note: The field S is of order $\sqrt{nw}$ near the horizon.

$\rightarrow$ string coupling $\sim S^{-1/2} \sim (nw)^{-1/4}$ near the horizon.

$\rightarrow$ in the limit of large nw we can ignore the string loop corrections to the effective action and focus on the tree level contribution.

However this is no longer the case if we want to study the non-leading corrections to the entropy (in inverse powers of nw).

We need to take into account quantum corrections to the string effective action, and then repeat the whole analysis.

The procedure is difficult due to the presence of 'holomorphic anomaly', but the answer was guessed by Cardoso, de Wit, Mohaupt.

Up to non-perturbative corrections involving powers of $e^{-\pi\sqrt{nw}}$, S_{BH} is given by:

$$S_{BH} = \pi\frac{nw}{S_0} + 4\,\pi\,S_0 - 12\ln\left[2\,S_0\right]$$

where S_0 is the solution of the equation:

$$-\pi\frac{nw}{S_0^2} + 4\pi - \frac{12}{S_0} = 0$$

From this for large nw, we get

$$S_{BH} = 4\pi\sqrt{nw} - 12\ln\sqrt{nw} + \mathcal{O}(1)$$

$$S_{BH} = 4\pi\sqrt{nw} - 12\ln\sqrt{nw} + \mathcal{O}(1)$$

Compare with

$$S_{stat} = 4\pi\sqrt{nw} - \frac{27}{2}\ln(\sqrt{nw}) + \mathcal{O}(1)$$

The two expressions do not seem to agree.

However, while considering finite 'size' corrections, the definition of the statistical entropy depends crucially on the choice of the ensemble.

Even if the relation between S_{BH} and S_{stat} extends beyond the leading order, it can only hold for some particular choice of ensemble. (Ooguri, Strominger, Vafa; Dabholkar)

For example consider the following alternative definition of statistical entropy. AS

First define a 'free energy' through a kind of 'grand canonical' ensemble:

$$e^{\mathcal{F}(\mu)} = \sum_{N=0}^{\infty} d_{N-1}\, e^{-\mu(N-1)}$$

then define a statistical entropy through the relation:

$$\widetilde{S}_{stat} = \mathcal{F}(\mu) + \mu\, nw$$

where μ solves the equation:

$$\frac{\partial \mathcal{F}}{\partial \mu} = -nw$$

In the limit of large nw the entropy $S_{stat} \equiv \ln d_{nw}$ computed from the microcanonical ensemble agrees with $\widetilde{S}_{stat}$ computed from this 'grand canonical' ensemble.

But non-leading corrections to S_{stat} and $\widetilde{S}_{stat}$ differ.

It is not *a priori* clear which statistical entropy is to be compared with S_{BH}, but we shall go ahead and compare S_{BH} with $\widetilde{S}_{stat}$.

$$\begin{aligned} e^{-\mathcal{F}(\mu)} &= \sum_{N=0}^{\infty} d_{N-1}\, e^{-\mu(N-1)} \\ &= 16\, e^{\mu} \prod_{n=1}^{\infty} \left(1 - e^{-n\,\mu}\right)^{-24} \end{aligned}$$

For small μ we get

$$-\mathcal{F}(\mu) = \frac{4\pi^2}{\mu} + 12 \ln \frac{\mu}{2\pi} + \ln(16) + \mathcal{O}\left(e^{-\frac{4\pi^2}{\mu}}\right)$$

Taking a Legendre transform of $\mathcal{F}(\mu)$ we get

$$\widetilde{S}_{stat} = 4\pi\sqrt{nw} - 12 \ln \sqrt{nw} + \mathcal{O}(1)$$

Compare with

$$S_{BH} = 4\pi\sqrt{nw} - 12 \ln \sqrt{nw} + \mathcal{O}(1)$$

One can in fact do better and show that up to an additive constant of ln(16) and non-perturbative correction involving powers of $e^{-\pi\sqrt{nw}}$, S_{BH} and $\widetilde{S}_{stat}$ agree exactly.

This is done by comparing the equations determining $\widetilde{S}_{stat}$ and S_{BH} up to exponentially suppressed terms.

$$
\begin{aligned}
\widetilde{S}_{stat} &= -\mathcal{F}(\mu) + \mu\, nw \\
&= \frac{4\pi^2}{\mu} + 12 \ln \frac{\mu}{2\pi} + \mu\, nw
\end{aligned}
$$

with μ determined from the equation:

$$
\frac{\partial \mathcal{F}}{\partial \mu} = nw \quad \to \quad -\frac{4\pi^2}{\mu^2} + \frac{12}{\mu} + nw = 0
$$

Compare this with the equation determining S_{BH}:

$$
S_{BH} = \pi \frac{nw}{S_0} + 4\,\pi\, S_0 - 12 \ln [2\, S_0]
$$

where S_0 is the solution of the equation:

$$
-\pi \frac{nw}{S_0^2} + 4\pi - \frac{12}{S_0} = 0
$$

These two sets of equations are identical up to an additive constant of $\ln(16)$ in $\widetilde{S}_{stat}$ if we identify:

$$
\mu = \pi / S_0
$$

Is this a coincidence?

One way to check this would be to repeat the analysis for various other compactifications of the heterotic string theory on manifolds of type $K_5 \times S^1$ and check if $\widetilde{S}_{stat}$ agrees with S_{BH} in all cases.

Other Open problems:

1) Generalization to elementary string states in dimension > 4.

The generalization of the scaling argument exists (Peet)

2) Role of other higher derivative corrections

This analysis takes care of only part of the higher derivative corrections which come from supersymmetrizing the curvature square terms.

These terms are somewhat special in the sense that they come from holomorphic corrections to the generalized prepotential.

However since at $r \sim 1$ the curvature is of order 1, other higher derivative terms will also be important.

Is there some kind of non-renormalization theorem that tells us that only the holomorphic corrections affect the value of a?

3) Generalization to type II compactification

The scaling argument can be generalized to type II theory on $T^5 \times S^1$

$\longrightarrow$ the black hole entropy for fundamental string wrapped on S^1 with winding number w and n units of momentum has

$$S_{BH} = a' \sqrt{nw}$$

a' is some universal constant

On the other hand, counting of degeneracy of elementary string states give

$$S_{stat} = 2\sqrt{2}\,\pi \sqrt{nw}$$

Q. Can we calculate a' by the same method as in the case of heterotic string?

Unfortunately tree level type II theories have no curvature^2 corrections to the effective action.

Thus a computation similar to the one for heterotic string gives

$$a' = 0$$

Thus here if we want to reproduce the statistical entropy we must take into account other higher derivative corrections.

Q. What is the basic difference between heterotic and type II?

Most likely this method of computing black hole entropy gives some sort of ln(index) rather than ln(degeneracy).

This is not surprising in view of the fact that in our analysis we have taken into account only a very special class of terms (holomorphic) terms.

For heterotic string index may be of order degeneracy whereas for type II the index may vanish.

What exactly is the index that is being computed by our method?