Three Lectures in Esfahan

G W Gibbons

DAMTP, University of Cambridge

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The lectures will cover three topics I have worked on recently, hopefully of relevance to M/Sring theory. The topics are

- Measures on Initial conditions and coupling constants

- The Emergence of Time and the Complex Numbers

- Deformations of $Sim(n-2)$ and metrics with holonomy $Sim(n-2)$

A priori probability distributions , “Priors” are integral part of the scientific method.

They are needed to to assess the probability that a hypothesis is true given a set of observations or measurements.

They are also required to assess the reliablity and information content of predictions since if a large measure of hypotheses give the same observations, then those observations don’t tell us much.

Conversely if only small measure of hypotheses predict a set of well verified observations, one may have high confidence in those hypotheses.

However much confusion can result if different people use different priors, or more misleadingly, don't clearly articulate the that they priors they are using.

This problem is particularly prevalent in cosmology where there is at present no consensus on a suitable *a priori* measure on initial conditions.

In choosing *a priori* probabilities, it seems safest to assume as little as possible, consistent with making any progress. Indeed progress consists of using the results of observations and experiments to refine and update what are initially very flat *a priori* distributions so that they peak at some approximation for the real world.

In M/String theory all of physics is subsumed into history. Thus all coupling constants, and low energy laws of physics are determined by initial conditions. Issues of naturalness reduce to the naturalness of initial conditions.

Thus if M/String theory is to qualify as a science it requires convincing *a priori* measures.

In other words, it is necessary to examine with more care than has always been done, words like probability, typicality, likelihod, fine-tuning etc. Currenty there appears to be little agreement about what these words actually mean and how they should be used. Very much the same holds for similar notions in elementary particle physics and String theory.

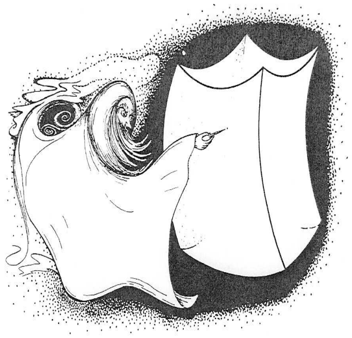

FIGURE 10. The Creator locating the tiny region of phase-space—one part in $10^{10^{123}}$—needed to produce a 10^{80}-baryon closed universe with a second law of thermodynamics in the form we know it.

An important consistency principle in this regard is due

to Humpty Dumpty

‘There’s glory for you!’

‘I don’t know what you mean by ”glory” ’, Alice said.

Humpty Dumpty smiled contemptuously. ‘Of course you don’t – till I tell you. I meant ”there’s a nice knock-down argument for you!”’

‘But ”glory” doesn’t mean ”a nice knock-down argument”,’Alice objected.

‘When I use a word,’Humpty Dumpty said, in rather a scornful tone, ‘it means just what I choose it to mean – neither more nor less.’

Take for example the words

Multiverse

and

Meta-Universe

They are frequently used in different ways sometimes by the same person on the same page of their book. An elision reminiscent of Beethoven at his finest.

Following Humpty Dumpty's lead, I will define **The Multiverse**

as the abstract and timeless set of all possible universes, i.e of all connected spacetimes satisfying the Einstein Equations.

This space, $M_{\text{Multiverse}}$ at least in mini-superspace examples, is even dimensional $\dim M_{\text{Multiverse}} = 2n - 2$ and carries, by virtue of it being a reduced phase space, a natural symplectic structure, i.e a closed 2-form ω and hence measure

$$\frac{1}{(n-1)!}(-1)^{\frac{1}{2}(n-1)(n-2)}\omega^{n-1} \tag{1}$$

This is the measure originally advocated by myself , Stephen Hawking and John Stewart and revisited recently by Neil Turok and myself *

*The Measure Problem in Cosmology. G.W. Gibbons, Neil Turok *Phys Rev* in press. .e-Print: hep-th/0609095

The main difficulty is that the total measure of $M_{\textsf{Multiverse}}$ diverges

$$\int_{M_{\textsf{Multiverse}}} \frac{1}{(n-1)!}(-1)^{\frac{1}{2}(n-1)(n-2)}\omega^{n-1} = \infty \qquad (2)$$

and Probabilities cannot be normalised. Neil and I have made a suggestion for solving this problem. If one accepts our suggestion, then the set of classical histories which inflate is exponentionally small: $\propto e^{-3N}$.

I will turn later to a possible interpretation of this statement.

The Meta-Universe

As introduced by Alex Vilenkin *

> The world view suggested by quantum cosmology is that inflating universes with all possible values we are a "typical" civilization living in this metauniverse

this is a single connected 4-dimensional spacetime $M_{\text{Meta-Universe}}$ possibly containing many causally disjoint regions. Points of $M_{\text{Meta-Universe}}$ are called spacetime events and the probability of a set $U \subset M_{\text{Meta-Universe}}$

*Predictions from Quantum Cosmology Alexander Vilenkin *Phys Rev Lett* **74** (1995) 846 of such events, a pocket universe or a causal diamond is taken to be proportional to their spacetime volume [†]

$$\int_U \sqrt{g|}d^4 x \tag{3}$$

Again the main problem is one of normalizability,

$$\int_{M_{\mathrm{Meta-Universe}}} \sqrt{g|}d^4 x = \infty\,. \tag{4}$$

As Wittgenstein might have said [‡] Wovon man nicht rechnen kann, darüber muß man schweigen

[†]The Geometry of Large Causal Diamonds and the No Hair Property of Asymptotically de-Sitter Spacetimes. G.W. Gibbons, S.N. Solodukhin *Phys.Lett* **B 652** :103-110(2007). arXiv:0706.0603 [hep-th] ; The Geometry of small causal diamonds. G.W. Gibbons, S.N. Solodukhin *Phys.Lett***B** 649:317-324,2007

[‡]L Wittgentstein, *Tractatus Logico-Philosophicus*

Why should one want a probablity theory for cosmology in the first place?

One motivation is to apply Bayesian Reasoning to Cosmological Observations.

That is the viewpoint I will take from now on.

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One encouraging feature of String theory is that the total volume of the various spaces of the moduli , which determine coupling constants etc with respect to natural metrics and measures are are of finite volume.

Thus in principle, the concept of fine tuning is meaningful in String theory.

More formally if

$P(O|U)$ is the probability of making an observation O in universe U, (Likelihood)

$P(U|O)$ is the probability we are in universe U having made an observation O (*A posteriori probability*)

$P(O)$ is the probability of making an observation O in *any* universe

$P(U)$ is the probability that the universe is U (*A priori probability*)

$P(O \cap U)$ is the probability of making an observation O and the universe is actually U

Then ....

$$P(U|O)P(O) = P(O \cap U) = P(O|U)P(U) \tag{5}$$

whence the Cosmic Bayes's Theorem tells us that

$$P(U|O) = \frac{P(O|U)P(U)}{\int_M P(O|U)P(U)dU} \tag{6}$$

where

the integral is over the Multiverse and dU is a measure on the multi-verse.

Thomas Bayes

quodque ſolum, certa nitri ſigna præbere, ſed plura concurrere debere, ut de vero nitro producto dubium non relinquatur.

LII. *An Eſſay towards ſolving a Problem in the Doctrine of Chances. By the late Rev. Mr.* Bayes, *F. R. S. communicated by Mr.* Price, *in a Letter to* John Canton, *A. M. F. R. S.*

Dear Sir,

Read Dec. 23, 1763.

I Now ſend you an eſſay which I have found among the papers of our deceaſed friend Mr. Bayes, and which, in my opinion, has great merit, and well deſerves to be preſerved. Experimental philoſophy, you will find, is nearly intereſted in the ſubject of it; and on this account there ſeems to be particular reaſon for thinking that a communication of it to the Royal Society cannot be improper.

He had, you know, the honour of being a member of that illuſtrious Society, and was much eſteemed by many in it as a very able mathematician. In an introduction which he has writ to this Eſſay, he ſays, that his deſign at firſt in thinking on the ſubject of it was, to find out a method by which we might judge concerning the probability that an event has to happen, in given circumſtances, upon ſuppoſition that we know nothing concerning it but that, under the ſame circum-

Philosophical Transactions of the Royal Society **53** (1763) 269-271

If we adopt Laplaces's Principle of Indifference then $P(U)$ is independent of U.

Any other choice of $P(U)$ is a proposal for the state of the the universe

We could even define Kolmogorov's measure of the information content of a proposal $P(U)$ via

$$\int P(U) \ln P(U) dU \tag{7}$$

which should be least for Laplace's Proposal

But what is dU?

Within the limitations of mini-superspace models, and modulo the issue of normaliabilty we have a complete solution. We reduce the model to a Hamiltonian system constrained to have vanishing Hamiltonian. It then follows that the Poincaré invariant $d^{n-1}pd^{n-1}q$ restricted to the multiverse of classical histories dose indeeed depend only on the history and not how it is described.

It is important to realise that the measure on the multiverse so defined carries no information about the direction of time nor any preferred instant of time. Many discussions of the plausibility or otherwise of certain initial conditions make explicit, or more dangerously implicit, assumptions about either or both.

In fact the method works not only for gravity plus scalar fields but one may also consider the addition of fluids as well.

The Einstein and matter equations provide a a Hamiltonian flow in a $2n$-dimensional phase space P, equipped with a symplectic form ω which may be written in local Darboux coordinates as

$$\omega = dp_i \wedge dq^i \,. \tag{8}$$

This gives the Liouville volume element on P

$$\frac{(-1)^n}{n!}\omega^n = d^n p d^n q \,. \tag{9}$$

However we need a measure on the space M of dynamical trajectories.

The Hamiltonian $\mathcal{H}$ is constrained to vanish and so the flow lines lie on a $2n-1$ dimensional Constraint submanifold

$$C = \mathcal{H}^{-1}(0) \,. \tag{10}$$

This is odd-dimensional but to take the so-called Symplectic or Marsden-Weinstein quotient , sometimes called the reduced phase space,

$$M = C/R = \mathcal{H}^{-1}(0)/R, \tag{11}$$

is straight forward, and moreover the quotient M , i.e *the space of classical histories satisfying the equations of motion or equivalently the space of physically distinct classical initial conditions* inherits a symplectic form. Thus provides a 'natural' measure on space of initial conditions as suggested by Gibbons Hawking and Stewart *.

*The use of the Liouvile measure in a different context had earlier been suggested by Henneau

The word ‘natural’ is being used here in the sense that the construction of the measure requires no more additional elements other than are present already in the equations of motion *. It contains no arbitrary cut-offs †. Moreover, by its construction, the measure is invariant under any additional canonical symmetries of the system.

The quotient M is that space of physically distinct initial conditions, in other words two sets of Cauchy data describing the same space-time, but taken at different times are identified. Thus $\{M, \omega\}$ may be thought of as the multiverse.

*Actually in our example $P = T^{\star}(Q)$

†Actually we will need an IR cutoff

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This initially seeming abstract construction may be made more intuitive using the concept of flux.

In general the symplectic form is a covariant second rank anti-symmetric tensor field with components in an arbitrary coordinate system

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

where the indices μ, ν take $2n$ values. The symplectic form is closed,

$$d\omega = 0\,.$$

This is equivalent to

$$\partial_{[\mu}\omega_{\nu\tau]} = 0.$$

Moreover if $\mathcal{H}$ is the Hamiltonian, then Hamilton's equations are

$$V^{\mu} = \omega^{\mu\nu}\partial_{\mu}\mathcal{H}$$

where $\omega^{\mu\nu}$ is the inverse of $\omega_{\mu\nu}$ and $V^{\mu} = \frac{dx^{\mu}}{dt}$ is tangent to the flow.

so we may re-write this as

$$\omega_{\mu\nu}V^{\nu} = \partial_{\mu}\mathcal{H}$$

It implies that

$$V^{\mu}\partial_{\mu}\mathcal{H} = 0$$

This means that V^{μ} lies in the level sets of the Hamilitonian $\mathcal{H} =$ constant .

Now let us choose coordinates such that

$$x^{2n} = \mathcal{H}.$$

The closure condition, restricted to spatial indices is

$$\partial_{[i}\omega_{jk]} = 0.$$

we have

$$V^{2n} = 0,$$

Thus $V^i\omega_{ij} = 0$.

It is simplest to see what this means in the example $n = 2$, for which $i = 1, 2, 3$.

Define a “ magnetic field “ by

$$B_i = \epsilon_{ijk}\omega_{ij}$$

then $\mathbf{B}$ is divergence free

$$\partial_i B_i = 0.$$

Morover

$$\epsilon_{ijk} B_j V_k = 0.$$

thus $\mathbf{B} \times \mathbf{V} = 0$,

or $\mathbf{V}$ is parallel to $\mathbf{B}$.

Now consider a bunch of trajectories , i.e a subset of the multiverse M which lie in the Constraint manifold $\mathcal{H} = 0$. Cut them with a transverse Cauchy surface Σ. The flux of the magnetic field through Σ

$$\int_{\Sigma} \mathbf{B}.d\Sigma \tag{12}$$

counts the number of universes in the bunch and is independent of which surface Σ we use, provided only that Σ intersects each trajectory once and only once

This is the measure advocated by Gibbons, Hawking and Stuart.

The argument in arbitrary dimension goes as follows.

Let

$$B_i = \epsilon_{ipqrs\ldots tu} \omega_{pq} \omega_{rs} \ldots \omega_{tu} \tag{13}$$

Then

$$\partial_i B_i = 0. \tag{14}$$

Thus

$$\omega_{[pq} \omega_{rs} \ldots \omega_{tu]} \propto \epsilon_{ipqrs\ldots tu} B_i \tag{15}$$

and so

$$V_p \epsilon_{ipqrs\ldots tu} B_i = 0, \tag{16}$$

which implies that

$$V_{[i}B_{j]} = 0, \tag{17}$$

that is, B_i is parallel to V_i.

Consider a single minimally-coupled scalar field ϕ with potential $V(\phi)$ in a homogeneous and isotropic (FRW) Universe with line element

$$-N^2 dt^2 + a^2(t)\gamma_{ij}dx^i dx^j, \tag{18}$$

where γ_{ij} is a metric on a space of constant (three-dimensional) scalar curvature $k = 0$ or ± 1.In units in which $(8\pi G) = 1$, the Einstein-scalar action is

$$\mathcal{S} = \int dtN \left(-3a(N^{-2}a'^2 - k) + \frac{1}{2}a^3 N^{-2}\phi'^2 - a^3 V(\phi) \right), \tag{19}$$

where primes denote t derivatives.

Varying the action with respect to the lapse function N yields the usual Friedmann equation

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{k}{a^2}, \tag{20}$$

where dots denote proper time derivatives, with $d\tau = Ndt$, and $H = \dot{a}/a$ is the expansion rate or Hubble parameter. Varying with respect to ϕ yields the scalar field equation

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi}. \tag{21}$$

Taking the time derivative of (20) and using (21) then yields

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 + \frac{k}{a^2}. \tag{22}$$

Finally, varying with respect to a yields a linear combination of (20) and (22).

Equation (22) will be of particular interest to our later discussion. For $k \leq 0$, the Hubble parameter H never increases, so no classical trajectory can cross a H=constant hypersurface more than once.

The canonical momenta conjugate to a, ϕ and N are

$$p_a = -6a\dot{a} = -6a^2 H, \qquad p_\phi = a^3\dot{\phi}, \qquad p_N = 0, \tag{23}$$

and the Hamiltonian is

$$\mathcal{H} = N\left(-\frac{p_a^2}{12a} + \frac{1}{2}\frac{p_\phi^2}{a^3} + a^3 V(\phi) - 3ak\right), \tag{24}$$

which vanishes by the equation of motion for p_N. We can use this to eliminate one of the four canonical variables a, p_a, ϕ, p_ϕ. In view of the monotonically decreasing property of H, mentioned earlier, and because $V(\phi)$ is, in general, complicated, we choose to eliminate p_ϕ, obtaining

$$p_\phi = \pm\sqrt{-\frac{1}{6}p_a^2 a^2 - 2a^6 V(\phi) + 6a^4 k}. \tag{25}$$

If $a = e^{\lambda}$, routine calculations give

$$V^i = (\dot{\phi}, \dot{H}, \dot{\lambda}) = (\pm\sqrt{6H^2 - 2V + \frac{6k}{a^2}}, V - 3H^2 - \frac{2k}{a^2}, H). \tag{26}$$

$$\omega = e^{3\lambda}(-6d\lambda \wedge dH \pm 3\frac{6H^2 - 2V + 4ke^{-2\lambda}}{\sqrt{6H^2 - 2V + 6ke^{-2\lambda}}} d\phi \wedge d\lambda \tag{27}$$

$$\pm \frac{6H}{\sqrt{6H^2 - 2V + 6ke^{-2\lambda}}} d\phi \wedge dH)\,. \tag{28}$$

$$\mathbf{B} = \frac{\pm 3e^{3\lambda}}{\sqrt{6H^2 - 2V + 6ke^{-2\lambda}}}\mathbf{V} \tag{29}$$

Note that we are not using Darboux-coordinates, but a physically more convenient choice.

As before, the equations of motion imply that

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 + \frac{k}{a^2} \tag{30}$$

Thus if $k \leq 0$, a good Cauchy surface is $H =$ constant, The measure is

$$\int 3e^{3\lambda} \frac{6H^2 - 2V + 4ke^{-2\lambda}}{\sqrt{6H^2 - 2V + 6ke^{-2\lambda}}} d\phi \wedge d\lambda \tag{31}$$

The range of ϕ is usually finite but the measure clearly diverges at large λ , i.e. large scale factors. This is essentially the problem pointed out by Hawking and Page.

However, an important point is that if $k = 0$, then a or equivalently λ, is neither geometrically meaningful nor physically measurable.

This suggest fixing λ and integrating only over ϕ. Later will do something better. but for the time being let's pursue this idea in the simplest $k = 0$ case.

If $k=0$, the equations of motion become

$$\dot{\phi}=\sqrt{2}\sqrt{3H^2-V}\,, \qquad \dot{H}=V-3H^2\,, \tag{32}$$

$$\dot{\lambda}=H\,. \tag{33}$$

The $\phi-H$ motion decouples and one obtains an autonomous system

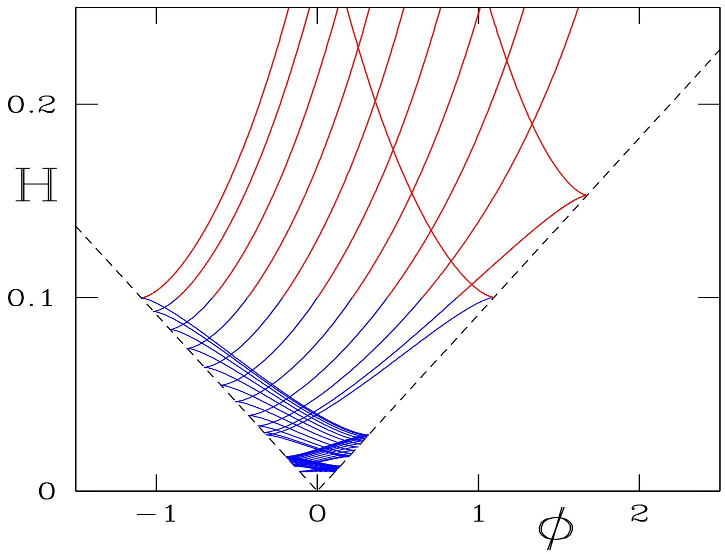
$$\sqrt{2}\frac{dH}{d\phi}=\pm\sqrt{3H^2-V}\,. \tag{34}$$

If $a=e^{\lambda}=e^{-N}$, where N is the number of e-folds then

$$\sqrt{2}\frac{dN}{d\phi}=\frac{H}{\sqrt{3H^2-V}}\,. \tag{35}$$

Slow roll is

$$H\approx\sqrt{\frac{V}{3}}(1+\frac{1}{12}(\frac{V'}{V})^2) \tag{36}$$



The set of trajectories in the (ϕ, H) plane for $k = 0$, and $V = \frac{1}{2}m^2\phi^2$ with $m^2 = .05$ in reduced Planck units. The measure surface is taken at $H = 0.1$, and the trajectories plotted are equally spaced in ϕ on that surface. Only the trajectories with positive $dH/d\phi$ are shown: those with negative $dH/d\phi$ are obtained by mirror reflection about $\phi = 0$.

Along a Cauchy surface $\Sigma : H =$ constant the measure is

$$\int e^{3\lambda} d\lambda d\phi |\frac{dH}{d\phi}|_{\Sigma}. \tag{37}$$

Since the number of e-folds N can be determined as a function of the initial value of ϕ along Σ , we can convert the probability distribution over ϕ to one over N.

Perturbing the formula

$$\sqrt{2}\frac{dN}{d\phi} = \frac{H}{\sqrt{3H^2 - V}} \tag{38}$$

one discovers that

$$\frac{d\delta H}{dN} = 3\delta H \tag{39}$$

which implies that N is an extremely sensitive function of the value of ϕ on Σ.

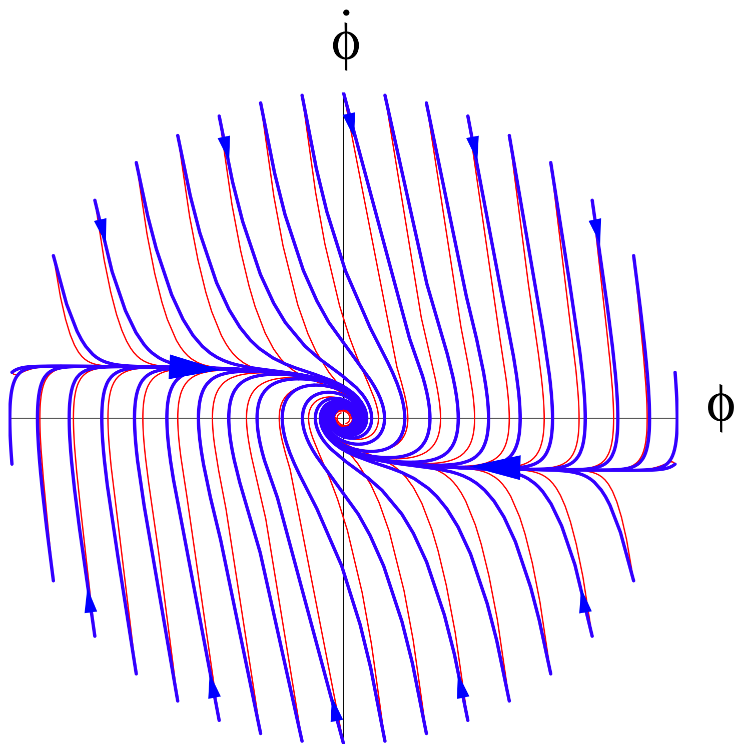
$$\delta H_\Sigma = |\frac{dH}{d\phi}|_\Sigma \delta\phi_\Sigma = e^{-3N}\delta H. \tag{40}$$

If we take δH to be sufficient for slow roll to break down, the measure on N becomes (setting $\lambda = 0$) ,

$$\int dN C(N) e^{-3N}, \tag{41}$$

where $C(N)$ is a slowly varying function of N. Thus, from this point of view, the probability of inflation is vanishing small!

This result may appear puzzling since in earlier work the fact that slow roll is an attractor to the future seemed to imply that inflation



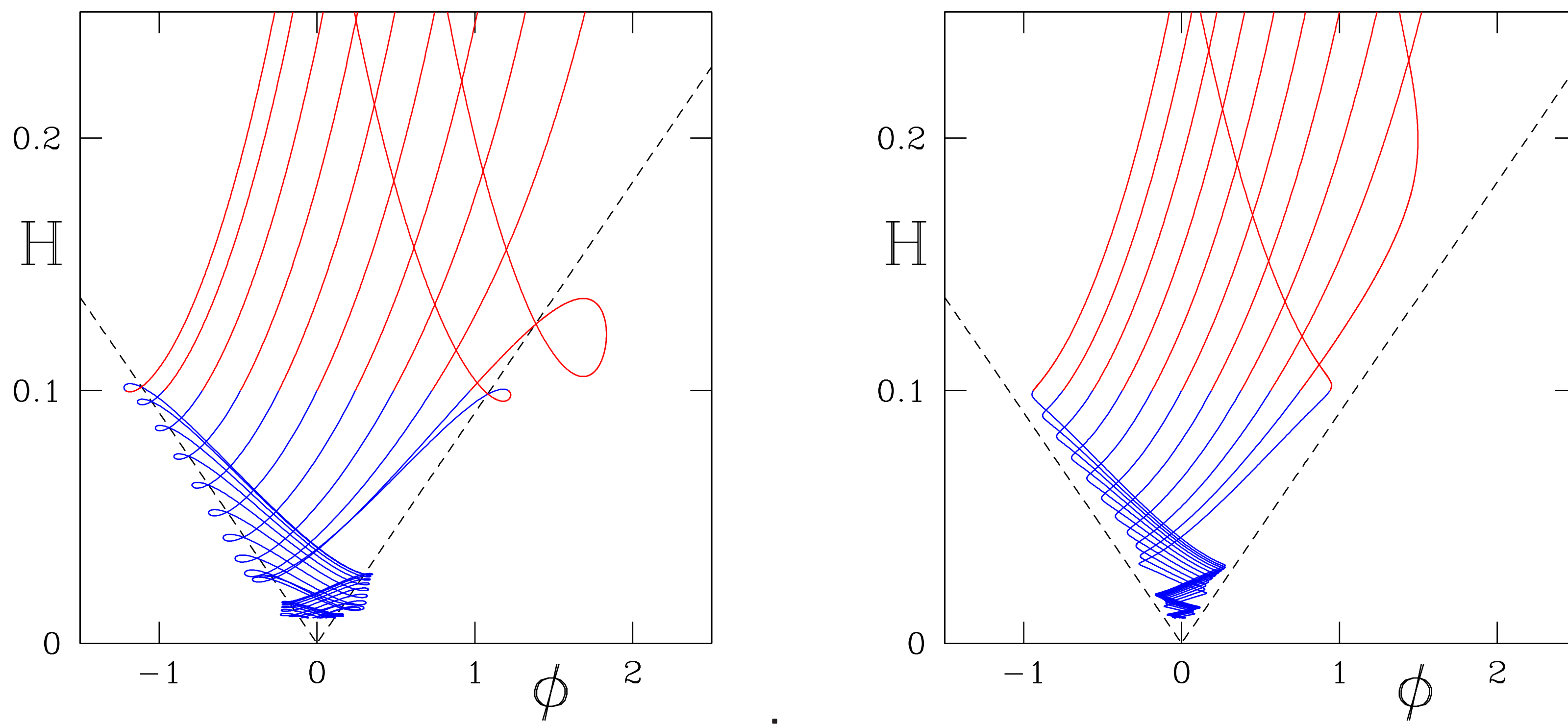
is generic

The $\phi - \dot{\phi}$ phase plane

However slow roll is a repellor to the past, and initial conditions ensuring that in the past there was *sufficient* inflation must be extremely finely tuned.

Note that these argument looks as if they are asymmetric with respect to reversal of time. However the measure we use is independent of time orientatation. This illustrates how dangersous it is to let unacknowedged assumptions slip in.

They also seem to depend on the radius of the circle centred on the origin. Choosing a very large circle gives one result. Choosing a very small circle gives the opposite result.

If $k \neq 0$ we have  .

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At large a the situation should not be significantly different from the $k = 0$ case. However, if $k \neq 0$, rescaling a is not an exact symmetry.

One possibility is to integrate over all scale factors smaller than a certain fixed value $a_{\mathsf{max}} = e^{\lambda_{\mathsf{max}}}$.and ignore those larger than a_{max}, on the ground that they all give the same presently observed universe *
The integral

$$\int_{-\infty}^{\lambda_{\mathsf{max}}} e^{3\lambda} d\lambda \frac{6H^2 - 2V + 4ke^{-2\lambda}}{\sqrt{6H^2 - 2V + 6ke^{-2\lambda}}} d\phi \tag{42}$$

certainly converges.

*We can't integrate over all scale factors bigger than a_{max} because this would diverge

There is a certain analogy here with state counting in elementary quantum mechanics. Given a Hamiltonian $\mathcal{H}(q,p)$, the number of states with energy less than or equal to E, $N(<E)$ (the cumulative density of states) is given semi-classically by

$$\int_{\mathcal{H}\leq E} dpdq \tag{43}$$

One then estimates the entropy as

$$S = \ln(N(<E))\,. \tag{44}$$

In our case the obvious guess would be

$$S \approx 3\ln a_{\mathrm{max}}, \tag{45}$$

which is not completely ridiculous.

Note that it is not obviously related to the entropy of de-Sitter space-time givne by $\frac{1}{4}$ the area of the cosmological horizon.

For one degree of freedom

$$\int_{\mathcal{H}\leq E} dpdq = \oint_{\mathcal{H}=E} pdq \tag{46}$$

This suggest our integration process is topological, and in some situations will be independent of the nature of the universe at small scales. In our case we are on a cotangent bundle $P = T^{\star}Q$. and ω is exact

$$\omega = d\theta\,, \qquad \theta = p_i dq^i, \tag{47}$$

the canonical one-form θ is *globally* defined. Thus

$$\omega^k = d(\theta \wedge \omega^{k-1}) \tag{48}$$

If $M \supset D = \{p, q | a < a_{\text{max}}\}$, then

$$\int_D \omega^{n-1} = \int_{\partial D} \theta \wedge \omega^{n-2} \tag{49}$$

where the r.h.s. is over a surface in M given by $a = a_{\text{max}}$ *.

*This looks rather Chern-Simons like

Extension to Perfect Fluids (wk.in progress with T Damour and N Turok)

$$X = -\nabla\psi^2. \tag{50}$$

Consider a Lagrangian$L = L(X)$. The energy momentum is that of a an irrotational perfect fluid

$$T_{\mu\nu} = (\rho + P)U^\mu U^\nu + P g_{\mu\nu} \tag{51}$$

$$U_\mu = \frac{\partial_\mu \psi}{|\partial\psi|}, \qquad P = L\,, \qquad \rho = 2XL_X - L\,. \tag{52}$$

e.g. radiation * $L = X^4 = (\nabla\psi)^4$

Constructing the Hamiltonian is now straightforward: the measure is related to the conserved entropy of the fluid!

*conformally invariant theory

In fact the entropy current is a Noether current for shift symmetry $\psi \to \psi +$ constant

If one breaks the shift symmetry by coupling to the inflaton ϕ one obtains a Hamiltonian treatment of a dissipative fluid which provides a toy model describing the generation of entropy at the end of inflation.

Wigner Distribution Functions (wk.in progress with Hartle, Hertog and Turok) If inflation is no a priori probable we may need to alter our prior. This can be done by introducing a Wave function for the Universe $\Psi = \Psi(q^i) = \Psi(a = e^\lambda, \phi)$ satisfying (at least in some formal sense the Wheeler-De-Witt equation

$$H\Psi(a = e^\lambda, \phi) = 0\,. \tag{53}$$

To obtain a probability distribution $P = P(U) = P(p, q)$ we need to calculate the Wigner distribution by

$$W(p_j, q^i) = \frac{1}{\pi^2} \int_{\mathbb{R}^n} d^2y \overline{\psi(q^i + y^i)} \psi(q^i - y^i) \exp\left(2iy^j p_j\right). \tag{54}$$

This makes sense because e^λ, ϕ are naturally (flat) Cartesian coordinates on the configuration space (Wheeler's superspace).

Natural measures or, prors on coupling constants are essential for any rational discussion of fine tuning and anthropic arguments. In some cases this is unproblematic. For example in anthropic arguments based on the axion * one needs a measure on the phase $\equiv S^1 \equiv U(1)$. In other cases, such as Yang-Mill's couplings, mass matrices or CP violating parameters this is not so obvious. As mentioned above, String theory seems to improve things.

*Dimensionless constants, cosmology and other dark matters. Max Tegmark, Anthony Aguirre, Martin Rees, Frank Wilczek *Phys.Rev* **D73** 023505,2006. arXIv: [astro-ph/0511774]

Electric-Magnetic duality

In string theory one typically couples an axion and dilaton to a $U(1)$ field and maintain invariance of the equations of motion, but not the action, under $SL(2,R)$ classically, or a $SL(2,Z)$ quantum mechanically, if one takes the Lagrangian function

$$-\frac{1}{4}(\tau_2 F_{\mu\nu}F^{\mu\nu} - \tau_1 F_{\mu\nu} \star F^{\mu\nu})\,. \tag{55}$$

The electromagnetic field transform as

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \to S \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}\,. \tag{56}$$

The relation to the conventional coupling constants g a theta angle θ is

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\,. \tag{57}$$

There is a natural metric on coupling constant space

$$ds^2 = \frac{1}{\tau_2}|d\tau|^2\,, \tag{58}$$

S-duality acts as

$$\tau \to \frac{-c + a\tau}{d + -b\tau}\,. \tag{59}$$

To maintain the Dirac quantisation condition one needs $a, b, c, d \in Z$. Moreover two theories related by an $SL(2, Z)$ transformation are physically indistinguishable.

Normally these coupling constants span the entire upper half plane but in the present case they are restricted to the double coset, i.e. $D = SL(2,Z)\backslash SL(2,R)/SO(2)$, (also called the fundamental domain of the modular group), which is non-compact but nevertheless has finite area. A representative domain is $\tau_2 > 0$, $|\tau_1| < \frac{1}{2}$ and $|\tau| > 1$. There are two orbifold points, one at $\tau = i$, with deficit angle π and one at $\frac{1}{2} + i \pm \frac{\sqrt{3}}{2}$ with deficit $\frac{4\pi}{3}$.

The total area is finite

$$\int\int_D \frac{d\tau_1 d\tau_2}{\tau_2^2} = \frac{\pi}{3}\,. \tag{60}$$

The calculation above may be understood geometrically and illustrates how compatifications in string theory gives rise to normalisable measures on almost all but one “modulus”.

A unimodular metric on the 2-torus may be written as

$$ds^2 = \frac{1}{\tau_2}|dy^1 + \tau dy^2|^2 \,, \tag{61}$$

with

$$\tau = \tau_1 + i\tau_2 \,. \tag{62}$$

The De-Witte metric on the space of such metrics is

$$\mathrm{Tr} g^{-1} dg g^{-1} dg = 2\frac{d\tau d\bar{\tau}}{\tau_2^2} \tag{63}$$

Under

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \tag{64}$$

with $ab - cd = 1$, one finds that

$$\tau \to \frac{b + d\tau}{a + c\tau}. \tag{65}$$

Thus to maintain invariance one needs to compensate by the inverse

$$\tau \to \frac{-c + a\tau}{d + -b\tau}\,. \tag{66}$$

To maintain the torus periods one needs $a, b, c, d \in Z$.

The space of moduli is thus of finite total measure.

Deep theorems in geometry appear to show that the same is true for the space of Calabi-Yau's.

*

*Weil-Petersson volumes of the moduli spaces of CY manifolds. Andrey Todorov hep-th/0408033, Finiteness of volume of moduli spaces. Michael R. Douglas Zhiqin Lu hep-th/0509224 , On the Weil-Petersson volume and the first Chern class of the moduli space of Calabi-Yau manifolds. Zhiqin Lu, Xiaofeng Sun Commun.Math.Phys.261:297-322,2006. math/0510021 On the geometry of moduli space of polarized Calabi-Yau manifolds. Michael Douglas Zhiqin Lu . Submitted to Publ.Res.Inst.Ma math/0603414

The Kobayashi-Maskawa Matrix (wk with Ben Allanach and N Turok)

In the standard theory of CP violation the quark mass eigenstates are related by an $SU(3)$ matrix

$$U = G_1 C G_2 \,, \tag{67}$$

The $U(1)^2$ matrices G_1 and G_2 correspond the the freedom to phase the quarks , and are given by

$$G_1 = \begin{pmatrix} \exp \frac{2ip}{\sqrt{3}} & 0 & 0 \\ 0 & \exp i(-\frac{p}{\sqrt{3}} + q) & 0 \\ 0 & 0 & \exp -i(\frac{p}{\sqrt{3}} + q) \end{pmatrix} \tag{68}$$

$$G_2 = \begin{pmatrix} \exp i\frac{(r}{\sqrt{3}} + t) & 0 & 0 \\ 0 & \exp i(\frac{r}{\sqrt{3}} - t) & 0 \\ 0 & 0 & \exp -i\frac{2r}{\sqrt{3}} \end{pmatrix} , \tag{69}$$

The Kobayashi-Maskawa matrix C is an element of $SU(3)$, or strictly speaking the double coset $U(1)^2 \backslash SU(3)/U(1)^2$.

Explicity

$$\begin{pmatrix} cycz & cysz & sy \exp -iw \\ -cxsz - sxsycz \exp iw & cxcz - sxsysz \exp iw & sxcy \\ sxsz - cxsycz \exp iw & -sxcz - cxsysz \exp iw & cxcy \end{pmatrix} \qquad (70)$$

Thus p, q, r, t, x, y, z, w are coordinates on the group $SU(3)$ and the measurable physical coupling constants entering the Kobayashi-Maskawa matrix x, y, z, w may be regarded as coordinates on the double coset $U(1)^2 \backslash SU(3)/U(1)^2$. If $w \neq 0$, then the Lagrangian is CP violating.

There is a natural bi-invariant metric on $SU(3)$ given by

$$\begin{aligned} ds^2 &= -\mathrm{Tr}\, U^{-1} dU\, U^{-1} dU && (71) \\ &= \mathrm{Tr}\, dU\, dU^\dagger && (72) \\ &= dU_{jk}\, d\bar{U}_{jk}\,. && (73) \end{aligned}$$

The restriction to the Kobayashi-Maskawa matric is, according to Ozsvath and Schucking *

$$ds^2 = 2\{dx^2 + dy^2 + dz^2 + 2\sin y \cos w dx dz + \sin^2 y dw^2\}\,. \qquad (74)$$

Interestingly, this restricted metric is claimed to be in fact invariant under an action of $U(1)^3$. Ozsvath and Schucking showed that suitable coordinate change exists such that

$$ds^2 = 2\{du^2 + dx^2 + dz^2 + 2\cos u dx dz + \sin^2 u dv^2\} \qquad (75)$$

*I Ozsvath, *Working with Englebert* in *On Einstein's Path: Essays in Honour of Engelbert Schucking* ed A Harvey, Springer, New York (1996) 339-351

The induced normalizable but unnormalised measure coming from (74) is rather non-obvious:

$$\sqrt{g}\, dxdydzdw = \sin y (1 - \sin^2 y \cos^2 w)^{\frac{1}{2}} dxdydzdw\,. \tag{76}$$

Translation into Particle Data Book Notation

The matrix C is usually called V with elements

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{77}$$

The angles are defined as

$$z = \theta_{12} \qquad x = \theta_{23} \qquad y = \theta_{13}\,, \qquad w = \delta\,. \tag{78}$$

In this notation the invariant introduced by Jarlskog is

$$J = c_{12} c_{23} {c_{13}}^2 s_{12} s_{23} s_{13} \sin\delta\,. \tag{79}$$

Given this probablity distribution, we can now talk about the statistical properties of J , in other words how "natural" is CP violation and presumably begin to answer the question

$\boxed{\text{How likely is baryosynthesis and hence our existence.?}}$

Killing vectors of KM-metric

If $K = K^\mu \partial_\mu$ is a Killing vecor field then The condition for an isometry is

$$\pounds_K g_{\mu\nu} = K^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu K^\lambda g_{\lambda\nu} + \partial_\nu K^\lambda g_{\mu\lambda} = 0\,. \tag{80}$$

For a metric on a four-dimensional space this gives ten equaations for four unknowns and so is heavily overconstrained.

Since ∂_x and ∂_z are Killing and commuting and we are seeking a third Killing vector commuting with both of them, them it is reasonable to assume the thid is of the form

$$K = A(y,w)\partial_y + B(y,w)\partial_w\,. \tag{81}$$

Now we have

$$\pounds_K g_{yy} = 2\partial_y A = 0\,, \tag{82}$$

whence

$$A = A(w)\,. \tag{83}$$

Now

$$\pounds_K g_{ww} = 2A \sin w \cos w + \partial_w B \sin^2 w = 0\,, \tag{84}$$

whence

$$A \cos w + \partial_w B \sin y = 0\,. \tag{85}$$

Furthermore

$$\pounds_K g_{xz} = 2A \cos y \cos w - B \sin y \sin w = 0\,, \tag{86}$$

whence

$$A \cos y \cos w - B \sin y \sin w = 0\,. \tag{87}$$

and

$$\pounds_K g_{wy} = \partial_w A + \sin^2 w \partial_y B = 0\,. \tag{88}$$

whence

$$\partial_w A + \sin^2 w \partial_y B = 0\,. \tag{89}$$

The remaining six equations appear to give no further conditions. In any case, we find from (83) and (87) that

$$B = \cot w A(w) \cot y\,, \tag{90}$$

Proceeding we find

$$K = \frac{\partial}{\partial v} = \sin w \partial_y + \cos w \cot y \partial_w \,. \tag{91}$$

Since

$$\pounds_K g_{xz} = 0 = K(\sin y \cos w) \,, \tag{92}$$

it is reasonable to define

$$\sin y \cos w = \cos u \tag{93}$$

where u is constant along the orbits of the third Killing field.

Now we recognise the metric on the quotient $X \backslash U(1) \times U(1)$ where $U(1) \times U(1) = T^2$ is generated by ∂_z and ∂_z is the round metric on the hemisphere $\frac{1}{2}S^2$, (since w lise in the first quadrant)

$$ds^2 = dy^2 + \sin^2 y dw^2 \,. \tag{94}$$

This is obtained by embedding it into Eucidean 3-space E^3 as $(\sin y \cos w, \sin y$

One is therefore led to consider a different embedding:

$$(\sin y \cos w, \sin y \sin w, \cos w) = (\cos u, \sin u \cos v, \sin u \sin v) \tag{95}$$

so that while ∂_w generates a rotation around the third axis, ∂_w generates a rotation around the first axis, and while $\cos y$ gives the projection along the third axis, $\cos u$ gives the projection along the first axis.

Substition then gives the metric form given by Ozsvath (**??**).

The (non-normalised) measure is thus

$$\mu = \sin^2 u \, du dv dx dz \tag{96}$$

Geometrically we can think of our moduli space $X = U(1)^2 \backslash SU(3)/U(1)^2$ as a trivial 2-torus bundle over the hemi-sphere $\frac{1}{2}S^2$. The coordinates of the torus are x, z and from the metric it follows that $\mathbf{e}_x = \partial_x$ and $\mathbf{e}_y = \partial_y$ are a normalised basis and that the angle between them is $\cos^{-1}(\cos^2 u)$ which varies between 0 at the north pole and $\frac{\pi}{2}$ at the equator. Jarlskog's CP violating quantity J We may express the Jarlskog invariant J in terms of the coordinates (x, z, u, v) . One has

$$J = \frac{1}{4} \sin(2z) \sin(2x) \sin^3 \cos u \sin^2 v \, . \tag{97}$$

Steffen Gielen has calculated numerically (in the coordinates (x, z, y, w)) that while $\langle J \rangle = 0$,

$$\langle J^2 \rangle = 5.34 \times 10^{-4}, \qquad \langle J^4 \rangle = 1.57 \times 10^{-6}, \tag{98}$$

and that the standard deviation is

$$\Delta J^2 = \sqrt{\langle J^4 \rangle - \langle J^2 \rangle^2} = 1.13 \times 10^{-3}, \tag{99}$$

and

$$\Delta J = \sqrt{\langle J^2 \rangle} = 2.31 \times 10^{-2}. \tag{100}$$

On the other hand, experimentally

$$J = 3.8^{+0.16}_{-0.18} \times 10^{-5}. \tag{101}$$

Thus, the observed CP violating parameter J is between two and three orders of magnitude smaller than one would expect by chance. In fact in [**?**] (p. 11) it is stated that the maximum value of J is $\frac{1}{6\sqrt{3}} \approx 0.1$ which is ascribed to the strong heirarchy exhibited by the CKM matrix elements.

Mass matrices The natural (positive definite or Riemannian) metric on symmetric, or more generally Hermitian $n \times n$ matrices is

$$ds^2 = \mathrm{Tr}\, dM dM^\dagger \,. \tag{102}$$

and is invariant under conjugation by $U(n)$.

This is the standard metric used in Wigner's random matrix theory, an atempt to explain nuclear levels in tems of a randomly chosen Hamiltonian. Of course to normalise the induced Riemannian measure, one frequently multiplies it by a Gaussian factor, but that makes little difference to the considerations which follow.

If we demand that the mass matrices M are positive semi-definite they form a convex cone in $C^{\frac{1}{2}n(n+1}$. Fixing the trace or the trace of the square gives a compact space with a well defined measure One may write

$$M = S\Delta S^{\dagger} \tag{103}$$

where S is unitary and Δ is diagonal with entries λ_i. The case of 3×3 real symmetric matrices is described in detail by Giulini [*]. . The general Hermitian case is described by Zinn-Justin and Zuber [†]

[*] D. Giulini, A Euclidean Bianchi model based on S**3 / (D(8)*, *J. Geom. Phys.* **20** (1995) 149 [arXiv:gr-qc/9508040]

[†] P. Zinn-Justin and J. B. Zuber, On some integrals over the U(N) unitary group and their large N limit, *J. Phys.* **A 36** (2003) 3173 [arXiv:math-ph/0209019].

As far as masses are concerned we are not interested in the angles of the unitary group, but rather the eigen-values. Thus we simply average over the angles and obtain an induced metric on the eigenvalues. If these are non-negative we are confined to the positive orthant ${R_+}^n$ of R^n. Imposing a trace condition then limits one to a hyperplane plane $\sum_i \lambda_i = 1$, or a hypersphere $\sum_i \lambda_i^2 = 1$. The former case means that the measure is defined on an $(n-1)$-plex Σ_{n-1}. (i.e inside a (Dalitz style) triangle for the physically relevant 3×3 case).

The induced measure coming from the metric on ${R_+}^n$ is

$$\prod_{i<j} |\lambda_i - \lambda_j| d^n\lambda \tag{104}$$

Thus co-incident or near co-incincident masses are strongly disfavoured according to this measure. Given the quark masses and their error bars, it would be possible to calculate precisely the degree of fine tuning.

The Conclusions are that

- It remains a non-trvial problem to place priors on the space of initial conditions, coupling constants or of quantum states, except for cases with a finite number of classical degrees of freedom or Hilbert space dimensions.

- String theory, in which coupling constants are subsumed into initial conditions gives rise to some improvements but not as yet a solution.

- The absence of such priors challenges are our basic notions of science as a rational activity.