System under study: heterotic string theory compactified on $T^5 \times S^1$.

We consider a BPS string state carrying w units of winding and n units of momentum along S^1.

The degeneracy of this states $\sim e^{4\pi\sqrt{nw}}$ for large n, w.

$\rightarrow S_{stat} = \ln(\text{degeneracy}) \simeq 4\pi\sqrt{nw}$

The goal is to see if we get the same expression for the Bekenstein-Hawking entropy by considering a BPS black hole of same mass and charge.

Analysis of the BPS black hole solution of the low energy effective field theory gives zero area of the event horizon.

However, analysis of the solution near the horizon shows that for large n and w the α' corrections to the supergravity action are important near the horizon although string loop corrections are small.

After taking into account a class of higher derivative terms in the action which represent holomorphic correction to the prepotential, we found that the modified black hole entropy is given by:

$$S_{BH} = 4\pi\sqrt{nw}$$

$\rightarrow$ matches the statistical entropy.

However, even within this approximation we encountered a subtlety during our analysis.

After appropriate symmetry transformation the black solution under study can be brought to the general form:

$$
\begin{aligned}
\widehat{ds}^2_{string} &= -\frac{f_1(r)}{f_3(r)}\,d\tau^2 + \frac{f_2(r)}{f_3(r)}\,(dr^2 + r^2 d\Omega_2^2) \\
\widehat{S} &= f_3(r)\,, \\
\widehat{T} &= f_4(r)\,, \\
\widehat{F}^{(1)}_{r\tau} &= f_5(r)\,, \\
\widehat{F}^{(2)}_{r\tau} &= f_6(r)\,.
\end{aligned}
$$

$f_1, \ldots f_6$ are universal functions without any parameter.

Field equations, BPS conditions and requirement of correct asymptotic behaviour gives constraints on the functions $f_1, \ldots f_6$.

These constraints on $f_1, \ldots f_6$ may be summarized as:

$$
\begin{aligned}
f_1(r) &= e^{h(r)}\,, \\
f_2(r) &= e^{-h(r)}\,, \\
f_3(r) &= \frac{2}{r}\,\frac{1}{\sqrt{1+4\,(h'(r))^2}}\,, \\
f_4(r) &= \frac{1}{\sqrt{1+4\,(h'(r))^2}}\,, \\
f_5(r) &= \frac{1}{2}\,\partial_r\left(e^{h(r)}\,\sqrt{1+4\,(h'(r))^2}\right)\,, \\
f_6(r) &= \frac{1}{2}\,\partial_r\left(e^{h(r)}\,\sqrt{1+4\,(h'(r))^2}\right)\,.
\end{aligned}
$$

h satisfies the differential equation:

$$
\begin{aligned}
& h'\left(1+4\,(h')^2\right)+r\,h'' \\
= \; & \frac{r^2}{8}\,e^{-h}\left(1+4\,(h')^2\right)^{3/2}-\frac{r}{4}\left(1+4\,(h')^2\right)\,.
\end{aligned}
$$

At large r the equation for h admits a solution:

$$h = \ln \frac{r}{2}$$

$f_1, \ldots f_6$ calculated from this gives us back the supergravity results.

For small r the equation for h admits a solution:

$$h = 2 \ln \frac{r}{2}$$

This gives the correct formula for S_{BH}.

However, $h(r)$ satisfies a second order differential equation.

A second order differential equation has two integration constants.

Thus there is no guarantee *a priori* that a solution that has the small r behaviour $h = 2 \ln(r/2)$ will approach the asymptotic form $h = \ln(r/2)$ at large r.

Study small fluctuations about the solution $h = \ln(r/2)$ at large r.

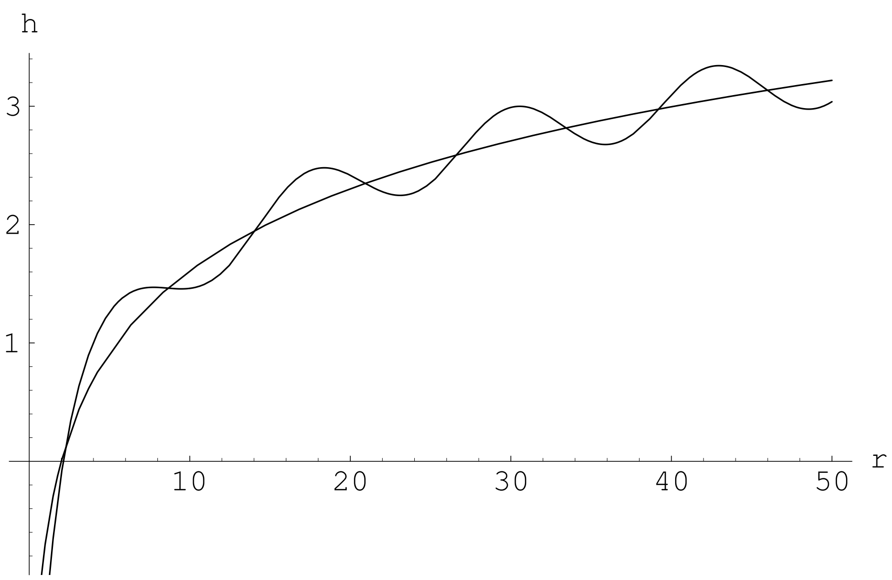
Result:

$$h \simeq \ln\frac{r}{2} + A\cos\left(\frac{r}{2} + B\right) + \mathcal{O}(A^2)$$

A, B: integration constants

Thus for a generic initial condition we expect the solution to oscillate about $h = \ln(r/2)$.

Numerical results show that this is exactly what happens. (AS; Hubeny, Maloney, Rangamani)



In order to interprete this result we need to analyse the origin of the oscillatory solutions around $h = \ln(r/2)$ for large r.

The f_i's computed from $h = \ln(r/2)$ represent a flat (locally) background for large r.

(All field strengths fall off to zero as $r \to \infty$.)

Thus for small A

$$h \simeq \ln\frac{r}{2} + A\cos\left(\frac{r}{2} + B\right)$$

represents solution of the linearized equations of motion for various fields around flat background.

The r dependence of the oscillatory part indicates as if we have fields of $\text{mass}^2 = -\frac{1}{4}$.

How is this possible?

Origin of the negative mass^2 modes:

In the presence of higher derivative terms in the action, typically there are additional solutions of the equations of motion even at the linearized level.

Example: Take a scalar field ψ with action:

$$\frac{1}{2}\int d^4x\, \psi\, \partial_\mu\partial^\mu \left(1 - M^{-2}\, \partial^\mu\partial_\mu\right) \psi\,.$$

The equations of motion for ψ has solutions:

$$\psi = Ae^{ik.x}$$

with

$$k^2 = 0 \qquad \text{or} \qquad k^2 = -M^2$$

Similarly, in the presence of higher derivative terms, the equations of motion of the string effective action will also have these additional oscillatory solutions even at the linearized level.

$\to$ responsible for the oscillations seen in our analysis.

Quantization of these additional solutions will give rise to additional states in the spectrum which are not present in the string spectrum.

Solution (Zwiebach):

We must try to remove these higher derivative terms by field redefinition.

In the scalar field example we take:

$$\widetilde{\psi} = \left(1 - M^{-2}\partial_\mu\partial^\mu\right)^{1/2}\psi$$

This gives the standard kinetic term for $\widetilde{\psi}$.

This maps $\psi = A\, e^{ik.x}$ with $k^2 = -M^2$ to

$$\widetilde{\psi} = 0$$

The generalization of this construction to gauge field, metric etc. will remove the higher derivative terms from the action at the quadratic level and map the additional oscillatory solutions to zero.

For example, for the metric, this will require defining a new metric

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + aR_{\mu\nu} + b\,R\,g_{\mu\nu} + \ldots$$

The coefficients a, b, ... have to be chosen appropriately to remove higher derivative terms from the quadratic term in the action.

These new fields are the correct ones to be used in describing string theory.

Once we use these right field variables, the oscillatory part of the solution will get mapped to zero.

As a result our solution should approach the correct asymptotic form at large r.

Can we explicitly carry out this field redefinition and verify this?

This requires reformulating the supergravity action in terms of a new set of variables.

This has not been done yet.

Presumably when we use the correct field variables, the second order differential equation for h will be replaced by an ordinary equation with unique solution.

Generalization to other heterotic string compactifications.

Heterotic on $K_5 \times S^1$.

K_5: any manifold / orbifold, possibly with background gauge fields etc., that preserves at least $N = 2$ supersymmetry.

($N = 2$ supersymmetry is needed to get the BPS states.)

Consider a heterotic string wrapped on S^1 with winding number w and carrying n units of momentum along S^1.

In the limit of large nw, the statistical entropy associated with this state is still given by:

$$S_{stat} \simeq 4\pi\sqrt{nw}$$

(This is controlled by the central charge of the conformal field theory describing the heterotic string compactification.)

$\longrightarrow$ does not depend on the choice of K_5.

The classical solution describing this heterotic string involves background fields along S^1 and the non-compact directions.

The tree level effective field theory involving these fields is independent of the choice of K_5 to all orders in α'.

As a result the classical solution describing the black hole solution does not depend on the choice of K_5.

$\rightarrow$ we get the same entropy of the black hole:

$$S_{BH} = 4\pi\sqrt{nw}$$

$\rightarrow$ The agreement between S_{stat} and S_{BH} continues to hold.

AS

Some open problems

1. We have seen that after appropriate symmetry transformations, the near horizon limit of the classical black solution representing an elementary string is independent of any external parameter or the choice of compactification.

Thus string propagation in this background is described by a universal conformal field theory.

A detailed analysis of this CFT is likely to generate new insight into the black holes that they describe.

2. One could try to carry out a similar analysis for heterotic string compactified on $T^n \times S^1$ for other values of n.

This requires studying entropy of higher dimensional black holes.

The argument showing that the S_{BH} has the form $a\sqrt{nw}$ can be generalized to higher dimensions. (Peet)

Can we compute a by taking into account the higher derivative corrections?

This might be possible if we can find supersymmetrization of the curvature^2 term in (9+1) dimensions.

We could then compactify this theory on T^n and study black hole solutions describing elementary string states.

3. The analysis described here takes care of only part of the higher derivative corrections which come from supersymmetrizing the curvature square terms.

These terms are somewhat special in the sense that they come from holomorphic corrections to the generalized prepotential.

However since at $r \sim 1$ the curvature is of order 1, other higher derivative terms will also be important.

Is there some kind of non-renormalization theorem that tells us that only the holomorphic corrections affect the value of a?

4. Generalization to type II compactification

The scaling argument can be generalized to type II theory on $T^5 \times S^1$

$\longrightarrow$ the black hole entropy for fundamental string wrapped on S^1 with winding number w and n units of momentum has

$$S_{BH} = a' \sqrt{nw}$$

a' is some universal constant

On the other hand, counting of degeneracy of elementary string states give

$$S_{stat} = 2\sqrt{2}\,\pi\,\sqrt{nw}$$

Q. Can we calculate a' by the same method as in the case of heterotic string?

Unfortunately tree level type II theories have no curvature^2 corrections to the effective action.

Thus a computation similar to the one for heterotic string gives

$$a' = 0$$

Thus here if we want to reproduce the statistical entropy we must take into account other higher derivative corrections.

Q. What is the basic difference between heterotic and type II?

Most likely this method of computing black hole entropy gives some sort of ln(index) rather than ln(degeneracy).

This is not surprising in view of the fact that in our analysis we have taken into account only a very special class of terms (holomorphic) terms.

For heterotic string index may be of order degeneracy whereas for type II the index may vanish.

What exactly is the index that is being computed by our method?

5. Finite charge corrections:

One of the advantages of working with the elementary string states is that we know their degeneracy very precisely.

The degeneracy d_{nw} of BPS states carrying charge quantum numbers (n, w) is determined from the formula

$$\sum_{N=0}^{\infty} d_{N-1} q^N = 16 \prod_{n=1}^{\infty} (1 - q^n)^{-24}$$

For large nw this gives:

$$d_{nw} \sim \exp(4\pi\sqrt{nw})$$

However we can calculate the corrections to this formula.

$$S_{stat} = \ln(d_{nw}) = 4\pi\sqrt{nw} - \frac{27}{2}\ln(\sqrt{nw}) + \mathcal{O}(1)$$

Question: Can we reproduce these corrections by keeping track of non-leading contribution to S_{BH}?

Note: The field S is of order $\sqrt{nw}$ near the horizon.

$\longrightarrow$ string coupling $\sim S^{-1/2} \sim (nw)^{-1/4}$ near the horizon.

$\longrightarrow$ in the limit of large nw we can ignore the string loop corrections to the effective action and focus on the tree level contribution.

However this is no longer the case if we want to study the non-leading corrections to the entropy (in inverse powers of nw).

We need to take into account quantum corrections to the string effective action, and then repeat the whole analysis.

There is however an ambiguity in carrying out this comparison.

The definition of various thermodynamic quantities is idependent of the ensemble we use for large charges, but depends on the ensemble when we considers non-leading corrections.

It is not *a priori* clear which definition of statistical entropy we should compare with S_{BH}. (Ooguri, Strominger, Vafa)

For example consider the following alternative definition of statistical entropy. AS

First define a 'free energy' through a kind of 'grand canonical' ensemble:

$$e^{\mathcal{F}(\mu)} = \sum_{N=0}^{\infty} d_{N-1}\, e^{-\mu(N-1)}$$

then define a statistical entropy through the relation:

$$\widetilde{S}_{stat} = \mathcal{F}(\mu) + \mu\, nw$$

where μ solves the equation:

$$\frac{\partial \mathcal{F}}{\partial \mu} = -nw$$

In the limit of large nw the entropy $S_{stat} \equiv \ln d_{nw}$ computed from the microcanonical ensemble agrees with $\widetilde{S}_{stat}$ computed from this 'grand canonical' ensemble.

But non-leading corrections to S_{stat} and $\widetilde{S}_{stat}$ differ.

It turns out that in the particular example of heterotic string compactification on torus, $\widetilde{S}_{stat}$ is the quantity that agrees with S_{BH}.

What needs to be done is to do similar calculations in many other examples and come up with a hypothesis.