Sim(n-2): Deformations, Holonomy and Quantum Corrections

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This lecture is based on three recent papers.

- General very special relativity is Finsler geometry. G.W. Gibbons Joaquim Gomis, C.N. Pope Published in *Phys.Rev*.**D76** :081701,2007. e-Print: arXiv:0707.2174 [hep-th]

- Time-dependent multi-centre solutions from new metrics with holonomy SIM(n-2) G.W. Gibbons, C.N. Pope e-Print: arXiv:0709.2440 [hep-th]

- Metrics With Vanishing Quantum Corrections, A.A. Coley G.W. Gibbons, S. Hervik and C.N. Pope e-Print: arXiv:0803.2438 [hep-th]

Reduced holonomy groups typically arise when the manifold admits a tensor or spinor field which is invariant under parallel translation. For Riemannian metrics this may be a spinor field as in Berger's List of the 1950'2.

- $SU(k) \subset SO(2k)$: Ricci flat Kähler

- $Sp(k) \subset SO(4k)$: Hyperkähler

- $G_2 \subset SO(7)$

- $Spin(7) \subset SO(8)$.

Bilinears in spinors $\bar{\psi}\gamma_{\mu_1\mu_2\dots\mu_p}\psi$ then give rise to covariantly constant p-form fields However in the Riemannian case parallel transport can never leave invariant a vctor field n, except in the trivial case when the metric splits as the metric product with a one-dimensional factor. Even less can it leave invariant a direction field, i.e. a vector field n^ν up to scale so that n^μ is a recurrent vector field

$$\nabla_\mu n^\nu = B_\mu n^\nu \qquad \Leftrightarrow \qquad \nabla_X n = B(X),\ n \tag{1}$$

for some recurrence one-form B_μ.

To see why, recall that ∇ preserves length whence $n_\nu \nabla_\mu n^\nu = 0$, so if $B_\mu \neq 0$, then

$$n_\nu n^\nu = 0\,, \tag{2}$$

Thus n^ν must be a lightlike vector field.

If such a vector field exists, the holonomy must be contained in $Sim(n-2) = R \rtimes E(n-2)$ the semi-direct product group of isometries and dilatations of the $(n-2)$-plane E^{n-2}. This may be familiar from Wigner's little group of a light-like vector which is $E(n-2)$.

In fact $Sim(n-2)$ is the maximal subgroup of the Lorentz group $SO(n-1,1)$.

In the special case that $B_\mu = 0$ the null vector n is covariantly constant and we obtain a (if the Ricci tensor vanishes) a Brinkmann wave known since 1923. The holonomy is then contained in $E(n-2)$ and for gravitational waves in the translation subgroup R^{n-2}.

In that case there is a covariantly constant spinor and the solutions are BPS, of which more later.

If $n = 4$ we can be more explicit since $Spin(3,1) = SL(2,C)$ Elements of $Sim(2)$ are of the form

$$\begin{pmatrix} \lambda & a \\ 0 & \lambda^{-1} \end{pmatrix} . \tag{3}$$

The action on the Weyl spinor is

$$\psi = \begin{pmatrix} u \\ 0 \end{pmatrix} \rightarrow \lambda \begin{pmatrix} u \\ 0 \end{pmatrix} = \lambda\psi \tag{4}$$

and we need $\lambda = 0$ if ψ is to be covariantly constant. Four-vectors correspond to Hermitean matrices

$$x = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \tag{5}$$

and

$$\psi\psi^\dagger = \begin{pmatrix} |u|^2 & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

and the null vector $n = (1, 0, 0, 1)$ is invariant in direction but scales like $|\lambda|^2$.

The eight dimensional subgroup $ISim(2) = Sim(2) \ltimes R^4 \subset ISO(3,1)$ of the Poincaré group $ISO(3,1) = E(3,1)$ i.e. $Sim(2)$ with spacetime translations added and dubbed

Very Special Relativity

is Cohen and Glashow's proposal for the spacetime symmetry group in the presence of a particular kind of

Lorentz violation without spurion fields,

that is with no vacuum expectation value for any vector or tensor field.

One may study deformations of $ISim(2)$ using Lie algebra cohomology techniques.

Gomis and Pope and I found that, unlike the case of the Poincaré group which deforms to one of the two De-Sitter groups, one cannot deform $ISim(2)$ to incorporate a cosmological constant.

However an interesting deformation $DISim_b(2)$ does exist, and this leads naturally to Finsler Geometry

$$L(v) = (v_\mu n^\mu)^b (\sqrt{-\eta_{\sigma\tau} v^\sigma v^\tau})^{(1-b)} \tag{7}$$

This is one motivation for studying metrics with $Sim(n-2)$ holonomy.

By rescaling n^ν we can arrange that

$$\nabla_\mu n^\nu = \kappa n^\nu n_\mu \,. \tag{8}$$

It follows that n_μ is geodesic and expansion-free and both normal and tangent (but not transverse !) to a null hypersurface = const , i.e. is vorticity free . The metric then takes the form first written down by A.G Walker (of Robertson Walker fame) in 1950

$$ds^2 = 2du \left[dv + H(v, u, x^i) du + A_i(u, x^k) dx^i \right] + g_{ij}(u, x^k) dx^i dx^j , \tag{9}$$

$$n = \frac{\partial}{\partial v} \,. \tag{10}$$

Note that if $H(v, u, x^i)$ is independent of v then $n = \frac{\partial}{\partial v}$ is a covariantly constant null vector and we are back to the case of a Brinkmann wave studied in the 1920's.

We now impose the Einstein condition $R_{\mu\nu} = \Lambda g_{\mu\nu}$.

This leads to

$$H(u, v, x^i) = \Lambda v^2 + vH_1(u, x^i) + H_0(u, x^i) \tag{11}$$

We now find that once we have a, possibly u-dependent, solution of

$$R_{ij} = \Lambda g_{ij} \tag{12}$$

the remaining equations form a linear system in the sense that they can be integrated successively using the inverses of linear operators!

In other words we need to solve

$$
\begin{aligned}
\nabla^2 H_0 - \frac{1}{2}F^{ij}F_{ij} - 2A^i\partial_i H_1 - H_1\nabla^i A_i + 2\Lambda A^i A_i - 2\nabla^i \dot{A}_i & \\
+\frac{1}{2}\dot{g}^{ij}\dot{g}_{ij} + g^{ij}\ddot{g}_{ij} + \frac{1}{2}g^{ij}\dot{g}_{ij}H_1 &= 0\,, \quad (13)\\
\nabla^j F_{ij} + \partial_i H_1 - 2\Lambda A_i + \nabla^j \dot{g}_{ij} - \partial_i(g^{jk}\dot{g}_{jk}) &= 0\,, \quad (14)\\
\nabla^2 H_1 - 2\Lambda\nabla^i A_i + \Lambda g^{ij}\dot{g}_{ij} &= 0\,. \quad (15)
\end{aligned}
$$

If $\Lambda = 0$ and $n = 4$ we recover the metrics of Goldberg and Kerr of 1961. If $\Lambda < 0$ and $n = 4$ we find new metrics.

These generalise and correct metrics of Ghanam and Thompson. Set $\Lambda = -2$

$$ds_4^2 = 2dudv + \frac{dx^2 + dy^2}{2x^2} - [2v^2 + H(x,y)]du^2\,, \tag{16}$$

where $H(x,y)$ is an arbitrary harmonic on the upper half plane

If $n = 5$ our solutions include Kaluza-Klein electropoles with NUT charges depdending on two arbitrary harmonic functions on E^3.

$$ds_4^2 = -H^{-1/2}(dv + A)^2 + H^{1/2} dx^i dx^i \,, \tag{17}$$

with

$$\nabla \cdot \mathbf{A} = 0 \,, \qquad \nabla \times \mathbf{A} = \nabla V \,, \qquad H = U + \tfrac{1}{2} V^2 \,, \tag{18}$$

$$\nabla^2 U = 0 \,, \qquad \nabla^2 V = 0 \,. \tag{19}$$

Performing a discrete electric-magnetic duality gives one metrics representing Kaluza-Klein magnetic monopoles with NUT charges and angular momentum.

Taking time dependent harmonic functions which allow boost symmetry, gives time dependent metrics in four dimensions representing collisions of Klauza-Klein magnetic monopoles, which generalise some recent colliding brane metrics.

A special case of our metrics are vaccum pp-waves. These have $\Lambda = 0$ and $g_{ij} = \delta_{ij}$. It has been known for some time that these have no quantum corrections

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G. W. Gibbons, Quantized Fields Propagating In Plane Wave Space-Times, *Commun. Math. Phys.* **45** (1975) 191.

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So what about metrics with $Sim(n-2)$ holonomy?

The corrected Einstein equations are assumed to take the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} - \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{20}$$

where $T_{\mu\nu}$ is conserved and constructed from the the metric and Riemann tensor and its covariant derivatives.

We ask whether a classical metric $g_{\mu\nu}$ (possibly rescaled by a constant factor h) solves the full quantum corrected equations e.o.m. This requires that

$$T_{\mu\nu}(h g_{\rho\sigma}) = F(h) g_{\mu\nu} \tag{21}$$

for some function $F(h)$ and that h may be chosen to satisfy

$$\Lambda(h-1) = 2F(h)\,. \tag{22}$$

A sufficient condition is that it hold for *any* symmetric conserved tensor constructed from the classical metric $g_{\mu\nu}$ and its derivatives. We call this condition *Weak Universality* because, subject to there being real solutions of the algebraic equation, we can simply rescale the metric to get a solution of *any* set of corrected field equations.

An example of a weakly universal metric is a maximally symmetric space, such as de Sitter spacetime, for which

$$R_{\mu\nu\sigma\tau} = c(g_{\mu\sigma}g_{\nu\tau} - g_{\nu\sigma}g_{\mu\tau})\,, \tag{23}$$

where c is necessarily a constant, and hence

$$R_{\mu\nu\sigma\tau;\lambda_1;\ldots;\lambda_k} = 0 \tag{24}$$

for arbitrary integers k. It follows that c must satisfy

$$T_{\mu\nu} = f(c)g_{\mu\nu}\,, \tag{25}$$

for some function $f(c)$, and if a value of c can be found satisfying

$$f(c) + \frac{(n-1)(n-2)}{4}c = 0\,, \tag{26}$$

we have a solution.

For pp-waves waves the Ricci tensor $R_{\mu\nu}$ vanishes, and hence the classical value of Λ vanishes. Horowitz and Steif showed, when evaluated on a pp- wave background, all other conserved tensors $T_{\mu\nu}(g_{\rho\sigma})$ (except of course the metric itself) will vanish. Thus, in distinction to the case of maximally symmetric spaces such as de Sitter spacetime, no rescaling of the metric is required when passing from the classical to the quantum-corrected metric.

Such metrics we call *Strongly Universal*

To make the notion of strong universality precise we need to consider, for a general spacetime, the vector space (over real constants) of all symmetric conserved second rank tensors constructed from the metric and its derivatives, modulo constant multiples of the metric itself. If, when restricted to a classical metric $g_{\mu\nu}$ such as a pp wave, all such tensors vanish, then we say that the classical metric is *Strongly Universal*.

Bleecker has defined *Critical Metrics*,if they solve the field equation derived from any diffeomorphism invariant action functional, which in our case means

$$\frac{1}{4}\int\sqrt{|g|}\,(R-2\Lambda)+I(g) \tag{27}$$

such that

$$T_{\mu\nu}=-\frac{2}{\sqrt{|g|}}\frac{\delta I(g)}{\delta g^{\mu\nu}}. \tag{28}$$

where $\int\sqrt{|g|}I(g)$ is an effective action. All of our heirarchy of conditions are stronger than the mere vanishing of $I(g)$ on shell. cf corrections to Calabi-Yau's which take the metric outside the Calabi-Yau class

Note however that universal metrics are necessarily critical.

We have examined the new metrics and find that in four dimensions

- All metrics with $Sim(2)$ holonomy are weakly universal and hence critical

- Goldberg Kerr (Ricci flat) metrics with $Sim(2)$ holonomy are strongly universal

Note that these result are not a consequence of supersymmetry.