Precision Counting of Black Hole Microstates

Motivation:

Low energy limit of string theory gives rise to gravity coupled to other fields.

These theories typically have black hole solutions.

Thus string theory gives a framework for studying classical and quantum properties of black holes.

One of the important properties characterizing a black hole is the Bekenstein-Hawking entropy S_{BH}.

In the low energy limit

$$S_{BH} = A/(4G_N)$$

For a wide class of extremal (BPS) black holes

$$S_{BH} = S_{stat}, \qquad S_{stat} \equiv \ln(Degeneracy)$$

Strominger,Vafa

This gives a good understanding of this entropy from microscopic viewpoint.

Originally the comparison between black hole and statistical entropy was carried out in the limit of large charges.

In this limit the curvature at the horizon is small and hence we can ignore higher derivative corrections to the effective action in computing the black hole entropy.

On the microscopic side we can use appropriate asymptotic formula for the degeneracy of states to calculate the statistical entropy.

Given this success, it is natural to carry out our study of black holes to finer details.

What are the effects of higher derivative corrections to the black hole entropy?

How are these related to the finite charge corrections to the statistical entropy?

In order to address this problem we need to address two issues.

First of all we need to learn how to take into account the effect of the higher derivative terms on the computation of black hole entropy.

We also need to know how to calculate the statistical entropy to greater accuracy.

In the first 2-3 lectures we shall mainly address the second problem in the context of $\mathcal{N} = 4$ supersymmetric string theories.

Later I shall try to make contact with the known results for black holes.

Plan

1. General properties of dyon partition function in $\mathcal{N} = 4$ SUSY string theories

   This will contain review of known results without derivation.

2. Special case: Heterotic string theory on T^6

   This will illustrate how these general properties are verified in this special case.

3. Relation to black holes

General properties of dyon partition function in $\mathcal{N} = 4$ SUSY string theories

Banerjee,A.S,,Srivastava, arXiv:0802.0544

A generic $\mathcal{N} = 4$ supersymmetric string theory in $D = 4$ has several U(1) gauge fields.

$R \equiv$ no.of $U(1)$ gauge fields. $(R \geq 6)$

6 graviphotons + $(R - 6)$ matter multiplets

A generic state carries (electric,magnetic) charges (Q, P)

Q, P: R-dimensional vectors

There are also two sets of moduli scalar fields:

a complex scalar modulus τ: $\Im(\tau) > 0$

$6(R-6)$ real scalars labelled by $R \times R$ matrix M subject to the constraint

$$M^T = M, \quad M^T L M = L$$

L: a matrix with 6 eigenvalues 1 and $(R-6)$ eigenvalues -1

These, together with the metric and the U(1) gauge fields, give all the massless bosonic fields.

This theory has quarter BPS states carrying charges (Q, P).

They are invariant under 4 of the 16 supercharges.

Their masses are determined using the BPS formula:

$$m(\vec{Q},\vec{P})^2 \;=\; \frac{1}{\Im(\tau)}(Q-\bar{\tau}P)^T(M+L)(Q-\tau P) +2\left[(Q^T(M+L)Q)(P^T(M+L)P) -(P^T(M+L)Q)^2\right]^{1/2}$$

$d(Q,P)$: number of quarter BPS states with charge (Q,P) weighted by $(-1)^F(2h)^6/6!$

F: fermion number, h: helicity Kiritsis

$d(Q,P)$ vanishes for non-BPS states but is not zero for quarter BPS states.

For a supermultiplet $d(Q,P) = (-1)^{2\langle h\rangle}$

$\langle h\rangle$: average helicity of the supermultiplet

Thus we expect that as we vary the moduli continuously $d(Q,P)$ will not change.

$\rightarrow$ a non-vanishing and protected index

At a generic point in the moduli space only $d(Q,P)$ number of states is forced to remain BPS.

Thus $\ln d(Q,P)$ should be compared with entropy of a BPS black hole.

The above story is not completely correct.

On some codimension 1 subspaces of the moduli space a BPS state may become marginally unstable against decay into a pair of BPS states.

$$\begin{aligned} & m_{BPS}(Q, P; \tau, M) \\ = \; & m_{BPS}(Q_1, P_1; \tau, M) + m_{BPS}(Q_2, P_2; \tau, M) \end{aligned}$$

$$Q = Q_1 + Q_2, \quad P = P_1 + P_2$$

On these walls there is no gap separating the BPS state of charge (Q, P) from the continuum.

$d(Q, P)$ can jump across these walls.

$$(Q, P) \Rightarrow (Q_1, P_1) + (Q_2, P_2)$$

For quarter BPS $\Rightarrow$ half-BPS + half-BPS

$$Q_1 \parallel P_1, \qquad Q_2 \parallel P_2$$

$$(Q, P) \Rightarrow (\alpha Q + \beta P, \gamma Q + \delta P) + (\delta Q - \beta P, -\gamma Q + \alpha P)$$

$$\alpha\delta = \beta\gamma, \qquad \alpha + \delta = 1$$

$\alpha, \beta, \gamma, \delta$ must be consistent with charge quantization laws.

$\rightarrow$ can take discrete values.

For fixed M, the walls in the τ plane are circles or straight lines.

The moduli space is divided up into domains bounded by these walls of marginal stability.

$d(Q,P)$ depends not only on (Q,P) but also on the domain in which the moduli lie.

Consider the ith wall bordering a domain:

$$(Q,P) \Rightarrow (\alpha_i Q+\beta_i P, \gamma_i Q+\delta_i P)+(\delta_i Q-\beta_i P, -\gamma_i Q+\alpha_i P)$$

$$\alpha_i\delta_i = \beta_i\gamma_i, \qquad \alpha_i + \delta_i = 1$$

We shall label a domain by:

$$\vec{c}: \{(\alpha_i, \beta_i, \gamma_i, \delta_i)\}$$

Note: $\{(\alpha_i, \beta_i, \gamma_i, \delta_i)\}$ are discrete parameters due to charge quantization.

T-duality transformation:

$$Q \to \Omega Q, \quad P \to \Omega P, \quad M \to \Omega M \Omega^T$$

$\Omega \in$ a discrete subgroup of $O(6, R-6)$.

$$(Q,P) \Rightarrow (\alpha_i Q + \beta_i P, \gamma_i Q + \delta_i P) + (\delta_i Q - \beta_i P, -\gamma_i Q + \alpha_i P)$$

$(\alpha_i, \beta_i, \gamma_i, \delta_i)$ are T-duality invariant.

S-duality transformation:

$$Q \to aQ+bP, \quad P \to cQ+dP, \quad \tau \to (a\tau+b)/(c\tau+d)$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ a discrete subgroup of SL(2,R).

$$(Q,P) \Rightarrow (\alpha_i Q+\beta_i P, \gamma_i Q+\delta_i P)+(\delta_i Q-\beta_i P, -\gamma_i Q+\alpha_i P)$$

$$\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

Review

We considered an $\mathcal{N}=4$ supersymmetric string theory with R U(1) gauge fields.

$\rightarrow$ a state a characterized by R dimensional electric charge vector Q and R dimensional magnetic charge vector P.

$d(Q,P)$: an index that counts the number of quarter BPS supermultiplets carrying charge (Q,P), weighted by $(-1)^{2\langle h\rangle}$.

Once we take into account all interactions we expect that only $d(Q,P)$ supermultiplets will remain BPS.

$d(Q,P)$ does not change under continuous variation of the moduli but could jump as we cross walls of marginal stability on which

$$\begin{aligned} m_{BPS}(Q,P) &= m_{BPS}(\alpha Q+\beta P, \gamma Q+\delta P) \\ &\quad +m_{BPS}(\delta Q-\beta P, -\gamma Q+\alpha P) \end{aligned}$$

$$\alpha\delta = \beta\gamma, \qquad \alpha+\delta = 1$$

$\alpha, \beta, \gamma, \delta$ take discrete values consistent with charge quantization.

We characterize a domain bounded by walls of marginal stability by the $(\alpha, \beta, \gamma, \delta)$ values of all the walls bordering the domain.

$$\vec{c} = \{\alpha_i, \beta_i, \gamma_i, \delta_i\}$$

T-duality transformation:

$$Q \to \Omega Q, \quad P \to \Omega P, \quad M \to \Omega M \Omega^T$$

$(\alpha_i, \beta_i, \gamma_i, \delta_i)$ are T-duality invariant.

S-duality transformation:

$$Q \to aQ + bP, \quad P \to cQ + dP, \quad \tau \to (a\tau + b)/(c\tau + d)$$

$$\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

T-duality invariance $\to d(Q,P;\vec{c}) = d(\Omega Q, \Omega P; \vec{c})$

Thus $d(Q,P;\vec{c})$ should depend on Q and P only through the T-duality invariant combinations

$$Q^2 \equiv Q^T L Q, \quad P^2 \equiv P^T L P, \quad Q\cdot P \equiv Q^T L P$$

$L \equiv O(6, R-6)$ invariant metric

$\vec{u}$: collection of other T-duality invariants

(e.g. gcd($Q \wedge P$)) Dabholkar, Gaiotto, Nampuri

$$d(Q,P;\vec{c}) = f(Q^2, P^2, Q\cdot P; \vec{u}; \vec{c})$$

Define dyon partition function as

$$\begin{aligned}\Psi(\rho,\sigma,v;\vec{u},\vec{c}) &= \sum_{Q^2,P^2,Q\cdot P} (-1)^{Q\cdot P+1} f(Q^2,P^2,Q\cdot P;\vec{u};\vec{c}) \\ &\qquad \exp\left[i\pi(\sigma Q^2+\rho P^2+2vQ\cdot P)\right] .\end{aligned}$$

This sum typically converges in some domain in the complex (ρ,σ,v) space.

The domain of convergence depends on $\vec{c}$.

$\to$ constraints on $\Im(\rho)$, $\Im(\sigma)$, $\Im(v)$

In all known examples,

Ψ is independent of $\vec{c}$.

Inverse Fourier transform:

$$
\begin{aligned}
& f(Q^2, P^2, Q\cdot P; \vec{u}, \vec{c}) \\
\propto \;& (-1)^{Q\cdot P+1} \int_{\mathcal{C}(\vec{c})} d\rho d\sigma dv \, \Psi(\rho, \sigma, v; \vec{u}) \\
& \qquad \exp\left[-i\pi(\sigma Q^2 + \rho P^2 + 2vQ\cdot P)\right] .
\end{aligned}
$$

$\mathcal{C}(\vec{c})$: a three dimensional subspace (contour) at fixed $\Im(\rho)$, $\Im(\sigma)$, $\Im(v)$ where the original sum converges.

Note: The dependence of f on $\vec{c}$ comes only through the choice of the contour.

Thus we have a map from the domains in the moduli space labelled by $\vec{c}$ to the domains in $(\Im(\rho), \Im(\sigma), \Im(v))$ space specifying the choice of the contour $\mathcal{C}(\vec{c})$.

As we cross a wall of marginal stability in the moduli space we pick a new contour $\mathcal{C}(\vec{c})$.

Thus the jump in the index across a wall of marginal stability is given by the residue at the pole picked up during contour deformation.

Consider decay into a pair of half-BPS states

$$(Q, P) \Rightarrow (\alpha Q + \beta P, \gamma Q + \delta P), + (\delta Q - \beta P, -\gamma Q + \alpha P)$$

$$\alpha\beta = \gamma\delta, \qquad \alpha + \delta = 1$$

In all known examples the jump in the index across this wall is controlled by the pole of $\Psi(\rho, \sigma, v; \vec{u})$ at

$$\rho\gamma - \sigma\beta + v(\alpha - \delta) = 0 \, .$$

Consequences of S-duality

$$(Q, P) \to (Q', P') = (aQ + bP, cQ + dP)$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ a discrete subgroup of $SL(2, R)$.

Under this $\vec{u} \to \vec{u}'$, $\vec{c} \to \vec{c}\,'$.

$$f(Q^2, P^2, Q \cdot P; \vec{u}; \vec{c}) = f(Q'^2, P'^2, Q' \cdot P'; \vec{u}'; \vec{c}\,')$$

$$\to \quad \Psi(\rho, \sigma, v; \vec{u}; \vec{c}) = \Psi(\rho', \sigma', v'; \vec{u}'; \vec{c}\,')$$

$$\rho' \equiv d^2\rho + b^2\sigma + 2bdv, \quad \sigma' \equiv c^2\rho + a^2\sigma + 2acv,$$
$$v' \equiv cd\rho + ab\sigma + (ad + bc)v$$

$$\Psi(\rho, \sigma, v; \vec{u}; \vec{c}) = \Psi(\rho', \sigma', v'; \vec{u}'; \vec{c}')$$

$\vec{c}$ independence of Ψ gives

$$\Psi(\rho, \sigma, v; \vec{u}) = \Psi(\rho', \sigma', v'; \vec{u}')$$

For transformations in the subgroup of S-duality group that leaves $\vec{u}$ unchanged,

$$\Psi(\rho, \sigma, v; \vec{u}) = \Psi(\rho', \sigma', v'; \vec{u})$$

In all known cases the inverse partition function $\Psi(\rho, \sigma, v; \vec{u})^{-1}$ transforms as a modular form under a subgroup of the genus two modular group.

Define $\Omega \equiv \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$

Then

$$\Psi((A\Omega + B)(C\Omega + D)^{-1}; \vec{u}) = \det(C\Omega + D)^{-k}\, \Psi(\Omega; \vec{u})$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \subset Sp(2, \mathbb{Z})$$

$$AD^T - BC^T = I, \quad AB^T = BA^T, \quad CD^T = DC^T$$

$\vec{u}$ preserving S-duality transformations form a small subgroup of G with $B = C = 0$.

Special case: Heterotic on T^6

In this case $R = 28$.

16 of the gauge fields come from the original rank 16 gauge group in ten dimensions.

12 of them come from the components $g_{m\mu}$ and $B_{m\mu}$.

m: along the 6 directions of the torus

μ: along the (3+1)D space-time

Thus we have 28-dimensional charge vectors Q and P taking values in the Narain lattice.

M is a 28×28 matrix satisfying

$$M^T = M, \qquad MLM^T = L$$

L is a 28×28 matrix with 6 eigenvalues 1 and 22 eigenvalues -1.

M carries information about g_{mn}, B_{mn} and the components of the sixteen (9+1)D gauge fields along T^6.

Total number = 36 + 16 × 6 = 132

Our first task will be to determine the T-duality invariants.

Consider a pair of charge vectors (Q, P)

Continuous T-duality invariants:

$$Q^2 = Q^T L Q, \quad P^2 = P^T L P, \quad Q \cdot P = Q^T L P$$

What are the discrete T-duality invariants $\vec{u}$?

$\vec{u}$ is a collection of four integers

$$(r_1, r_2, r_3, u_1)$$

$$r_1, r_2, r_3, u_1 \in \mathbb{Z}, \quad r_1, r_2, r_3 > 0, \quad 1 \leq u_1 \leq r_3$$

$$\gcd(r_1, r_2) = 1, \qquad \gcd(u_1, r_3) = 1.$$

Definition of r_1, r_2, r_3, u_1:

If Q is r_1 times a primitive vector, and P is r_2 times a primitive vector, define

$$e_1 = Q/r_1, \qquad e_2 = P/r_2, \qquad r_1, r_2 \in \mathbb{Z}$$

Now take the vector $e_2 - se_1$, $s \in \mathbb{Z}$.

Adjust s to bring $e_2 - se_1$ to the form of t times a primitive vector e_3 with as large an integer t as possible.

Define

$$r_3 = t_{max}, \qquad u_1 = s_{max}$$

This defines r_1, r_2, r_3, u_1.

What about $r \equiv \mathrm{gcd}(Q \wedge P)$?

One finds that

$$r = r_1 r_2 r_3$$

An S-duality transformation on (Q, P) leaves r fixed, and

acts transitively on the set (r_1, r_2, r_3, u_1) subject to the condition $r \equiv r_1 r_2 r_3$ fixed.

$\Gamma^0(r) \subset SL(2, \mathbb{Z})$ leaves (r_1, r_2, r_3, u_1) fixed.

$$\Gamma_0(r): \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad b \in r\mathbb{Z}$$

S-duality acts transitively on the set (r_1, r_2, r_3, u_1) subject to the condition $r \equiv r_1 r_2 r_3$ fixed.

Recall:

$$\Psi(\rho, \sigma, v; \vec{u}) = \Psi(\rho', \sigma', v'; \vec{u}')$$

under S-duality.

$\longrightarrow$ we need to calculate Ψ for one representative $\vec{u}$ for each r.

e.g.

$$r_1 = r, \quad r_2 = r_3 = u_1 = 1$$

Review

We are analyzing heterotic string theory on T^6.

– has 28 U(1) gauge fields

$\rightarrow$ Q and P are 28 dimensional.

Continuous T-duality invariants:

$$Q^2 = Q^T L Q, \quad P^2 = P^T L P, \quad Q \cdot P = Q^T L P$$

Discrete T-duality invariants $\vec{u}$: collection of four integers

$$(r_1, r_2, r_3, u_1)$$

$$r_1, r_2, r_3, u_1 \in \mathbb{Z}, \quad r_1, r_2, r_3 > 0, \quad 1 \leq u_1 \leq r_3$$

$$\gcd(r_1, r_2) = 1, \qquad \gcd(u_1, r_3) = 1.$$

$$\gcd(Q \wedge P) = r_1 r_2 r_3 \equiv r$$

An S-duality transformation on (Q, P) acts transitively on the set (r_1, r_2, r_3, u_1) subject to the condition $r \equiv r_1 r_2 r_3$ fixed.

Recall:

$$\Psi(\rho, \sigma, v; \vec{u}) = \Psi(\rho', \sigma', v'; \vec{u}')$$

under S-duality.

$\rightarrow$ we need to calculate Ψ for one representative $\vec{u}$ for each r.

e.g.

$$r_1 = r, \quad r_2 = r_3 = u_1 = 1$$

$$r = 1 \leftrightarrow r_1 = r_2 = r_3 = u_1 = 1$$

In this case explicit computation of Ψ is possible using a dual description.

– type IIB on $K3 \times S^1 \times \widetilde{S}^1$

The configuration: Shih, Strominger, Yin

1) One Kaluza-Klein monopole along $\widetilde{S}^1$

2) Q_5 D5-brane wrapped on $K3 \times S^1$

3) Q_1 D1-branes wrapped on S^1

4) $-k$ units of momentum along S^1

5) J units of momentum along $\widetilde{S}^1$

– BMPV black hole at the center of Taub-NUT

After translated to the heterotic description, this gives

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2k, \quad Q \cdot P = J$$

$$r = \text{gcd}(Q_1, Q_5, J)$$

We take

$$\text{gcd}(Q_1, Q_5) = 1$$

Thus

$$r = 1$$

We calculate the partition function in weakly coupled IIB theory and then extend it to other domains using S-duality invariance.

In the weakly coupled type IIB description the low energy dynamics of the system is described by three weakly interacting pieces:

1) The closed string excitations around the Kaluza-Klein monopole

2) The dynamics of the D1-D5 center of mass coordinate in the Kaluza-Klein monopole background

3) The relative motion between the D1 and the D5-brane

The dyon partition function is obtained as the product of the partition function of these three subsystems.

Recall that:

ρ is the variable dual to $P^2/2 = Q_5(Q_1 - Q_5)$

σ is the variable dual to $Q^2/2 = k$

v is the variable dual to $Q \cdot P = J$.

We calculate the partition function of each subsystem as a function of (ρ, σ, v) and then take their product.

Low energy dynamics of KK monopole:

$$e^{-2\pi i\sigma}\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma})^{-24}\right\}$$

D1-D5 center of mass motion in KK monopole background:

$$\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma})^{4}\,(1-e^{2\pi in\sigma+2\pi iv})^{-2}\,(1-e^{2\pi in\sigma-2\pi iv})^{-2}\right\}$$
$$\times\, e^{-2\pi iv}\,(1-e^{-2\pi iv})^{-2}$$

Relative motion between the D1 and D5 branes:

$$e^{-2\pi i\rho}\prod_{\substack{l,b,k\in Z\\ k\geq 0,l>0}}\left\{1-\exp(2\pi i(k\sigma+l\rho+bv))\right\}^{-c(4lk-b^2)}$$

Dijkgraaf, Moore, Verlinde, Verlinde

Definition of $c(n)$:

$$F(\tau, z) \equiv 8 \left[\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right]$$

$$F(\tau, z) = \sum_{b \in \mathbb{Z}, n} c(4n - b^2)\, q^n\, e^{2\pi i z b}$$

After taking the product we get

$$\Psi = e^{-2\pi i \rho} \prod_{\substack{l,b,k \in Z \\ k \geq 0, l \geq 0, b < 0 \text{ for } k = l = 0}} \left\{ 1 - \exp(2\pi i (k\sigma + l\rho + bv)) \right\}^{-c(4lk - b^2)}$$

$$\Psi(\rho, \sigma, v; r = 1) = 1/\Phi_{10}(\rho, \sigma, v)$$

Φ_{10}: weight 10 Igusa cusp form of $Sp(2, \mathbb{Z})$.

Dijkgraaf, Verlinde, Verlinde

The counting that leads to the partition function also tells us how we should expand it to extract $d(Q, P)$.

First expand in powers of $e^{2\pi i\rho}$ and $e^{2\pi i\sigma}$.

Then expand each term in powers of $e^{\pm 2\pi i v}$.

Corresponds to the contour choice

$$\Im(\rho), \Im(\sigma) >> |\Im(v)| > 0$$

This prescription suffers from a 2-fold ambiguity.

$\Im(v) > 0$ and $\Im(v) < 0$.

Consider the factor

$$e^{-2\pi iv}(1-e^{-2\pi iv})^{-2}$$

from the D1-D5 center of mass dynamics.

Can be expanded as

$$\sum_{j=1}^{\infty} j\, e^{2\pi ijv} \quad \text{or} \quad \sum_{j=1}^{\infty} j\, e^{-2\pi ijv}$$

It turns out that these two prescription give $d(Q,P)$ in two different domains in the moduli space, both lying inside the weak coupling limit of IIB.

Pope; Gauntlett, Kim, Park, Yi

The two domains in the τ plane

$\rightarrow$ correspond to the contour choice:

$$\Im(\rho), \Im(\sigma) >> \Im(v) > 0$$

and

$$\Im(\rho), \Im(\sigma) >> -\Im(v) > 0$$

In other domains we have different choices of the three dimensional integration contour $\mathcal{C}$.

Note that $d(Q, P)$ is different on the two sides of the wall but the partition function Ψ is described by the same analytic function of ρ, σ, v.

There are infinite number of other domains but we can determine the form of Ψ in these domains using S-duality invariance.

S-duality invariance $\rightarrow$

$$\Psi(\rho,\sigma,v;\vec{c}) = \Psi(\rho',\sigma',v';\vec{c}\,')$$

$$\rho' \equiv d^2\rho + b^2\sigma + 2bdv, \quad \sigma' \equiv c^2\rho + a^2\sigma + 2acv,$$
$$v' \equiv cd\rho + ab\sigma + (ad+bc)v$$

Choose $\vec{c}$ to be any one of the two domains in which we have computed Ψ.

Explicit computation shows that

$$\Psi(\rho,\sigma,v;\vec{c}) = \Psi(\rho',\sigma',v';\vec{c})$$

Thus

$$\Psi(\rho',\sigma',v';\vec{c}\,') = \Psi(\rho',\sigma',v';\vec{c})$$

$\rightarrow$ Ψ is independent of the domain $\vec{c}$.

$r > 1$ cases

Direct computation will require choosing

$$\text{gcd}(Q_1, Q_5, J) = r$$

$\rightarrow$ dynamics of D1-D5 system not well understood.

Instead we try to guess the form of the partition function using various consistency requirements.

Take the representative

$$r_1 = r, \quad r_2 = r_3 = u_1 = 1$$

A generic S-duality transformation takes it to other (r_1, r_2, r_3, u_1) with $r_1 r_2 r_3 = r$.

Subgroup of S-duality preserving $(r, 1, 1, 1)$ is $\Gamma^0(r)$.

Thus we expect the partition function to be invariant under $\Gamma^0(r)$.

The guess for $\Psi(\rho, \sigma, v; r, 1, 1, 1)$

Banerjee,AS,Srivastava

$$\sum_{\substack{s \in \mathbb{Z}, s|r \\ \bar{s} \equiv r/s}} s \frac{1}{\bar{s}^3} \sum_{k=0}^{\bar{s}^2-1} \sum_{l=0}^{\bar{s}-1} \Phi_{10}\left(\rho, s^2\sigma + \frac{k}{\bar{s}^2}, sv + \frac{l}{\bar{s}}\right)^{-1}$$

Effectively

$$d(Q, P; r) = \sum_{s|r} s\, d(Q/s, P; r = 1)$$

A possible derivation has been suggested recently using D1-D5 moving in the background of multiple KK monopole.

Dabholkar, Gomes, Murthy

It satisfies various consistency conditions.

1. It is invariant under S-duality group $\Gamma^0(r)$.

2. We can recover the known dyon spectrum in $\mathcal{N} = 4$ SUSY Yang-Mills theory by going to appropriate regions in the moduli space where we have enhanced non-abelian gauge symmetry.

3. In the large charge limit we get the correct form of the black hole entropy.

4. Jump in the spectrum across walls of marginal stability agrees with the one computed from black holes.

Comparison with black hole entropy

For this we need to study the behaviour of $d(Q,P)$ for large charges.

Goal: develop a systematic procedure for determining the asymptotic expansion of $d(Q,P)$ in inverse powers of the charges.

$$d(Q,P;\vec{u},\vec{c}) \propto (-1)^{Q\cdot P+1} \int_{\mathcal{C}(\vec{c})} d\rho d\sigma dv\, \Psi(\rho,\sigma,v;\vec{u}) \\ \exp\left[-i\pi(\sigma Q^2 + \rho P^2 + 2vQ\cdot P)\right] .$$

To extract the large charge behaviour we deform the contour to the region

$$\Im(\sigma), \Im(\rho), \Im(v) \sim \frac{1}{\text{charge}}$$

The deformed contour does not give contribution growing as $\exp(\text{charge}^2)$.

Thus the exponentially growing contribution relevant for computation of black hole entropy comes from the poles the contour crosses during this deformation.

We need to identify the pole that contributes to the leading asymptotic expansion.

In all known examples the leading asymptotic growth comes from a pole at

$$\rho\sigma - v^2 + v = 0$$

Dijkgraaf,Verlinde,Verlinde
Cardoso,de Wit, Kappelli,Mohaupt

Result of picking up residue at this pole:

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

for some function $F(\rho,\sigma)$.

Next we do the ρ and σ integral using saddle point approximation.

Define $W(\vec{J})$ through

$$e^{W(\vec{J})} = \int d\rho d\sigma e^{-F(\rho,\sigma)+J_1\rho+J_2\sigma}$$

Then

$$e^{W(\vec{0})} = d(Q,P)$$

Define $\widehat{\rho}$, $\widehat{\sigma}$, $\Gamma(\widehat{\rho},\widehat{\sigma})$ through

$$\widehat{\rho} = \frac{\partial W(\vec{J})}{\partial J_1}, \quad \widehat{\sigma} = \frac{\partial W(\vec{J})}{\partial J_2}$$

$$\Gamma(\widehat{\rho},\widehat{\sigma}) = J_1\widehat{\rho} + J_2\widehat{\sigma} - W(\vec{J})$$

$$\widehat{\rho} = \partial W(\vec{J})/\partial J_1, \quad \widehat{\sigma} = \partial W(\vec{J})/\partial J_2$$

$$\Gamma(\widehat{\rho}, \widehat{\sigma}) = J_1 \widehat{\rho} + J_2 \widehat{\sigma} - W(\vec{J})$$

Then

$$J_1 = \partial \Gamma / \partial \widehat{\rho}, \quad J_2 = \partial \Gamma / \partial \widehat{\sigma}$$

If $\partial \Gamma / \partial \widehat{\rho} = \partial \Gamma / \partial \widehat{\sigma} = 0$ at $(\widehat{\rho}, \widehat{\sigma}) = (\widehat{\rho}_0, \widehat{\sigma}_0)$ then

$$\Gamma(\widehat{\rho}_0, \widehat{\sigma}_0) = -W(\vec{0}) = -\ln d(Q, P)$$

If $\partial\Gamma/\partial\hat{\rho} = \partial\Gamma/\partial\hat{\sigma} = 0$ at $(\hat{\rho},\hat{\sigma}) = (\hat{\rho}_0,\hat{\sigma}_0)$ then

$$\Gamma(\hat{\rho}_0,\hat{\sigma}_0) = -W(\vec{0}) = -\ln d(Q,P)$$

Thus $\ln d(Q,P)$ is the value of $-\Gamma(\hat{\rho},\hat{\sigma})$ at its extremum.

$-\Gamma(\hat{\rho},\hat{\sigma})$ can be called the statistical entropy function.

On the other hand Γ can be calculated by summing over 1PI Feynman diagrams in the 0-dimensional quantum field theory with action $F(\rho,\sigma)$.

Loop expansion parameter: Inverse charge

Example: Heterotic string theory on T^6

Define (a,S) through

$$\widehat{\rho} = i/(2S), \qquad \widehat{\sigma} = i(a^2 + S^2)/(2S)$$

$$\begin{aligned} -\Gamma(a,S) &= \frac{\pi}{2}\Big[\left(\frac{Q^2}{S} + \frac{P^2}{S}(S^2 + a^2) - 2\,\frac{a}{S}\,Q\cdot P\right) \\ &\quad + 128\,\pi\,\phi(a,S)\Big] + \mathcal{O}(Q^{-2}, P^{-2}) \end{aligned}$$

$$\phi(a,S) = -\frac{3}{16\pi^2}\Big(\ln S + 4\ln|\eta(a+iS)|\Big)$$

Statistical entropy = value of $-\Gamma$ at its extremum with respect to a and S.

How good is the asymptotic formula?

Q^2	P^2	$Q \cdot P$	$d(Q,P)$	S_{stat}	$S_{stat}^{(0)}$	$S_{stat}^{(1)}$
2	2	0	50064	10.82	6.28	10.62
4	4	0	32861184	17.31	12.57	16.90
6	6	0	16193130552	23.51	18.85	23.19
6	6	1	11232685725	23.14	18.59	22.88
6	6	2	4173501828	22.15	17.77	21.94
6	6	3	920577636	20.64	16.32	20.41
6	6	4	110910300	18.52	14.05	18.40

How does this result compare with the entropy of a BPS black hole carrying the same set of charges?

In the presence of higher derivative corrections we must use Wald's formula for black hole entropy.

For extremal black holes this can be implemented via the entropy function formalism.

Algorithm:

1. Write down the most general near horizon background consistent with the symmetries of $AdS_2 \times S^2$.

– parametrized by sizes of AdS_2 and S^2, constant vev of the scalar fields, and near horizon electric and magnetic fields.

2. Evaluate the lagrangian density in this near horizon background.

3. Take the Legendre transform with respect to the near horizon electric field, identifying the conjugate variables as electric charges.

– the black hole entropy function.

4. Extremize the entropy function with respect to the sizes of AdS_2 and S^2 and the near horizon scalar field vevs.

This gives the black hole entropy.

If we take $\mathcal{N} = 4$ supergravity and extremize the corresponding entropy function with respect to all the parameters except the vev (a, S) of the axion-dilaton field, the result is

$$\frac{\pi}{2}\Big[\left(\frac{Q^2}{S} + \frac{P^2}{S}(S^2 + a^2) - 2\,\frac{a}{S}\,Q \cdot P\right)\Big]$$

This coincides with the leading contribution to the statistical entropy function.

Extremization with respect to a and S

$\to$ identical results for black hole and statistical entropy.

Addition of Gauss-Bonnet correction to the supergravity action

$\rightarrow$ additional corrections to the black hole entropy function:

$$-12\pi^2 \left(\ln S + 4 \ln |\eta(a+iS)| \right)$$

This is identical to the order charge^0 correction to the statistical entropy function.

After extremization we get the same result for black hole and statistical entropy.

However it is not clear why the other order α' corrections to the supergravity action do not contribute to black hole entropy.

Recall that $d(Q, P)$ changes across walls of marginal stability.

Can we see these changes on the black hole side?

In the large charge limit these changes are exponentially suppressed compared to the leading term.

Thus we would expect that the asymptotic expansion of S_{BH} should not change as we move across the walls of marginal stability.

However there is still an exponentially suppressed change across walls of marginal stability.

Can we see this on the black hole side?

It turns out that this jump in $d(Q,P)$ is associated with 2-centered solutions to the supergravity equations of motion together with higher derivative corrections. Denef,Moore

Each of these centers has the near horizon geometry of a small black hole.

(black holes whose entropy is zero at the leading order but is non-zero after taking into account higher derivative corrections.)

Consider such a 2-centered solution with the first center carrying charge (Q_1, P_1) and the second center carrying charges (Q_2, P_2).

If we consider the wall of marginal stability associated with the decay

$$(Q, P) \Rightarrow (Q_1, P_1) + (Q_2, P_2)$$

then the two centered solution exists only on one side of the wall of marginal stability.

As we cross the wall of marginal stability the solution disappears.

Denef; Bates and Denef; Denef and Moore

Thus from the black hole side the change in the index can be identified as the index associated with the 2-centered solution.

Explicit computation gives

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)$$
$$\left\{\sum_{L_1|(Q_1,P_1)} d_h\left(\frac{Q_1}{L_1},\frac{P_1}{L_1}\right)\right\}\left\{\sum_{L_2|(Q_2,P_2)} d_h\left(\frac{Q_2}{L_2},\frac{P_2}{L_2}\right)\right\}$$

$d_h(q,p)$: index of half-BPS states carrying charges (q,p).

How does this compare with the microscopic result?

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)$$
$$\left\{ \sum_{L_1|(Q_1,P_1)} d_h\left(\frac{Q_1}{L_1}, \frac{P_1}{L_1}\right)\right\} \left\{ \sum_{L_2|(Q_2,P_2)} d_h\left(\frac{Q_2}{L_2}, \frac{P_2}{L_2}\right)\right\}$$

In all known cases this agrees exactly with the jump in the index $d(Q,P)$ computed by evaluating the residue of the integrand at the appropriate pole of the partition function $\Psi(\rho,\sigma,v)$.

Thus we see the black holes not only capture the leading asymptotic behaviour of $d(Q,P)$ for large charges, but also capture information about exponentially small corrections to $d(Q,P)$.

Although we have illustrated our results in the context of heterotic string theory compactified on T^6, similar results have been found in a wide class of $\mathcal{N} = 4$ supersymmetric string theories.

1. In the limit of large charges the black hole entropy computed from supergravity action + gauss-bonnet correctons agree with the statistical entropy to first non-leading order.

2. The jump in the index across walls of marginal stability, computed from the residue at the appropriate poles of the partition function, agree with the ones calculated from the index of 2-centered black hole solutions.