The emergent nature of time and the complex numbers in quantum cosmology

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Introduction:

Recently I was asked to speak at a conference on the Origins of Time's Arrow *

This gave me the opportunity to discuss ideas I have had for some time about the nature of time in quantum gravity.

I have long believed that time in quantum mechanics is closely related to the

the use of a complex, rather than say real, Hilbert space.

*http://www.nyas.org/ebrief/miniEB.asp?eBriefID=687#meetingrep

This becames particularly clear to me when considering quantum field theory in time dependent backgrounds, such as in cosmology when the notion of positive frequency ceases to be well defined. One also has to face this problem in quantum cosmology. This suggests to me that at a fundamental level quantum mechanics may be really real with not one, but a multitude of complex structures.

In other words, worry about The Arrow of Time, especially in cosmology. I think that before worrying about that, we should ask What is the nature of time? in Quantum Cosmology.

Both Quantum Mechanics and General Relativity have something to say about this.

But what they say is not quite compatible

The plan of this lecture is as follows

- Nature of time in quantum Mechanics
- Mathematical Interlude on Real and Complex Vector Spaces
- Real Quantum Mechanics and the Precautionary Principle
- Examples from Quantum Cosmology
- Examples from Spinors and Unification

- Colliding Branes

- Conclusion

Because the nature of time in Quantum Mechanics is less familar and less frequently discussed than it is in General Relativity I shall begin by recalling how time is intimately connected with the complex (Hilbert Space) structure of quantum mechanics.

In other words, the use of complex numbers and hence of complex amplitudes in Quantum Mechanics is intimately bound up with how Quantum States evolve in time.

$$i\frac{d\Psi}{dt} = H\Psi \tag{1}$$

In particular there can be no evolution if Ψ is real *.

*Conversely, as shown by Dyson in his three-fold way, if H is time-reversal invariant one may pass to a real (boson) or quaternionic (fermion) basis

If one anaylses the Logical Structure of Quantum Mechanics one finds that it consists of two different types of statements:

- **I** Timeless statements about states, propositions, the Principle of Superposition, probabilities, observables etc

- **II** Statements about how states and observables change, Schrödinger's equation and Unitarity etc.

The upshot of an analysis of $\mathbf{I}$ (so called Quantum Logic *) is that pure states are points in a Projective Space over $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ †.

$$\Psi \equiv \lambda \Psi \,, \qquad \lambda \in \mathbf{R}, \mathbf{C} \text{ or } \mathbf{H} \tag{2}$$

*Von-Neumann, Jordan, Wigner, Jauch, Piron, Stueckelberg, Adler etc

†We ignore the exceptional case of the octonions

Now any vector space over $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ is a vector space V over $\mathbf{R}$ with some additional structure, so let's use real notation. For clarity, I propose a short Mathematical Interlude

In standard quantum mechanics, (pure) states are rays in a Hilbert space $\mathcal{H}_{\mathsf{qm}}$ which is a vector space over the complex numbers carrying a Hermitian positive definite inner product $h(U,V)$ such that

$$(i) \qquad h(U,\lambda V) = \lambda h(U,V)\,, \qquad \forall \lambda \in \mathbf{C}\,. \tag{3}$$

$$(ii) \qquad h(U,V) = \overline{h}(V,U). \tag{4}$$

$$(iii) \qquad h(U,U) > 0. \tag{5}$$

It follows that $h(U,V)$ is antilinear in the first slot

$$h(\lambda U,V) = \overline{\lambda} h(U,V)\,, \qquad \forall \lambda \in \mathbf{C}\,. \tag{6}$$

In Dirac's bra and ket notation elements of $\mathcal{H}_{\mathrm{qm}}$ are written as kets:

$$V \leftrightarrow |V\rangle \tag{7}$$

and elements of the $\mathbf{C}$- dual space $\mathcal{H}^{\star}_{\mathrm{qm}}$, the space of $\mathbf{C}$- linear maps $\mathcal{H}_{\mathrm{qm}} \longrightarrow \mathbf{C}$ as bras: and there is an anti-linear map from $\mathcal{H}_{\mathrm{qm}}$ to $\mathcal{H}^{\star}_{\mathrm{qm}}$ given by

$$U \longrightarrow \langle U| \tag{8}$$

such that

$$h(U,V) = \langle U \,|\, V\rangle \,, \tag{9}$$

thus

$$\langle U| = h(U, \,\cdot\,) \,. \tag{10}$$

In components

$$|V\rangle = V^i\,|i\rangle \tag{11}$$

and

$$\langle U| = \langle j|\,\bar{U}^{\bar{j}}, \tag{12}$$

$$\langle U\,|\,V\rangle = h(U,V) = h_{\bar{i}j}\bar{U}^{\bar{i}}V^j, \tag{13}$$

where

$$h_{\bar{i}j} = \langle j\,|\,i\rangle\,, \tag{14}$$

and

$$\overline{h_{\bar{i}j}} = h_{\bar{j}i}\,. \tag{15}$$

Complex Vector spaces as Real Vector spaces For simplicity of exposition one may imagine that $\mathcal{H}_{\text{qm}}$ as finite dimensional $\dim_{\mathbf{C}} \mathcal{H}_{\text{qm}} = n < \infty$. Since a complex number is just a pair of real numbers, any Hermitian vector space may be regarded as a real vector space V of twice the dimension $\dim_{\mathbf{R}} V = 2n$ with something added , a complex structure J, i.e a real-linear map such that

$$J^2 = -1 \,, \tag{16}$$

and a positive definite metric . g such that J is an isometry, i.e.

$$g(JX, JY) = g(X, Y) \,. \tag{17}$$

It follows that V is also a symplectic vector space, with symplectic form

$$\omega(X,Y) = g(JX,Y) = -\omega(Y,X)\,, \tag{18}$$

and J acts canonically, i.e.

$$\omega(JX,JY) = \omega(X,Y)\,. \tag{19}$$

Alternatively given J and the symplectic form ω we obtain the metric g via

$$g(X,Y) = \omega(X,Jy)\,. \tag{20}$$

The standard example is the complex plane $\mathbf{C} = \mathbf{R}^2$ where if

$$\mathbf{e}_1 = (1, 0)\,, \qquad \mathbf{e}_2 = (0, 1)\,, \tag{21}$$

$$J(\mathbf{e}_1) = \mathbf{e}_2\,, \qquad J(\mathbf{e}_2) = -\mathbf{e}_1 \tag{22}$$

or as a matrix

$$J = \begin{pmatrix} 0 & -1, \\ 1 & 0 \end{pmatrix} \tag{23}$$

and thus

$$J(x\mathbf{e}_1 + y\mathbf{e}_2) = x\mathbf{e}_2 - y\mathbf{e}_1 \tag{24}$$

which is the same in the usual notation as

$$i(x + iy) = -y + ix\,, \tag{25}$$

where $1 \leftrightarrow (1, 0)$ and $i \leftrightarrow (0, 1)$.

A complex structure J can be thought of as a rotation of ninety degrees in n orthogonal two planes. To specify it therefore it suffices to specify the (unordered) set of planes and the *sense* of rotation in each 2-plane.

A Real vector space as a Complex Vector space

Given the original real vector space, how are the complex numbers actually introduced? We start with V and pass to its complexification, the tensor product

$$V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C} \,. \tag{26}$$

Note that $\mathrm{dim}_{\mathbf{R}} V_{\mathbf{C}} = 4n = 2\mathrm{dim}_{\mathbf{C}} V_{\mathbf{C}}$,

We now extend the action of J to $V_{\mathbf{C}}$, so it commutes with $i \in \mathbf{C}$:

$$J\alpha X = \alpha J X \,, \qquad \forall \quad \alpha \in \mathbf{C} \,, X \in \mathbf{C} \,. \tag{27}$$

We may now diagonalize J over $\mathbf{C}$ and write

$$V_{\mathbf{C}} = W \oplus \overline{W} \tag{28}$$

where

$$JW = iW\,, \qquad J\overline{W} = -i\overline{W}\,. \tag{29}$$

Clearly $\dim_{\mathbf{R}} W = 2n = 2\dim_{\mathbf{C}} W = \dim_{\mathbf{R}} V$, and W may be thought of as V in complex notation.

Thus if $X \in V$, we have that

$$X = \frac{1}{2}(1 - iJ)X + \frac{1}{2}(1 + iJ)X, \tag{30}$$

with $\frac{1}{2}(1 - iJ)X \in W$ and $\frac{1}{2}(1 - iJ)X \in \overline{W}$. Vectors in W are referred to as type $(1,0)$ or holomorphic and vectors in $\overline{W}$ as type $(0,1)$ or anti-holomorphic.

The metric on $V_{\mathbf{C}}$

If V admits a metric for which J acts by isometries, we may extend the metric g to all of $V_{\mathbf{C}} = W \oplus \overline{W}$ by linearity over $\mathbf{C}$, we find that

$$(i) \quad g(\bar{U}, V) = \overline{g}(U, V) \tag{31}$$

$$(ii) \quad g(U, \bar{U}) > 0, \tag{32}$$

$$(iii) \quad g(U, V) = 0\,, \forall U, V \in W\,, \text{and}\,, \forall U, V \in \overline{W}\,. \tag{33}$$

We now return to the problem at hand. We use an index notation for vectors v^a, in our real vector space V with $a = 1, 2, \ldots \mathrm{dim}_R V = 2n$. The metric is written as $g_{ab} = g_{ba}$, the symplectic form as $\omega_{ab} = -\omega_{ba}$ and complex structure as $J^a{}_b$.

Observables are symmetric bilinear forms:

$$\langle \Psi O \Psi \rangle = \Psi^a O_{ab} \Psi^b \,, \qquad O_{ab} = O_{ba} \tag{34}$$

$a = 1, 2, \ldots, n = \mathrm{dim}_{\mathbf{R}} V$. Mixed states ρ are positive definite observables dual to the observables

$$\langle O \rho \rangle = \rho^{ab} O_{ab} = \mathrm{Tr}\,(\rho O)\,, \qquad \rho^{ab} = \rho^{ba} \tag{35}$$

There is a privileged density matrix the completely ignorant density matrix which we may think of as a metric* g_{ab} on V and use it to normalise our states

$$\langle \Psi | \Psi \rangle = g_{ab} \Psi^a \Psi^b \,, \qquad g_{ab} = g_{ba}\,, \qquad \mathrm{Tr}\rho = g^{ab} \rho_{ab} \tag{36}$$

*strictly speaking the inverse

The upshot of a conventional analysis of **II** (Dirac called it Transformation Theory) is that states change by by means of linear maps which preserves the metric (i.e. preserves complete ignorance)

$$\Psi^a \to S^a{}_b \Psi^b\,, \qquad g_{ab} S^a{}_c S^b{}_d = g_{cd} \tag{37}$$

Thus $S \in SO(n, \mathbf{R})$, $n = \dim_{\mathbf{R}} V$. Infinitesimally

$$S^a{}_b = \delta^a_b + T^a{}_b + \dots \tag{38}$$

where the Operator $T^a{}_b$ gives a two-form when the index is lowered

$$g_{ab} T^b{}_c := T^\flat_{ac} = -T^\flat_{ca} \tag{39}$$

But Dirac taught us that, just as in Hamiltonian mechanics, to every (Hermitian) Operator there is an Observable and *vice versa*. How can this be? Our vector space V over $\mathbf{R}$ needs some extra structure, in fact a complex structure $J^a{}_b$ or privileged operator which also preserves the metric (i.e. preserves complete ignorance).

$$g_{ab}\, J^a{}_c J^b{}_d = g_{cd}\,, \tag{40}$$

Then

$$J^a{}_b J^b{}_c = -\delta^a_c \Longrightarrow \omega_{ab} = -\omega_{ba} \tag{41}$$

where the symplectic two-form $\omega_{ab} = g_{ac} J^c{}_b$ may be used to lower indices and obtain a symmetric tensor for every (Hermitian) observable (i.e. one that generates a transformation preserves the symplectic form)

$$\omega_{ab} T^b{}_c := T_{\flat ac} = +T_{\flat ca} \tag{42}$$

We can think of this more group theoretically *. In regular Quantum Mechanics V is a Hermitian vector space its transformations should be unitary, but

$$U(\frac{n}{2}, \mathbf{C}) = SO(n, \mathbf{R}) \cap GL(\frac{n}{2}, \mathbf{C}) \tag{43}$$

where $GL(\frac{n}{2}, \mathbf{C}) \subset GL(n, \mathbf{R})$ is the subgroup preserving J, and $SO(n, \mathbf{R}) \subset GL(n, \mathbf{R})$ is the subgroup preserving the metric g.One also has

$$U(\frac{n}{2}, \mathbf{C}) = SO(n, \mathbf{R}) \cap Sp(n, \mathbf{R}) \tag{44}$$

where $Sp(n, \mathbf{R}) \subset GL(n, \mathbf{R})$ is the subgroup preserving the symplectic form ω, and of course

$$U(\frac{n}{2}, \mathbf{C}) = SO(n, \mathbf{R}) \cap GL(\frac{n}{2}, \mathbf{C}) \tag{45}$$

*Or recall what we know about Kähler mfds. Quantum Mechanics makes use of a Kählerian vector space

A precautionary principle

The main message of this talk is that given a vector space V over $\mathbf{R}$ *it may have no complex structure* (n must obviously be even!) or if it is does, *the complex structure may not be unique* (they are typically members of infinite families)

Thus on four dimensional Euclidean space $\mathbf{E}^4$ they belong (modulo a choice of orientation) to a two-sphere $S^2 = SO(4)/U(2)$.

More generally, every quaternion vector space has such a 2-sphere's worth of complex structures * , i.e. *a 2-sphere's worth of of times*!

*cf Hyper-Kähler manifolds such as K3

To bring out the fact that in physics we use many different complex structures for many different reasons it is occasionally helpful to indicate explicitly by the symbol i_{qm} the very particular complex structure on the Hilbert space $\mathcal{H}_{\mathsf{qm}}$ of the standard model and so that Schrödinger's equation really reads

$$i_{\mathsf{qm}}\frac{d\Psi}{dt} = H\Psi\,. \tag{46}$$

Therefore it seems wise to adopt a course of action, particularly at the classical level before quantization, in which one proceeds as far as possible by considering all physical quantities and their related mathematical structures to be real until one is forced to introduce complex notation and i_{qm} at the point where one introduces quantum mechanics.

In other words, in what follows, I plan to follow, in so far as is possible, Hamilton's course of action

The author acknowledges with pleasure that he agrees with M. CAUCHY, in considering every (so-called) Imaginary Equation as a symbolic representation of two separate Real Equations: but he differs from that excellent mathematician in his method generally, and especially in not introducing the sign $\sqrt{-1}$ until he has provided for it, by his Theory of Couples, a possible and real meaning, as a symbol of the couple $(0, 1)$.

Dyson's Three-fold way In this language, Dyson's observation is that in standard quantum mechanics an anti-linear involution Θ acting on rays may be normalized to satisfy

$$\Theta^2 = \pm 1\,, \tag{47}$$

where the plus sign corresponds to an even spin state and the odd sign to an odd spin state. To say that Θ is anti-linear is to say that it anti-commutes with the standard complex structure $i_{\rm qm}$, ${i_{\rm qm}}^2 = -1$

$$\Theta i_{\rm qm} + i_{\rm qm}\Theta = 0\,. \tag{48}$$

Now for the plus sign Θ, is a projection operator and we get what is called a *real structure* on the original complex Hilbert space and if the

Hamiltonian is time-reversal invariant, then we may use the projection operator to project onto the subspace of real states. On the other hand for the minus sign we construct

$$K = \Theta\, i_{\mathrm{qm}} \tag{49}$$

and find that $\Theta, i_{\mathrm{qm}}, K$ satisfy the algebra of the quaternions.

With this prepartion we can immediately see that

THE MUCH DISCUSSED QUESTION OF WHETHER BLACK HOLE EVAPORATION IS UNITARY IS MEANINGLESS IF THERE THERE IS NO COMPLEX STRUCTURE, AND ILL-POSED IF THERE IS MORE THAN ONE in Quantum Field Theory in Curved Spacetime the main problem is that there is no unique definition of "positive frequency ". In the free theory, $V = \mathcal{H}_{\text{one particle}}$ is the space of real-valued solutions of wave equations. V is naturally (and covariantly) a symplectic (boson), or orthogonal (fermion)* vector space

$$\omega(f,g) = \int (\dot{f}g - \dot{g}g) d^3x = -\omega(g,f) \tag{50}$$

$$g(\psi,\chi) = \int (\psi^t \chi) d^3x = g(\chi,\psi) \tag{51}$$

*we use real (Majorana) commuting spinors for convenience: there use is not essential

To quantise we complexify and decompose

$$V_{\mathbf{C}} = \mathbf{C} \otimes V = V^{+} \oplus V^{-} \tag{52}$$

This decomposition (which defines a complex structure)* is not unique. THIS NON-UNIQUENESS CORRESPONDS PHYSICALLY TO THE POSSIBILITY OF PARTICLE PRODUCTION AND IS AN ESSENTIAL PART OF OUR CURRENT UNDERSTANDING OF BLACK HOLE EVAPORATION AND INFLATIONARY PERTURBATIONS

*Ashtekar & Magon, Woodhouse

We can make this more concrete by the observation that a general Bogoliubov transformation which does nor preserve positive frequency does not therefore preserve the complex structure: it belongs to $Sp(2n, \mathbf{R})$ but not $GL(n, \mathbf{C})$. Another way to say it is that if particle production takes place there is a clash of complex structures:

$$J_{\text{initial}} \neq J_{\text{final}}. \tag{53}$$

In the language of Geometric Quantization: a choice of complex structure is a choice of polarization.

At this point it may be instructive to recall why time is not an operator and why commutation relations of the form

$$[\hat{x}^{\mu}, \hat{P}_{\nu}\ ] = i\delta^{\mu}_{\nu} \tag{54}$$

don't apply in quantum field theory in Minkowski spacetime. If they did, then they would have, up to natural equivalence, to be represented in the standard Stone-Von-Neumann fashion on $L^2(\mathbf{E}^{3,1})$. But then the energy $\hat{P}^0$, could not be bounded below. Thus $L^2(\mathbf{E}^{3,1})$ is not the quantum mechanical Hilbert space. Rather, as stated above, it is the space of positive frequency solutions of the Klein-Gordon or Dirac equations. These are is much more subtle objects and certainly not uniquely defined in a curved spacetime manifold $\{\mathcal{M}, g\}$, unlike $L^2(M, \sqrt{-g}d^4\,x)$, the obvious generalization of $L^2(\mathbf{E}^{3,1})$, which is unambiguous even in a curved spacetime.

In Hartle and Hawking's Wave Function for The Universe

$$\Psi(h_{ij}, \Sigma) = \int d[g] e^{-I_{\text{euc}}(g)}, \quad h_{ij} = g_{ij}|_{\Sigma = \partial M} \tag{55}$$

Is real valued. To get a notion of time one typically passes to a Lorentzian WKB approximation S_c

$$\Psi = A e^{iS_c} + \bar{A} e^{-iS_c} \tag{56}$$

but this is only a semi-classical approximation, in other words TIME, THE COMPLEX NUMBERS AND THE COMPLEX STRUCTURE OF QUANTUM MECHANICS EMERGE ONLY AS AN APPROXIMATION AT LATE TIMES

In fact in Euclidean Quantum Field Theory it is not sufficient just to compute correlators.

In order to recover Quantum Mechanics, rather than merely to indulge in an unphysical case of Statistical Mechanics, the correlators must exhibit Reflection Positivity. This guarantess the possibility of analytically continuing to real time.

This can be done for Riemannian backgrounds if they admit a suitable reflection map *

Most Riemannian metrics do not admit such a reflection map.

*e.g. static or time-symmetric metrics as for Real tunneling geometries, (Gibbons &Hartle, Jaffe etc)

My last example involves a Lorentzian Born From Nothing Scenario *. Essentially, one considers de-Sitter spacetime modded out by the antipodal map $dS/\mathbf{Z}_2$ (so-called elliptic interpretation).

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \frac{3}{\Lambda} \qquad \mathbf{Z}_2 : \; X^A \equiv -X^A . \tag{57}$$

Now the antipodal map preserves space orientation but reverse time orientation. But in quantum mechanics a time reversing transformation is represented by an anti-unitary operator Θ and if all states are invariant up to a factor

$$\Theta\Psi = \lambda\Psi \tag{58}$$

then only real linear combinations are allowed.

*Friedman, Gott & Li , Gratton etc

Thus Quantum Mechanics in $dS/\mathbf{Z}_2$ is Real Quantum Mechanics.

This jibes with the fact that under the action of the antipodal map is antisymplectic on the bosonic space of solutons V

$$\omega(\cdot,\cdot) \rightarrow -\omega(\cdot,\cdot)\,. \tag{59}$$

This renders imposing the CCR's impossible *

Compare regular time reversal

$$(p_i, q^i) \rightarrow \omega = (-p_i, q^i) \Longrightarrow dp_i \wedge dq^i \rightarrow -dp_i \wedge dq^i = -\omega \tag{60}$$

If there is no symplectic form then the Heisenberg commutation relations make no sense, one cannot geometrically quantise.

*Bernard Kay has implemented this argument more rigourously within an algebraic framework

This pathology arises quite generally for spacetimes which do not admit a Time Orientation, i.e. a smooth choice of future lightcone. In other words quantum field theory is not defined unless one may define an Arrow of Time * .

Amusingly CTC's seem to be quiet innocuous from this point of view. It seems that they can be compatible with qunatum mechanics.

An interesting question, discussed by Chamblin and myself, is whether this arrow is intrinsically defined, or whether both possibilities are on the same footing.

In other words, do there exist time-orientable spacetimes which have an intrinsic direction of time?

*Amusingly CTC's seem to be quiet innocuous from this point of view. It seems that they can be compatible with quantum mechanics.

The analogy here is with a quartz crystal which is either left-handed or right handed. This is because the point group contains no reflections or inversions.

For a spatial manifold Σ one asks: does Σ there exist an orientation reversing diffeomorphism. In other words is there a diffeomorphism taking Σ with one orientation to Σ with the opposite orientation?. For such mfds a Parity Map cannot be defined. Such "handed " manifolds are quite common, certain Lens Spaces and $\mathbf{CP}^2$ being examples *.

For spacetimes the analogous question is whether there exist a time reversing diffeo Θ?

We found some rather exotic examples, based on higher dimensional Taub-NUT spactimes for which no such diffeo Θ exists.

*see Hartle and Witt

Spacetime Signature and the Real Numbers

In $4+1$, and indeed $9+1$ and $10+1$, spacetime dimensions, it is possible, by choosing the spacetime signature appropriately, to develop spinor analysis *at the classical level entirely over the reals*. That is, to consistently use Majorana spinors whose components really are real. In four spacetime dimensions this requires the mainly plus signature convention (the opposite to that which Penrose uses). The complex numbers need only enter when one quantizes.

To see this in more detail we need some facts about Clifford Algebras

Given a vector space V * with metric g, of signature (s,t) where s counts the positive and t the negative signs, Clifford algebra Cliff$(s,t;\mathbf{R})$ is by definition the associative algebra over the *reals* generated by the relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,, \tag{61}$$

where γ is a basis for V. As a real algebra, the signature does make a difference. For example

$$\text{Cliff}(0,1;\mathbf{R}) \equiv \mathbf{C}\,, \tag{62}$$

while

$$\text{Cliff}(0,1;\mathbf{R}) \equiv \mathbf{R} \oplus \mathbf{R}\,. \tag{63}$$

*V is *not* $\mathcal{H}_{\text{qm}}$ thought of as real!

In fact Cliff$(0,1;\mathbf{R})$ is identical with what are often called ‘double numbers ’or ‘hyperbolic numbers’, i.e numbers of the form.

$$a + eb\,, \qquad a, b \in \mathbf{R}\,, \qquad e^2 = 1\,. \tag{64}$$

As an algebra, Cliff$(0,1;\mathbf{R})$ is not simple, $P_\pm = \frac{1}{2}(1 \pm e)$ are projectors onto two commuting sub-algebras.

In a matrix representation

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{65}$$

However if we pass to the complex Clifford over $\mathbf{C}$ we lose the distinction since

$$\text{Cliff}(0,1;\mathbf{C}) \equiv \text{Cliff}(0,1;\mathbf{C}) \equiv M_2(\mathbf{C})\,, \tag{66}$$

where $M_2(\mathbf{C})$ is the algebra of all complex valued two by two matrices.

It is precisely at this point that the precautionary principle comes in. We should not rush into adopting

$$\mathsf{Cliff}(3,1;\mathbf{C}) \equiv \mathsf{Cliff}(1,3;\mathbf{C}) \equiv M_4(\mathbf{C})\,, \tag{67}$$

but rather enquire what are the possible differences between the two signatures † . In fact

$$\mathsf{Cliff}(3,1;\mathbf{R}) \equiv M_4(\mathbf{R})\,, \qquad \mathsf{Cliff}(1,3;\mathbf{R}) \equiv M_2(\mathbf{H})\,, \tag{68}$$

where

$$\mathbf{H} \equiv \mathsf{Cliff}(0,2;\mathbf{R}) \tag{69}$$

†A similar point has been made recently by Schucking but he plumps for the quaternions

are the quaternions. Despite the differences the spin groups are identical

$$\mathrm{Spin}(3,1) \equiv \mathrm{Spin}(1,3) \equiv SL(2,\mathbf{C})\,, \tag{70}$$

but if discrete symmetries are taken into account they differ:

$$\mathrm{Pin}(3,1) \neq \mathrm{Pin}(1,3)\,. \tag{71}$$

This has important consequences in spacetimes which are time, space or spacetime non-orientable.

Majorana Spinors It is a striking and, I believe, a possibly rather significant fact that the signature $(3, 1)$ leads directly to a Majorana representation, in which all γ matrices are real. Certainly if one holds that $N = 1$ supersymmetry and $N = 1$ supergravity are important, this fact renders the mainly positive signature rather attractive. The precautionary principle would lead one to adopt the signature $(3, 1)$ and use a real notation for as long as one can, certainly at the classical level where one need never introduce complex numbers. Thus the basic entities are Majorana spinors ψ belonging to a four dimensional real vector space $\mathbf{M}$ with real, or real Grassmann number components ψ^a, $a = 1, 2, 3, 4$.

The charge conjugation matrix $C = -C^t$ satisfies

$$C\gamma_\mu C^{-1} = -\gamma_\mu^t\,, \qquad C\gamma_5 C^{-1} = -\gamma_5^t\,. \tag{72}$$

It serves as a Lorentz-invariant symplectic form on $\mathbf{M}$. Thus $\mathrm{Spin}(3,1) \subset \mathrm{Sp}(4;\mathbf{R}) \equiv \mathrm{Spin}(3,2)$.

Dirac Spinors Dirac spinors, consist of pairs of Majorana spinors ψ^i , $i = 1, 2$ which are elements of $\mathbf{R}^4 \oplus \mathbf{R}^4 \equiv \mathbf{R}^4 \otimes \mathbf{C}^2 \equiv \mathbf{R}^8$ If δ_{ij} is the metric and $\epsilon_{ij} = \delta_{ik} J^k{}_j$, the symplectic and $J^k{}_j$ the complex structure which rotates the two summands into each other, we can endow $\mathbb{D} \equiv \mathbf{R}^8$ with a symplectic form ω and a pseudo-riemannian metric g , and hence a pseudo-hermitean structure.

One can think of the Dirac spinors as elements of a four dimensional complex vector space $\mathbf{D} = \mathbf{M}_{\mathbf{C}} \equiv \mathbf{C}^4$, the complexification of the real space of of Majorana spinors $\mathbf{M}$.

Weyl Spinors To see where Weyl spinors fit in we observe that γ_5 acts as a complex structure converting $\mathbf{M} \equiv \mathbf{R}^4$ to $W \equiv \mathbf{C}^2$. In other words, we write

$$\mathbf{M} \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{D} = W \oplus \overline{W}\,, \tag{73}$$

Elements of W^2 are chiral spinors for which

$$\gamma_5 \psi_R = i\psi_R, \tag{74}$$

Elements of $\overline{W}$ are anti-chiral spinors for which

$$\gamma_5 \psi_L = -i\psi_L, \tag{75}$$

The projectors $\frac{1}{2}(1 - i\gamma_5)$ and $\frac{1}{2}(1 + i\gamma_5)$ project onto chiral and anti-chiral Weyl spinors respectively.

It is of course possible to treat Weyl spinors without the explicit introduction of complex numbers at the expense of introducing pairs of Majorana spinors ψ_1, ψ_2 subject to the constraint that

$$\gamma_5 \psi_1 = -\psi_2 \,, \qquad \gamma_5 \psi_2 = \psi_1 \,. \tag{76}$$

One then has

$$\psi_R = \psi_1 + i\psi_2 \,, \psi_L = \psi_1 - i\psi_2 \,. \tag{77}$$

Unification and Spin(10).

If the viewpoint advocated here is on the right track, one might expect that should be signs in what little information we have about possible unification schemes. A very popular one is based on the group $SO(10)$ and it is perhaps gratifying that it seems to fit with the philosophy espoused here.

In the standard electro-weak model, the neutrinos are purely left-handed and a description of the fundamental degrees of freedom in terms of Weyl spinors is often felt to be appropriate. One may then argue that this more more convenient with the mainly minus signature. However nothing prevents one describing it using Majorana notation and the mainly plus signature. Moreover the discovery of the non-zero neutrino masses and the so-called see-saw mechanism make it plausible that there is a right handed partner for the neutrinos and the fact that then each family would fit into a chiral (i.e. $\mathbf{16}$) representation of Spin(10) makes it perhaps more attractive to describe the fundamental fields in Majorana notation. This would tend to favour the use of the mainly plus signature.

To see this in more detail recall *

$$\mathsf{Cliff}(10,0;\mathbb{R}) \equiv M_{32}(\mathbb{R})\,. \tag{78}$$

Let Γ_a, $a = 1, 2, \ldots, 10$ be a representation of the generators by real 32×32 matrices and

$$\Gamma_{11} = \Gamma_1 \Gamma_2 \ldots \Gamma_{10}\,, \tag{79}$$

so that †

$$\Gamma_{11}^2 = -1\,. \tag{80}$$

*This is clear from the periodicity modulo eight of Clifford algebras $\mathsf{Cliff}(s+8,t) \equiv \mathsf{Cliff}(s,t) \otimes M_{16}(\mathbb{R})$ and the easily verified fact that the that $\mathsf{Cliff}(2,0;\mathbb{R}) \equiv M_2(\mathbb{R})$.

†The matrices Γ_a, Γ_{11} generate the M-theory Clifford algebra $\mathsf{Cliff}(10,1;\mathbf{R}) \equiv M_{32}(\mathbf{R}) \oplus M_{32}(\mathbf{R})$.

It is customary to describe the Spin(10) model in terms of 16 left handed spacetime Weyl fermions which are then placed in a single complex chiral $\mathbf{16}$, Ψ of Spin(10)

$$\Gamma_{11}\Psi = i\Psi\,, \tag{81}$$

but this is completely equivalent ,and notationally simpler to regard the 16 spacetime Weyl fermions as 32 spacetime Majorana fermions and then to regard Ψ as a 32 dimensional Majorana spinor of Spin(10) subject to the constraint

$$\Gamma_{11}\Psi = \gamma_5\Psi\,. \tag{82}$$

In yet more detail, we start with the 15 observed left handed Weyl fermions of the electro-weak theory with their weak hypercharges $Y = Q - t_3$, where Q is the electric charge and t_3 the third component of weak iso-spin

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, Y = \frac{1}{6} \qquad \begin{pmatrix} v_L \\ e_L \end{pmatrix}, Y = -\frac{1}{2} \tag{83}$$

$$u_L^c \, , Y = -\frac{2}{3} \qquad d_L^c \, , Y = \frac{1}{3} \qquad e_L^c \, , \, , Y = 1 \, . \tag{84}$$

The first row consists of 4 iso-doublets and the second row of 7 iso-singlets. The up and down quarks u_L and d_L are in a $\mathbf{3}$ of $SU(3)$ colour and their charge conjugates u_L^c, d_L^c are in a $\bar{\mathbf{3}}$ of $SU(3)$. In fact the, because effective group is $S(U(3) \times U(2)) \equiv SU(3) \times SU(2) \times$

$U(1)/\mathbf{Z}_3 \times \mathbf{Z}_2$, where $\mathbf{Z}_3$ and $\mathbf{Z}_2$ are the centres of $SU(3)$ and $SU(2)$ respectively. This is because the electric charge assignments are such that acting with $\mathbf{Z}_3 \times \mathbf{Z}_2 \equiv \mathbf{Z}_6$ can always be compensated by a $U(1)$ rotation.

Now $S(U(3) \times U(2))$ is a subgroup of $SU(5)$ and is well known one may fit all 15 left handed Weyl spinors in a $\mathbf{5}$ and a $\mathbf{10}$. However it is more elegant to adjoin the charge conjugate of the right handed neutrino, ν_L^c to make up a complex $\mathbf{10}$ of Spin(10). In fact the multiplets may be organized into multiplets of the Spin(6) $\times$ Spin(4) $\equiv SU(4) \times SU(2) \times SU(2)$ subgroup of Spin(10)

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}. \tag{85}$$

$$\begin{pmatrix} u_L^c \\ d_L^c \end{pmatrix} , \qquad \begin{pmatrix} \nu_L^c \\ e_L^c \end{pmatrix} . \tag{86}$$

In this formalism we have left-right symmetry with the first row consisting of 4 weak iso-doublets and the bottom row of 4 doublets of some other, as yet unobserved $SU(2)$. The quarks and leptons also form two Spin(6) $\equiv SU(4)$ quartets.

Colliding-Brane Cosmologies are now popular. Before collision each brane has its own complex quantum mechanics. The brane colliosion produces particles (see e.g. Fermions on colliding branes. Gary Gibbons (Cambridge U., DAMTP) , Kei-ichi Maeda (Cambridge U., DAMTP & Waseda U.) , Yu-ichi Takamizu (Waseda U.) . Oct 2006. 8pp. Published in Phys.Lett.B647:1-7,2007. e-Print: hep-th/0610286)

There is here a potential clash of different complex structures.

$$\{V_1, J_1\} \oplus \{V_2, J_2\} \rightarrow \{V_3, J_3\} \tag{87}$$

$$J_3 = J_1 \oplus J_2\,? \tag{88}$$

If

$$J_1 \oplus J_2 \neq J_3 \tag{89}$$

one gets a non-trivial Bogoliubov transformations and hence particle production

Our universe could have begun this way!

We have see in this lecture that

- Time and its arrow are intimately linked with the complex nature of quantum mechanics.

- It is not difficult to construct spacetimes for which no arrow of time exists and on which which backgrounds only real quntum mechanics is possble

- Only Riemannian mfds admitting a reflection map Θ allow the recovery of standard quantum mechanics

- Even if one can define an arrow of time it may not be possible to define an operator Θ which reverses it.

Why then do we have such a strong impression that time exists and that it has an arrow? When and how did the complex numbers get into quantum mechanics?

Like so many things in life: its all a matter of history. The universe "started " with very special initial conditions "when "neither time nor quantum mechanics were present. Both are emergent phenomena. Both are consequences of the special state we find ourselves in.

Constructing and understanding that state, and its alternatives is the on-going challenge of Quantum Cosmology.