Fermions in non-relativistic AdS/CFT correspondence

Amin Akhavan, Mohsen Alishahiha, A.D and Ali Vahedi

Sharif University of Technology
Institute for Studies in Theoretical Physics and Mathematics IPM

arXiv:0902.0276
Some Aspects Of Non-Relativistic Conformal Field Theory (NRCFT)

The simplest NRCFT is the free Schrodinger field:

\[(i \partial_t + \frac{1}{2m} \nabla^2)\psi(t, \vec{x}) = 0\]

\[S = \int dtd\vec{x} \psi^\dagger(t, \vec{x}) \left(i \partial_t + \frac{1}{2m} \nabla^2\right) \psi(t, \vec{x})\]

The symmetry group of Schrodinger equation is:

- ♠ Galilean group, translations, rotations, boosts, 
- ♠ Scale transformation,
- ♠ Special conformal transformation,

\[x' = \lambda x, t' = \lambda^2 t\]

\[x' = \frac{x}{1 + \alpha t}, t' = \frac{t}{1 + \alpha t}\]

This group (Schrodinger group) is the maximal symmetry group of Schrodinger equation.

NRCFT is a field theory which is invariant under Schrodinger group.
The Schrödinger algebra is:

\[
\begin{align*}
[M_{ij}, P_k] &= -i(\delta_{ik} P_j - \delta_{jk} P_i), \\
[M_{ij}, M_{kl}] &= -i\delta_{ik} M_{jl} + \text{perms}, \\
[D, P_i] &= -iP_i, \\
[C, P_i] &= iK_i,
\end{align*}
\]

It can be shown by using Ward identities that the form of two point function in any NRCFT is:

\[
\langle \psi_M(t, x) \bar{\psi}_{-M}(0, 0) \rangle = Ct^{-\Delta} e^{-iMx^2/2t}
\]
**Relation between Schrodinger group and Conformal group**

Massless Klein-Gordon equation in \((d + 1)D\) is invariant under Conformal Group of \((d + 1)D\), \(SO(2, d)\):

\[
S = \int d^{d+1}x \partial_\mu \phi^\dagger \partial^\mu \phi
\]

By writing action in Light-Cone coordinate:

\[
t = x^0 + x^{d+1} \quad \xi = x^0 - x^{d+1}
\]

action becomes:

\[
S = \int dt d\xi d^{d-1}x \left( \partial_t \phi \partial_\xi \phi + \partial_i \phi \partial^i \phi \right)
\]

By getting \(\xi\) direction periodic and imposing the condition that the field has definite momentum in \(\xi\) direction:

\[
\phi(t, \xi, x^i) = e^{iM\xi} \phi(t, x^i)
\]

we receive to Schrodinger action:

\[
S = \int dt d\vec{x} \phi^\dagger(t, \vec{x}) \left( i\partial_t + \frac{1}{2m} \nabla^2 \right) \phi(t, \vec{x})
\]
So Schrödinger group can be viewed as a subgroup of Conformal group in one higher dimension that consists of operators which do not mix modes in \( \xi \) direction, or in other words:

\[
[A, P_\xi] = 0 \iff A \in \text{Schroedinger Algebra}
\]

\[\text{Conformal Group in (d+1)D} \Rightarrow \text{Schrödinger Group in [(d-1)+1] D}\]
Since NRCFT can be derived from CFT in one higher dimension, and gravity dual of CFT is $AdS$ Space-time in one higher dimension, we expect that the gravity dual of NRCT be a space-time with two higher dimension:

$$AdS_{d+2} \Rightarrow CFT_{d+1} \Rightarrow NRCFT_d$$

metric with Schrodinger isometry:

$$ds^2 = -\frac{dt^2}{z^4} + \frac{2dtd\xi + d\overrightarrow{x}^2}{z^2} + dz^2$$

Isometry:
\( \overrightarrow{P} : \overrightarrow{x} \to \overrightarrow{x} + \overrightarrow{x}_0, \quad H : t \to t + t_0, \)
\( \overrightarrow{K} : \overrightarrow{x} \to \overrightarrow{x} - \overrightarrow{v} t, \quad \xi \to \xi - \overrightarrow{v}.\overrightarrow{x} \)
\( N : \xi \to \xi + \xi_0 \)
\( D : \overrightarrow{x} \to \lambda \overrightarrow{x}, \quad t \to \lambda^2 t, \quad z \to \lambda z \quad \xi \to \xi, \)
\( C : \overrightarrow{x} \to (1 - \alpha t) \overrightarrow{x}, \quad t \to (1 - \alpha t)t, \quad z \to (1 - \alpha t)z, \xi \to \xi - \frac{\alpha}{2} (\overrightarrow{x}^2 + z^2). \)

By using the AdS/CFT correspondence:

\[
Z_{AdS}[\phi_0] = \int_{\phi_0} D\phi \ exp(-I[\phi]) = Z_{CFT}[\phi_0] = \left< \exp\left( \int d^d x \ \psi \ \phi_0 \right) \right>
\]

\[
\left< \psi(x, t) \psi(0, 0) \right> \propto t^{-\Delta} e^{iM^2/2i}
\]
Fermion field

Two point function of fermionic field:

Field theory side:

As Schrödinger equation can be derived from massless Klein-Gordon equation in one higher dimension, Non-Relativistic equation for half spin particles (Levy-Leblond equation) can be derived from massless Dirac equation in one higher dimension:

\[ S = \int d^5 x \overline{\psi} i \gamma^\mu \partial_\mu \psi. \]

\[ t = \frac{1}{\sqrt{2}} (x^0 + x^4), \quad \xi = \frac{1}{\sqrt{2}} (x^0 - x^4). \]

\[ S = \frac{i}{\sqrt{2}} \int d^3 x d\xi dt \psi^\dagger \left( \gamma_\xi \gamma_t \partial_\xi + \gamma_t \gamma_\xi \partial_t - (\gamma_\xi + \gamma_t) \gamma_i \partial_i \right) \psi. \]
Consider a single mode with definite momentum in the null direction $\xi$. such that $\psi(t, \xi, x) = e^{iM_\xi} \psi_M(t, x)$.

$$S = \frac{i}{\sqrt{2}} \int d^3x dt \psi_+^\dagger_M \left( iM\gamma_\xi \gamma_t + \gamma_t \gamma_\xi \partial_t - (\gamma_\xi + \gamma_t) \gamma_i \partial_i \right) \psi_M.$$  

$$\left( \begin{array}{cc} 2E & -i\sqrt{2}\sigma_i k_i \\ i\sqrt{2}\sigma_i k_i & 2M \end{array} \right) \left( \begin{array}{c} \phi \\ \chi \end{array} \right) = 0,$$

$$\langle \psi_M(t, x) \bar{\psi}_-M(0, 0) \rangle = \frac{i}{\sqrt{2M}} \left( iM\gamma_t + \gamma_\xi \partial_t - \gamma_i \partial_i \right) G(t, x; 0, 0),$$

where

$$G(t, x; 0, 0) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\frac{k^2}{2M}t + k_\xi x_\xi} = c \left( \frac{M}{t} \right)^{3/2} e^{-\frac{iMx^2}{2t}}.$$
**Gravity side:**

we must solve the dirac equation in the background with Schrodiger symmetry:

\[
\left( r \Gamma \tilde{\xi} \partial_\xi + r \Gamma \tilde{\xi} \partial_t + r \Gamma \tilde{r} \partial_r + \frac{\mu^2}{2r} \Gamma \hat{\xi} \partial_\xi - \frac{d + 1}{2} \Gamma \hat{r} - m \right) \Psi(x, t, \xi, r) = 0,
\]

\[
\lim_{r \to 0} \Psi_M(k, r) \sim r^{\frac{d-2}{2} - \nu^+} \Gamma \tilde{\xi} v_M,
\]

\[
\lim_{r \to 0} \overline{\Psi}_M(k, r) \sim r^{\frac{d-2}{2} - \nu^+} \overline{u}_M \Gamma \tilde{\xi},
\]

\[
Z_{CFT} = \langle \exp \left[ \int d^d x \left( \overline{\Psi}_M \Gamma \tilde{\xi} v_M + \overline{u}_M \Gamma \tilde{\xi} \psi_M \right) \right] \rangle.
\]

\[
I_{AdS} = \int d\xi dt d^{d-1}x \sqrt{g} \overline{\Psi}(t, x, r, \xi) \Psi(t, x, r, \xi).
\]

\[
\left[ \overline{\Psi}_M^+(k, \epsilon) \psi_M^+(k, \epsilon) + \overline{\Psi}_M^-(k, \epsilon) \psi_M^-(k, \epsilon) \right] \approx i \mu^2 C \epsilon^{d + 1 - 2\nu^+} k^{-2\nu^+} \overline{u}_M(k) \Gamma \tilde{\xi} v_M(k),
\]

\[
\langle \psi_M(x, t) \overline{\psi}_M(0, 0) \rangle = C \epsilon^{-2\nu^+} \left( iM \Gamma \tilde{\xi} + \Gamma \tilde{\xi} \partial_t + \Gamma \tilde{r} \partial_r \right) \left( t^{-\Delta} e^{\frac{imx^2}{2t}} \right)
\]