Fermions in non-relativistic AdS/CFT correspondence

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• Some Aspects Of Non-Relativistic Conformal Field Theory (NRCFT)

The simplest NRCFT is the free schrodinger field:

$$(i\partial_t + \frac{1}{2m}\nabla^2)\psi(t,\overrightarrow{x}) = 0 \qquad S = \int dtd\overrightarrow{x}\,\psi^\dagger(t,\overrightarrow{x})\left(i\partial_t + \frac{1}{2m}\nabla^2\right)\psi(t,\overrightarrow{x})$$

The symmetry group of schrodinger equation is:

♠ Galilean group , $translations, rotations, boosts,$,

♠ Scale transformation, $x' = \lambda x \quad , t' = \lambda^2 t$,

♠ Special conformal transformation, $x' = \frac{x}{1+\alpha t} \quad , t' = \frac{t}{1+\alpha t}$

This group (Schrodinger group) is the maximall symmetry group of Schrodinger equation.

NRCFT is a field theory which is invariant under Schrodinger group.

The schrodinger algebra is:

$$
\begin{aligned}
&[M_{ij}, P_k] = -i(\delta_{ik}P_j - \delta_{jk}P_i), && [M_{ij}, K_k] = -i(\delta_{ik}K_j - \delta_{jk}K_i),\\
&[M_{ij}, M_{kl}] = -i\delta_{ik}M_{jl} + \text{perms}, && [P_i, K_j] = -iN\delta_{ij}\\
&[D, P_i] = -iP_i, \qquad [D, K_i] = iK_i, && [D, H] = -2iH,\\
&[C, P_i] = iK_i, \qquad [C, D] = -2iC, && [C, H] = -iD.
\end{aligned}
$$

It can be shown by using Ward identities that the form of two point function in any NRCFT is:

$$\langle\, \psi_M(t, x)\bar{\psi}_{-M}(0, 0)\,\rangle = Ct^{-\Delta}e^{-iMx^2/2t}$$

• Relation between Schrodinger group and Conformal group

Massless Klein-Gordon equation in $(d+1)$D is invariant under Conformal Group of $(d+1)$D ,$SO(2,d)$:

$$S = \int d^{d+1}x \partial_\mu \phi^\dagger \partial^\mu \phi$$

By writing action in Light-Cone coordinate:

$$t = x^0 + x^{d+1} \qquad \xi = x^0 - x^{d+1}$$

action becomes:

$$S = \int dt d\xi d^{d-1}x \;\; (\partial_t \phi \partial_\xi \phi + \partial_i \phi \partial^i \phi)$$

By getting ξ direction periodic and imposing the condition that the field has definite momentum in ξ direction:

$$\phi(t, \xi, x^i) = e^{iM\xi} \phi(t, x^i)$$

we receive to Schrodinger action:

$$S = \int dt d\overrightarrow{x} \phi^\dagger(t, \overrightarrow{x}) \left(i\partial_t + \frac{1}{2m} \nabla^2 \right) \phi(t, \overrightarrow{x})$$

So schrodinger group can be viewed as a subgroup of Conformal group in one higher dimension that consists of operators which do not mix modes in ξ direction,or in the other words :

$$[\mathbf{A}, P_{\xi}] = 0 \Leftrightarrow \mathbf{A} \in \text{Schrodinger Algebra}$$

Conformal Group in (d+1)D$\Rightarrow$ Schrodinger Group in [(d-1)+1] D

• Gravity description

Since NRCFT can be derived from CFT in one higher dimension,and gravity dual of CFT is AdS Space-time in one higher dimension , we expect that the gravity dual of NRCT be a space-time with two higher dimension:

$$AdS_{d+2} \Rightarrow CFT_{d+1} \Rightarrow NRCFT_d$$

metric with Schrodinger isometry:

$$ds^2 = -\frac{dt^2}{z^4} + \frac{2dtd\xi + d\overrightarrow{x}^2 + dz^2}{z^2}$$

Isometry:

$$\overrightarrow{P} : \overrightarrow{x} \rightarrow \overrightarrow{x} + \overrightarrow{x_0}, \qquad H : t \rightarrow t + t_0,$$
$$\overrightarrow{K} : \overrightarrow{x} \rightarrow \overrightarrow{x} - \overrightarrow{v}t \qquad , \quad \xi \rightarrow \xi - \overrightarrow{v}.\overrightarrow{x}$$
$$N : \xi \rightarrow \xi + \xi_0$$
$$D : \overrightarrow{x} \rightarrow \lambda \overrightarrow{x}, \qquad t \rightarrow \lambda^2 t, \qquad z \rightarrow \lambda z \qquad \xi \rightarrow \xi,$$
$$C : \overrightarrow{x} \rightarrow (1 - \alpha t)\overrightarrow{x}, \quad t \rightarrow (1 - \alpha t)t \quad , \quad z \rightarrow (1 - \alpha t)z \ , \xi \rightarrow \xi - \frac{\alpha}{2}(\overrightarrow{x}^2 + z^2).$$

By using the AdS/CFT correspondence:

$$Z_{AdS}[\phi_0] = \int_{\phi_0} D\phi \quad exp(-I[\phi]) = Z_{CFT}[\phi_0] = \left\langle exp\left(\int d^d x \quad \psi \ \phi_0 \right) \right\rangle$$

$$\left\langle \psi(x,t)\psi(0,0) \right\rangle \propto t^{-\Delta} e^{iM\frac{x^2}{2t}}$$

• Fermion field

Two point function of fermionic field:

Field theory side:

As Schrodinger equation can be derived from massless Klein-Gordon equation in one higher dimension ,Non-Relativistic equation for half spin particles (Levy-Leblond equation) can be derived from massless Dirac equation in one higher dimension:

$$S = \int d^5x \; \bar{\psi} i \gamma^\mu \partial_\mu \psi.$$

$$t = \frac{1}{\sqrt{2}}(x^0 + x^4), \qquad \xi = \frac{1}{\sqrt{2}}(x^0 - x^4).$$

$$S = \frac{i}{\sqrt{2}} \int d^3x d\xi dt \; \psi^\dagger \Big(\gamma_\xi \gamma_t \partial_\xi + \gamma_t \gamma_\xi \partial_t - (\gamma_\xi + \gamma_t) \gamma_i \partial_i \Big) \psi.$$

Consider a single mode with definite momentum in the null direction ξ. suc
that $\psi(t,\xi,x) = e^{iM\xi}\psi_M(t,x)$.

$$S = \frac{i}{\sqrt{2}}\int d^3x dt\ \psi^\dagger_{-M}\Big(iM\gamma_\xi\gamma_t + \gamma_t\gamma_\xi\partial_t - (\gamma_\xi + \gamma_t)\gamma_i\partial_i\Big)\psi_M.$$

$$\begin{pmatrix} 2E & -i\sqrt{2}\sigma_i k_i \\ i\sqrt{2}\sigma_i k_i & 2M \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0,$$

$$\langle\ \psi_M(t,x)\bar{\psi}_{-M}(0,0)\ \rangle = \frac{i}{\sqrt{2}M}\Big(iM\gamma_t + \gamma_\xi\partial_t - \gamma_i\partial_i\Big)G(t,x;0,0),$$

where

$$G(t,x;0,0) = \int \frac{d^3k}{(2\pi)^{3/2}}\ e^{i(\frac{k^2}{2M}t + k_i x_i)} = c\left(\frac{M}{t}\right)^{3/2} e^{-\frac{iMx^2}{2t}}$$

Gravity side:

we must solve the dirac equation in the background with Schrodiger symmetry :

$$\left(r\Gamma_{\hat{t}}\partial_\xi + r\Gamma_{\hat{\xi}}\partial_t + r\Gamma_{\hat{i}}\partial_i + r\Gamma_{\hat{r}}\partial_r + \frac{\mu^2}{2r}\Gamma_{\hat{\xi}}\partial_\xi - \frac{d+1}{2}\Gamma_{\hat{r}} - m \right) \Psi(x_i, t, \xi, r) = 0,$$

$$\lim_{r\to 0} \Psi_M(k,r) \sim r^{\frac{d}{2}-\nu^+}\Gamma_{\hat{\xi}}\mathbf{v}_M, \qquad \lim_{r\to 0} \overline{\Psi}_{-M}(k,r) \sim r^{\frac{d}{2}-\nu^+}\overline{\mathbf{u}}_{-M}\Gamma_{\hat{\xi}},$$

$$Z_{CFT} = \left\langle \ \exp\left[\int d^d x \left(\overline{\psi}_{-M}\Gamma_\xi \mathbf{v}_M + \overline{\mathbf{u}}_{-M}\Gamma_\xi \psi_M\right)\right] \right\rangle.$$

$$I_{AdS} = \int d\xi dt d^{d-1}x \sqrt{g}\, \overline{\Psi}(t,x,r,\xi)\Psi(t,x,r,\xi).$$

$$\left[\overline{\Psi}^+_{-M}(k,\epsilon)\Psi^+_M(k,\epsilon) + \overline{\Psi}^-_{-M}(k,\epsilon)\Psi^-_M(k,\epsilon)\right] \approx i\mu^2 C\epsilon^{d+1-2\nu^+}k^{-2\nu^+}\overline{\mathbf{u}}_{-M}(k)\Gamma_{\hat{\xi}}\mathbf{v}_M(k),$$

$$\langle \psi_M(x,t)\overline{\psi}_{-M}(0,0)\rangle = C\epsilon^{-2\nu^+}\left(iM\Gamma_{\hat{t}} + \Gamma_{\hat{\xi}}\partial_t + \Gamma_{\hat{i}}\partial_i \right)\left(t^{-\Delta} e^{\frac{iMx^2}{2t}} \right)$$