

System under study: **heterotic** string theory compactified on $T^5 \times S^1$.

We consider a **BPS** string state carrying w units of **winding** and n units of **momentum** along S^1 .

The **degeneracy** of this states $\sim e^{4\pi\sqrt{nw}}$ for **large** n, w .

$$\rightarrow S_{stat} = \ln(\text{degeneracy}) \simeq 4\pi\sqrt{nw}$$

The goal is to see if we get the **same expression** for the **Bekenstein-Hawking** entropy by considering a BPS black hole of **same mass** and **charge**.

Analysis of the **BPS black hole** solution of the **low energy** effective field theory gives **zero area** of the event horizon.

However, analysis of the solution **near** the **horizon** shows that for **large** n and w the α' **corrections** to the supergravity action are **important** near the horizon although string **loop** corrections are **small**.

Using various **symmetries** of the tree level effective action one can show that the form of the black hole entropy after including α' **corrections** have the form

$$S_{BH} = a \sqrt{nw}$$

a is a **numerical** constant.

Question: Can we **compute** a and show that it is 4π ?

A brief **history** of **subsequent developments**

1. **Strominger and Vafa** computed the statistical entropy S_{stat} of **BPS** black holes in **five dimensions** carrying **three** different types of **charges** by describing them as configurations of **D-branes**.

The corresponding black hole **solutions** have **finite area** event horizon and hence **finite** Bekenstein-Hawking entropy S_{BH} even **without** taking into account α' **corrections**.

In the limit of **large charges**

$$S_{BH} = S_{stat}$$

2. This result was **generalized** to many other examples including black holes in **four** dimensional **heterotic** string theories carrying both **electric** and **magnetic** charges, in the **limit** where all **charges** are **large**.

3. For a **special** class of these four dimensional black holes, **Maldacena, Strominger, Witten** computed the **subleading** (in $1/\text{charges}$) corrections to S_{stat} .

4. For these black holes, **subleading** corrections to S_{BH} were computed by **Cardoso, de Wit, Mohaupt + Kapelli** by taking into account a **special** class of **higher derivative** terms in the **action**.

It was found that **including** these **subleading** corrections we get

$$S_{BH} = S_{stat}$$

In computing the subleading corrections to S_{BH} we had to take into account **modification** of the **Bekenstein-Hawking** formula due to **Wald**.

5. Given the expression for the entropy of the black hole as a function of electric and magnetic charges, we can now set the **magnetic** charges to **zero** to compute entropy of black holes carrying **electric** charges (n, w) .

In the **leading** approximation the answer **vanishes**.

However the **full** expression including the **sub-leading** corrections do **not** vanish.

Result for **heterotic** string **wound** on S^1 :

$$S_{BH} = 4\pi\sqrt{nw} \quad \rightarrow \quad a = 4\pi$$

→ Exact **agreement** with S_{stat} . **Dabholkar**

Instead of reviewing the detailed analysis we shall now give a brief outline of the **steps** which are involved in the **computation** of a .

(Lopes Cardoso, de Wit, Kappeli, Mohaupt; Dabholkar; AS; Hubeny, Maloney, Rangamani)

Tree level heterotic string **effective action** contains a term

$$\frac{1}{16\pi} \int d^4x \sqrt{-\det g} S R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

Supersymmetrization of this term gives many **other terms**.

These constitute a **special** class of higher derivative terms which are '**holomorphic**'.

The **analysis** leading to the **computation** of a takes into account **only** these higher derivative corrections to the effective action.

Given this **modified action** we proceed as follows in order to determine the **modified solution** describing the heterotic string configuration carrying charges (n, w) .

1. First we note that the modified solution describing the heterotic string configuration under study must satisfy the **modified field equations** derived from the new action.

2. We can also use the fact that we are trying to describe a **BPS** state that is **invariant** under a certain set of space-time **supersymmetry** transformations.

As a result the **field configuration** describing this state must also be **invariant** under these space-time **supersymmetry** transformations.

These give **constraints** on the **field** configurations.

3. **Boundary condition**: At **large distance** where **higher derivative** corrections are **negligible**, the solution must approach the **leading** order solution found earlier.

Now recall the general form of the 'hatted solution':

$$\begin{aligned}\hat{d}s_{string}^2 &= -\frac{f_1(r)}{f_3(r)} d\tau^2 + \frac{f_2(r)}{f_3(r)} (dr^2 + r^2 d\Omega_2^2) \\ \hat{S} &= f_3(r), \\ \hat{T} &= f_4(r), \\ \hat{F}_{r\tau}^{(1)} &= f_5(r), \\ \hat{F}_{r\tau}^{(2)} &= f_6(r).\end{aligned}$$

Substitute these into the field equations / BPS conditions.

This gives constraints on f_1, \dots, f_6 .

For **large** r the solution must **match** the solution of the **low energy** effective field theory found earlier.

This gives, for **large** r ,

$$f_1(r) \simeq \frac{r}{2}, \quad f_2(r) \simeq \frac{2}{r}$$

$$f_3(r) \simeq \frac{2}{r}, \quad f_4(r) \simeq 1$$

$$f_5(r) \simeq \frac{1}{4}, \quad f_6(r) \simeq \frac{1}{4}$$

Define $h(r) = \ln(f_1(r))$.

Then the constraints on f_1, \dots, f_6 may be expressed as:

$$\begin{aligned}f_1(r) &= e^{h(r)}, \\f_2(r) &= e^{-h(r)}, \\f_3(r) &= \frac{2}{r} \frac{1}{\sqrt{1 + 4 (h'(r))^2}}, \\f_4(r) &= \frac{1}{\sqrt{1 + 4 (h'(r))^2}}, \\f_5(r) &= \frac{1}{2} \partial_r \left(e^{h(r)} \sqrt{1 + 4 (h'(r))^2} \right), \\f_6(r) &= \frac{1}{2} \partial_r \left(e^{h(r)} \sqrt{1 + 4 (h'(r))^2} \right).\end{aligned}$$

h satisfies the differential equation:

$$\begin{aligned}& h' \left(1 + 4 (h')^2 \right) + r h'' \\&= \frac{r^2}{8} e^{-h} \left(1 + 4 (h')^2 \right)^{3/2} - \frac{r}{4} \left(1 + 4 (h')^2 \right).\end{aligned}$$

At large r the equation for h admits a solution:

$$h = \ln \frac{r}{2}$$

f_1, \dots, f_6 calculated from this gives us back the supergravity results.

For **small** r the equation for h **admits** a solution:

$$h = 2 \ln \frac{r}{2}$$

Thus $f_2(r) = e^{-h} = 4/r^2$

This gives the **naive entropy** associated with the **hatted** solution:

$$a = \frac{\pi}{2} \lim_{r \rightarrow 0} (r^2 f_2(r)) = 2\pi$$

→ **finite area** of the event horizon but wrong answer!

However due to **higher derivative** terms in the action the **Bekenstein-Hawking formula** itself gets **modified** (Wald)

Wald's formula for spherically symmetric black holes:

$$S_{BH} = 2\pi \int_H d\theta d\phi \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \sqrt{\det h}$$

H : the horizon

\mathcal{L} : Lagrangian density

$$\epsilon^{rt} = -\epsilon^{tr} = \sqrt{-g^{rr} g^{tt}}$$

$$\epsilon^{\mu\nu} = 0 \text{ otherwise}$$

$h_{\alpha\beta}$: Metric along the horizon

Using this formula we get:

$$a = 4\pi$$

in exact agreement with the statistical entropy.

However our **analysis** is **not complete** yet.

$h(r)$ satisfies a **second order** differential equation.

It **admits** a solution $h = \ln(r/2)$ for **large r** that gives the correct **asymptotic** behaviour.

It **admits** a solution $h = 2 \ln(r/2)$ at **small r** that gives the **correct entropy**.

However a **second order** differential equation has two **integration constants**.

Thus there is **no guarantee** *a priori* that a solution that has the **small r** behaviour $h = 2 \ln(r/2)$ will approach the asymptotic form $h = \ln(r/2)$ at **large r** .

Study **small fluctuations** about the solution $h = \ln(r/2)$ at large r .

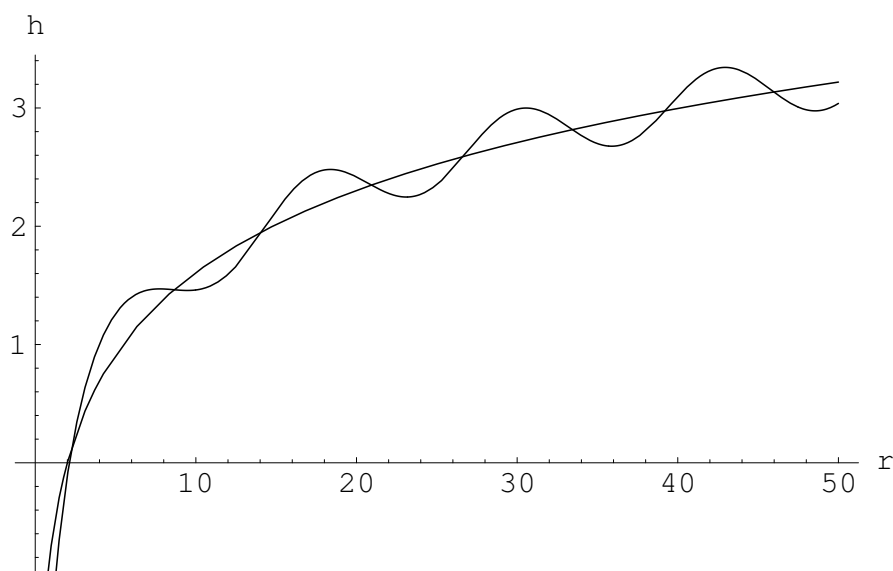
Result:

$$h \simeq \ln \frac{r}{2} + A \cos \left(\frac{r}{2} + B \right) + \mathcal{O}(A^2)$$

A, B : **integration constants**

Thus for a **generic** initial condition we expect the solution to **oscillate** about $h = \ln(r/2)$.

Numerical results show that this is exactly what happens. (**AS; Hubeny, Maloney, Rangamani**)



Interpretation of these oscillations:

In the presence of higher derivative terms in the action, typically there are additional solutions of the equations of motion even at the linearized level.

Example: Take a scalar field ψ with action:

$$\frac{1}{2} \int d^4x \psi \partial_\mu \partial^\mu (1 - M^{-2} \partial^\mu \partial_\mu) \psi.$$

The equations of motion for ψ has solutions:

$$\psi = A e^{ik \cdot x}$$

with

$$k^2 = 0 \quad \text{or} \quad k^2 = -M^2$$

Similarly, in the presence of higher derivative terms, the equations of motion of the string effective action will also have these additional oscillatory solutions even at the linearized level.

→ responsible for the oscillations seen in our analysis.

Quantization of these additional solutions will give rise to **additional states** in the **spectrum** which are **not present** in the **string** spectrum.

Solution (Zwiebach):

We must try to **remove** these **higher derivative** terms by **field redefinition**.

In the **scalar** field example we take:

$$\tilde{\psi} = \left(1 - M^{-2} \partial_{\mu} \partial^{\mu}\right)^{1/2} \psi$$

This gives the **standard** kinetic term for $\tilde{\psi}$ and maps $\psi = A e^{ik \cdot x}$ with $k^2 = -M^2$ to 0.

The **generalization** of this construction to **gauge field**, **metric** etc. will **remove** the **higher derivative** terms from the action at the **quadratic** level and **map** the additional **oscillatory** solutions to **zero**.

These **new fields** are the **correct** ones to be used in describing **string** theory.

Thus we see that once we use these **right field variables**, our solution should **approach** the **correct** asymptotic form at **large r** .

Presumably when we use the **correct** field variables, the **second order** differential equation for h will be replaced by an **ordinary equation** with **unique** solution.

Can we **explicitly** carry out this **field redefinition** and verify this explicitly?

This requires **reformulating** the **supergravity** action in terms of a **new** set of **variables**.

Generalization to other heterotic string compactifications.

Heterotic on $K_5 \times S^1$.

K_5 : any manifold / orbifold, possibly with background gauge fields etc., that preserves at least $N = 2$ supersymmetry.

($N = 2$ supersymmetry is needed to get the BPS states.)

Consider a heterotic string wrapped on S^1 with winding number w and carrying n units of momentum along S^1 .

In the limit of **large** nw , the statistical entropy associated with this state is still given by:

$$S_{stat} = 4\pi\sqrt{nw}$$

(This is controlled by the **central charge** of the conformal field theory describing the heterotic string compactification.)

→ does **not** depend on the choice of K_5 .

The **classical solution** describing this heterotic string involves background **fields** along S^1 and the **non-compact** directions.

The **tree level effective field theory** involving these fields is **independent** of the choice of K_5 to **all orders** in α' .

As a result the classical **solution** describing the black hole solution does **not** depend on the choice of K_5 .

→ we get the **same entropy** of the black hole:

$$S_{BH} = 4\pi\sqrt{nw}$$

→ The **agreement** between S_{stat} and S_{BH} continues to **hold**. **AS**

The full **ten** dimensional space **factorizes** as:

$$K_5 \times \mathcal{M}_5$$

with **dilaton** and other **fields** depending on the coordinates of \mathcal{M}_5 .

Thus the **conformal** field theory describing string propagation in this background is a **direct sum** of two conformal field theories:

1. The one associated with K_5 , and
2. The one associated with \mathcal{M}_5 .

Note: The **CFT** associated with \mathcal{M}_5 is a **universal** CFT without any parameters.

A detailed **analysis** of this **CFT** is likely to generate new **insight** into the black holes that they describe.

Finite charge corrections:

One of the **advantages** of working with the **elementary string** states is that we **know** their **degeneracy** very **precisely**.

The degeneracy d_{nw} of BPS states carrying charge quantum numbers (n, w) is determined from the formula

$$\sum_{N=0}^{\infty} d_{N-1} q^N = 16 \prod_{n=1}^{\infty} (1 - q^n)^{-24}$$

For **large** nw this gives:

$$d_{nw} \sim \exp(4\pi\sqrt{nw})$$

However we can calculate the **corrections** to this formula.

$$S_{stat} = \ln(d_{nw}) = 4\pi\sqrt{nw} - \frac{27}{2} \ln(\sqrt{nw}) + \mathcal{O}(1)$$

Question: Can we **reproduce** these corrections by keeping track of **non-leading** contribution to S_{BH} ?

Note: The field S is of order \sqrt{nw} near the horizon.

→ string coupling $\sim S^{-1/2} \sim (nw)^{-1/4}$ near the horizon.

→ in the limit of large nw we can ignore the string loop corrections to the effective action and focus on the tree level contribution.

However this is no longer the case if we want to study the non-leading corrections to the entropy (in inverse powers of nw).

We need to take into account quantum corrections to the string effective action, and then repeat the whole analysis.

The procedure is **difficult** due to the presence of '**holomorphic anomaly**', but the answer was **guessed** by Cardoso, de Wit, Mohaupt.

Up to **non-perturbative corrections** involving powers of $e^{-\pi\sqrt{nw}}$, S_{BH} is given by:

$$S_{BH} = \pi \frac{nw}{S_0} + 4\pi S_0 - 12 \ln [2 S_0]$$

where S_0 is the **solution** of the equation:

$$-\pi \frac{nw}{S_0^2} + 4\pi - \frac{12}{S_0} = 0$$

From this for **large** nw , we get

$$S_{BH} = 4\pi\sqrt{nw} - 12 \ln \sqrt{nw} + \mathcal{O}(1)$$

$$S_{BH} = 4\pi\sqrt{nw} - 12 \ln \sqrt{nw} + \mathcal{O}(1)$$

Compare with

$$S_{stat} = 4\pi\sqrt{nw} - \frac{27}{2} \ln(\sqrt{nw}) + \mathcal{O}(1)$$

The two expressions **do not** seem to **agree**.

However, while considering **finite 'size' corrections**, the **definition** of the **statistical entropy** depends crucially on the **choice** of the **ensemble**.

Even if the **relation** between S_{BH} and S_{stat} extends **beyond** the **leading order**, it can only hold for some **particular choice** of **ensemble**.
(Ooguri, Strominger, Vafa; Dabholkar)

For example consider the following **alternative** definition of **statistical entropy**. AS

First define a 'free energy' through a kind of 'grand canonical' ensemble:

$$e^{\mathcal{F}(\mu)} = \sum_{N=0}^{\infty} d_{N-1} e^{-\mu(N-1)}$$

then define a statistical entropy through the relation:

$$\tilde{S}_{stat} = \mathcal{F}(\mu) + \mu nw$$

where μ solves the equation:

$$\frac{\partial \mathcal{F}}{\partial \mu} = -nw$$

In the limit of large nw the entropy $S_{stat} \equiv \ln d_{nw}$ computed from the microcanonical ensemble agrees with \tilde{S}_{stat} computed from this 'grand canonical' ensemble.

But **non-leading** corrections to S_{stat} and \tilde{S}_{stat} differ.

It is not *a priori* clear which statistical entropy is to be compared with S_{BH} , but we shall go ahead and compare S_{BH} with \tilde{S}_{stat} .

$$\begin{aligned} e^{-\mathcal{F}(\mu)} &= \sum_{N=0}^{\infty} d_{N-1} e^{-\mu(N-1)} \\ &= 16 e^{\mu} \prod_{n=1}^{\infty} (1 - e^{-n\mu})^{-24} \end{aligned}$$

For small μ we get

$$-\mathcal{F}(\mu) = \frac{4\pi^2}{\mu} + 12 \ln \frac{\mu}{2\pi} + \ln(16) + \mathcal{O}\left(e^{-\frac{4\pi^2}{\mu}}\right)$$

Taking a Legendre transform of $\mathcal{F}(\mu)$ we get

$$\tilde{S}_{stat} = 4\pi \sqrt{nw} - 12 \ln \sqrt{nw} + \mathcal{O}(1)$$

Compare with

$$S_{BH} = 4\pi \sqrt{nw} - 12 \ln \sqrt{nw} + \mathcal{O}(1)$$

One can in fact **do better** and show that up to an **additive constant** of $\ln(16)$ and **non-perturbative** correction involving powers of $e^{-\pi\sqrt{nw}}$, S_{BH} and \tilde{S}_{stat} agree exactly.

This is done by **comparing** the **equations** determining \tilde{S}_{stat} and S_{BH} up to exponentially suppressed terms.

$$\begin{aligned}\tilde{S}_{stat} &= -\mathcal{F}(\mu) + \mu n w \\ &= \frac{4\pi^2}{\mu} + 12 \ln \frac{\mu}{2\pi} + \mu n w\end{aligned}$$

with μ determined from the equation:

$$\frac{\partial \mathcal{F}}{\partial \mu} = n w \quad \rightarrow \quad -\frac{4\pi^2}{\mu^2} + \frac{12}{\mu} + n w = 0$$

Compare this with the equation determining S_{BH} :

$$S_{BH} = \pi \frac{n w}{S_0} + 4\pi S_0 - 12 \ln [2 S_0]$$

where S_0 is the solution of the equation:

$$-\pi \frac{n w}{S_0^2} + 4\pi - \frac{12}{S_0} = 0$$

These two sets of equations are **identical** up to an **additive** constant of $\ln(16)$ in \tilde{S}_{stat} if we identify:

$$\mu = \pi / S_0$$

Is this a coincidence?

One way to check this would be to repeat the analysis for various other compactifications of the heterotic string theory on manifolds of type $K_5 \times S^1$ and check if \tilde{S}_{stat} agrees with S_{BH} in all cases.

Other Open problems:

1) **Generalization** to elementary string states in **dimension > 4** .

The generalization of the **scaling** argument **exists** (Peet)

2) Role of **other** higher derivative corrections

This analysis takes care of only **part** of the higher derivative **corrections** which come from supersymmetrizing the curvature square terms.

These terms are somewhat **special** in the sense that they come from **holomorphic** corrections to the generalized **prepotential**.

However since at $r \sim 1$ the **curvature** is of **order 1**, **other** higher derivative terms will also be **important**.

Is there some kind of **non-renormalization** theorem that tells us that only the **holomorphic** corrections **affect** the value of a ?

3) Generalization to **type II** compactification

The **scaling** argument can be generalized to **type II** theory on $T^5 \times S^1$

→ the black hole **entropy** for fundamental string **wrapped** on S^1 with **winding** number w and n units of **momentum** has

$$S_{BH} = a' \sqrt{nw}$$

a' is some **universal** constant

On the other hand, counting of **degeneracy** of elementary **string states** give

$$S_{stat} = 2\sqrt{2} \pi \sqrt{nw}$$

Q. Can we **calculate** a' by the same method as in the case of heterotic string?

Unfortunately tree level **type II** theories have **no curvature²** corrections to the effective action.

Thus a computation **similar** to the one for **heterotic** string gives

$$a' = 0$$

Thus here if we want to reproduce the statistical entropy we must take into account **other** higher derivative **corrections**.

Q. What is the basic **difference** between **heterotic** and **type II**?

Most likely this **method** of computing black hole **entropy** gives some sort of $\ln(\text{index})$ rather than $\ln(\text{degeneracy})$.

This is **not surprising** in view of the fact that in our analysis we have taken into account only a very **special** class of terms (**holomorphic**) terms.

For **heterotic** string **index** may be of **order degeneracy** whereas for **type II** the **index** may **vanish**.

What exactly is the **index** that is being **computed** by our **method**?