Kaluza-Klein Reductions and AdS/CFT

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Kaluza-Klein Theory

Pure Einstein gravity in $D = 5$ with an $S^1$ of radius $R$.

$$g_{MN} = \begin{pmatrix} e^\phi g_{\mu\nu} + e^{-2\phi} A_\mu A_\nu & e^{-2\phi} A_\mu \\ e^{-2\phi} A_\nu & e^{-2\phi} \end{pmatrix}$$

Let $y = y + 2\pi R$ be a coordinate on $S^1$, $x^\mu$ a $D = 4$ coordinate and expand:

$$\phi(x, y) = \sum \phi^n(x) e^{iny/R}$$
$$A_\mu(x, y) = \sum A^n_\mu(x) e^{iny/R}$$
$$g_{\mu\nu}(x, y) = \sum g^n_{\mu\nu}(x) e^{iny/R}$$

$D = 5$ Einstein equations $R_{MN} = 0 \Rightarrow$ the $D = 4$ fields have

$$\text{mass}^2 \sim n^2/R^2$$
There is a $U(1)$ gauge symmetry arising from $D = 5$ diffeos of the form:

$$x^\mu \rightarrow x^\mu, \quad y \rightarrow y + \Lambda(x)$$

which induces

$$A_\mu(x) \equiv A_\mu^{n=0}(x) \rightarrow A_\mu(x) - \partial_\mu \Lambda$$

Furthermore the fields with $n \neq 0$ have non-zero charge:

$$\phi^n \rightarrow e^{in\Lambda/R} \phi^n$$

$$A_\mu^n \rightarrow e^{in\Lambda/R} A_\mu^n$$

$$g_{\mu\nu}^n \rightarrow e^{in\Lambda/R} g_{\mu\nu}^n$$
KK Reduction I: Low-energy effective theory

Consider $E << 1/R$: ignore all modes with $mass \neq 0$ and just keep $n = 0$ massless modes: (write $\phi^0(x) = \phi(x)$ etc)

$$\nabla_\mu (e^{-3\phi} F_{\mu\nu}) = 0$$

$$\nabla^2 \phi + \frac{1}{4} e^{-3\phi} F^2 = 0$$

$$R_{\mu\nu} - \frac{3}{2} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} e^{-3\phi} F_{\mu\rho} F_{\nu}^{\rho} + \frac{1}{8} g_{\mu\nu} e^{-3\phi} F^2 = 0$$

Equations of motion can be derived from $D = 4$ low-energy effective action:

$$S = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{4} e^{-3\phi} F^2 \right].$$

$D = 4$ gravity, $U(1)$ gauge-field, scalar.
**Historical Aside:** Idea of extra dimensions for unification is not Kaluza in 1919 (or 1921) or Klein in 1926 but Nordstrom in 1914!

A. Nordstrom gravity I: $\nabla^2 \psi = -4\pi G \rho$

B. Nordstrom gravity II: $g_{\mu\nu} = e^{2\psi} \eta_{\mu\nu}, \ R = 24\pi GT$

C. Nordstrom in 1914: $D = 5$ Electromagnetism on $\mathbb{R}^{1,3} \times S^1$ with $A_M(x, y) = [A_\mu(x), \psi(x)]$ to unify gravity and electromagnetism.
KK Reduction II: consistent truncation

Break up $D = 4$ fields into a finite number $L$, including $g_{\mu\nu}$, and an infinite number $H$. Schematically:

\[
D^2 H \sim \sum_{l \geq 0} a_l L^l + \sum_{l \geq 0} b_l H^l + \sum_{k, l \geq 1} c_{kl} H^k L^l
\]
\[
D^2 L \sim \sum_{l \geq 0} d_l L^l + \sum_{l \geq 0} e_l H^l + \sum_{k, l \geq 1} f_{kl} H^k L^l
\]

Consistent to set $H = 0$ only if $a_l = 0$ and then one obtains equations of $D = 4$ theory of gravity with a finite number of fields whose solutions give rise to exact solutions of the $D = 5$ theory.

Dangerous terms are

\[
S = \int d^4x \sqrt{g}[R + ... + (DH)^2 + H \sum_k a_k L^k + ...]
\]
The standard $D = 5$ KK reduction is consistent 

$H \leftrightarrow$ heavy massive modes with $n \neq 0$. Charged 

$L \leftrightarrow$ massless modes with $n = 0$. Neutral 

Dangerous terms are absent because of the $U(1)$ gauge invariance. Observe that in general one must keep all of the neutral fields. e.g. it is inconsistent to set $\phi = 0$. 

Any solution of theory in low dimensions gives a solution of higher dimension theory. e.g. $D = 4$ vacuum solution $\mathbb{R}^{1,3}$, $\phi = A_\mu = 0$, “uplifts” to $\mathbb{R}^{1,3} \times S^1$ solution. 

Consistent truncations provide a powerful tool for solving higher dimensional Einstein Equations. Applications to AdS/CFT in string/M-theory.
Generalisations:

1. Start with a more general theory of gravity in any dimension and reduce on $T^n$: consistent to truncate to $U(1)^n$ invariant fields to get a lower dimensional theory with (at least) $U(1)^n$ symmetry.

2. Reduce on a group manifold $G$ with $G_L \times G_R$ invariant metric: consistent to truncate to $G_R$ invariant fields to get a lower dimensional theory with $G_L$ symmetry.

3. Reductions on spheres, supersymmetry and AdS/CFT...
Maximally supersymmetric solutions with spheres:

\[ D = 11 \text{ SUGRA} \]

\[ AdS_4 \times S^7, \quad G_4 = Vol(AdS_4) \]

Consistent KK reduction of \( D = 11 \) SUGRA on \( S^7 \) to \( D = 4 \)
\( N = 8 \) \( SO(8) \) gauged SUGRA.

\( D = 4 \) fields: \( g_{\mu\nu}, 28 \ A_\mu, 35_s + 35_c \phi \), fermions.

Comments

1. The truncation has been argued to exist, but it is not explicit. There are further truncations which are consistent eg keeping \( U(1)^4 \) gauge fields. Very useful for constructing solutions
2. There is no known group theory argument why this is true.
3. The truncation keeps both massless and massive fields.
Similarly:

**type IIB SUGRA**

\[ AdS_5 \times S^5, \quad F_5 = Vol(AdS_5) + Vol(S^5) \]

Consistent KK reduction of type IIB on \( S^5 \) to \( D = 5 \) \( SO(6) \) gauged SUGRA.

**\( D = 11 \) SUGRA**

\[ AdS_4 \times S^7, \quad G_4 = Vol(AdS_4) \]

Consistent KK reduction of \( D = 11 \) on \( S^4 \) to \( D = 7 \) \( SO(5) \) gauged SUGRA.
All of these solutions are dual to maximally supersymmetric SCFTs via the AdS/CFT correspondence:

\[ AdS_5 \times S^5 \leftrightarrow N=4 \text{ SYM in } d=4 \]
\[ AdS_4 \times S^7 \leftrightarrow N=8 \text{ SCFT in } d=3 \]
\[ AdS_7 \times S^4 \leftrightarrow (0,2) \text{ SCFT in } d=6 \]

Is there are a dual interpretation of the consistent truncation?

1. Fields kept are those of the superconformal current multiplet

\[ g_{\mu\nu} \leftrightarrow T_{ij} \]
\[ A^I_\mu \leftrightarrow J^I_i \]
\[ \ldots \]

2. The consistency of the truncation is roughly that for \( N \to \infty \)

\[ \langle OT \ldots T \rangle = 0, \quad O \neq T \]

Proven for d=2 SCFT  David, Sahoo, Sen
Conjecture JPG, O. Varela: for any supersymmetric $AdS \times_w M$ solution of string/M-theory there is a consistent KK truncation on $M$ to a gauged supergravity whose fields $g_{\mu\nu}, A^I_{\mu}\ldots$ are dual to the superconformal current multiplet: $T_{ij}, J^I_i,\ldots$.

Would like explicit KK reduction formulae as this allows one to uplift solutions of lower dimensional SUGRA to get explicit solutions in $D=10, 11$.

How?
1. Precise characterisation of general classes of $AdS$ solutions using $G$-structure techniques.

2. Using $G$-structure results to construct explicit ansatz.
Sasaki-Einstein Solutions

D3-branes at apex of $CY_3$ cones:

\[
ds^2 = AdS_5 \times SE_5 \\
F_5 = Vol(AdS_5) + Vol(SE_5)
\]

M2-branes at apex of $CY_4$ cones:

\[
ds^2 = AdS_4 \times SE_7 \\
G_4 = Vol(AdS_4)
\]

Every SE space has a “Reeb-vector” $\xi$ which is a Killing vector. This is dual to abelian $R$-symmetry. Choose local coordinates where $\xi = \partial_\psi$: 
Locally, the Sasaki-Einstein metric can be written

\[ ds^2(SE_{2n+1}) = ds^2(KE_{2n}) + (d\psi + a)^2 \]

where \( KE \) is a Kähler-Einstein space, with Kähler form \( J_{KE} \) and holomorphic \((n,0)\) form \( \Omega_{KE} \), and \( da = 2J_{KE} \)

*Reduction of \( D = 11 \) on \( SE_7 \) to minimal \( N = 2 \) \( D = 4 \) gauged SUGRA \( \text{JPG O. Varela} \)

\[ ds^2 = ds^2_4 + ds^2(KE_6) + (d\psi + a + A)^2 \]

\[ G_4 = Vol_4 + *F \land J_{KE} \]

Substitute into \( D = 11 \) equations of motion to get minimal \( D = 4 \) gauged SUGRA:

\[ S = \int d^4x \sqrt{g}[R - F^2 + 6] \]
Reduction of type IIB on $SE_5$ to minimal $D = 5$ gauge SUGRA. Bosonic fields are $g_{\mu\nu}$, $A_\mu$. Buchel, Liu

An application: Gutowski, Reall black hole solutions of $D = 5$ gauged supergravity can be uplifted on any $SE_5$ space to obtain solutions of type IIB SUGRA. State counting interpretation applies to general class of SCFTs.

There are other classes of supersymmetric $AdS_{d+1}$ solutions for various $d$ for both string and M-theory where conjecture is also proven JPG, E. O’Colgain, O.Varela
Generalisation to include additional massive “breathing” modes.

JPG, S. Kim, O. Varela, D. Waldram (Maldacena, Martelli, Tachikawa)

Consider $D = 11$ reduced on $SE_7$. KK ansatz:

$$ds_{11}^2 = ds_4^2 + e^{2v} \left[ e^{-2u} ds^2(KE_6) + e^{12u}(d\psi + a + A)^2 \right]$$

$$G_4 = f\text{vol}_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J_{KE} + dh \wedge J_{KE} \wedge (\eta + A_1)$$
$$+ 2hJ_{KE} \wedge J_{KE} + \sqrt{3} \left[ -i \frac{1}{4} D\chi \wedge \Omega_{KE} + \chi(\eta + A_1) \wedge \Omega_{KE} + \text{c.c.} \right],$$

This gives a consistent truncation to a $D = 4$ theory with

6 scalars: $u, v, h, \chi, H_3$
2 vectors: $dA, H_2$

The scalar $v$ is a “breathing mode” and $u$ is a “squashing mode”.
The truncation is consistent with $N = 2$ SUGRA. More precisely:

$N = 2$ supergravity multiplet: $g_{\mu\nu} + \text{vector}$
$N = 2$ vector multiplet: vector $+ 2$ scalars
$N = 2$ hypermultiplet: 4 scalars

Aside: can view this as a kind of flux compactification of $D = 11$ on $S^1 \times KE_6$:

If we consider the $AdS_4 \times SE_7$ vacuum solution, we can identify the multiplet of operators in the dual $N = 1$ SCFT in $d = 4$: massless graviton, massless vector $\leftrightarrow$ conformal current supermultiplet
massive vector plus 5 massive scalars $\leftrightarrow$ “breathing mode supermultiplet”; it’s a long vector multiple with $4 \leq \Delta \leq 6$. 
Similarly:

\[ AdS_5 \times SE_5 \] solutions of type IIB

There exists a consistent truncation of the \( SE_5 \) to \( D = 5 \) SUGRA that contains the breathing mode multiplet. Maldacena, Martelli, Tachikawa; JPG, S.Kim, O. Varela, D. Waldram

Interestingly the operator dual to the breathing mode is known in the SCFT and corresponds to \( \Delta = 8 \) operator \( \sim TrF^4 \).
Application 1. Holographic Superconductivity

Skew Whiffing: We can also consider the skew-whiffed solution \( AdS_4 \times SE_7, \ G_4 = -vol(AdS_4) \). This breaks all supersymmetry but is perturbatively stable. Can do a consistent KK reduction to get an \( N = 2 \) SUGRA theory with an \( AdS_4 \) vacuum that spontaneously breaks the supersymmetry.

For this case, the fields include metric, vector field plus charged scalar which have been used to study holographic superconductivity Gubser; Denef, Hartnoll
Application 2. Holographic solutions with non-relativistic conformal symmetry:

Invariant under Galilean transformations:
time and spatial translations, spatial rotations, Galilean boosts and a central “mass” operator
PLUS scale transformations: $x \rightarrow \mu x, \quad t \rightarrow \mu^z t$, where $z$ is the dynamical exponent

For any $SE_5$ space one can construct type IIB solutions with such symmetry with three spatial dimensions and $z = 2, 4$ Herzog et al, Maldacean et al; Ross et al

For any $SE_7$ space one can construct $D = 11$ solutions with such symmetry with two spatial dimensions and $z = 3$ JPG et al

Can be generalised to other $z \geq 4$ and $z \geq 3$, respectively JPG, Donos
IIB case:

\[ ds^2 = \frac{dr^2}{r^2} + r^2 \left[ 2dx^+dx^- + dx_1^2 + dx_2^2 \right] + ds^2(SE_5) + 2r^2Cdx^+ \]

\[ F_5 = 4r^3dx^+ \wedge dx^- \wedge dr \wedge dx_1 \wedge dx_2 + 4Vol(SE_5) \]

\[ - dx^+ \wedge [^*CY_3dC + d(r^4C) \wedge dx_1 \wedge dx_2] \]

with

\[ d*_{CY}dC = 0. \]

is supersymmetric. Choose

\[ C = r^\lambda \beta, \quad \Delta_{SE} \beta = \mu \beta, \quad d^\dagger \beta = 0, \quad \mu = \lambda(\lambda + 2) \]

then \( z = 2 + \lambda \) and since \( \mu \geq 8 \) we have \( z \geq 4 \).
e.g. $SE_5 = S^5$ case the spectrum of $\Delta_{S^5}$ on one-forms is known:

$$\mu = (s + 1)(s + 3), \quad s = 1, 2, 3, ...$$

Leading to solutions with

$$z = 4, \quad 15$$
$$z = 5, \quad 64$$
$$z = 6, \quad 175$$

etc.

The original $z = 4$ solution found via consistent truncation is when $C = r^2(\psi d\psi + a)$.
Conclusions

1. Conjecture: for any supersymmetric $AdS$ solution of string/M-theory there exists a consistent KK reduction to gravity supermultiplet. This is now a theorem in many cases.

2. For certain classes of $AdS$ solutions, based on Sasaki-Einstein spaces, we know that this can be extended to include additional massive modes, the breathing mode supermultiplet.

3. Helpful in constructing new solutions - non relativistic holography and holographic superconductivity.

4. What is dual AdS/CFT reason for all of these consistent truncations?
5. How general is the truncation with breathing mode supermultiplets? Consider $AdS_4 \times M_7$, $G_4 = Vol(AdS_4)$:
N=1 done; N=2 done; N=3 almost certainly true
Conjecture it is true for N=8 ($M_7 = S^7$)
Gravity supermultiplet:
metric: $g_{\mu\nu}$
vectors: 28
scalars: $35_s + 35_c$
Breathing mode supermultiplet:
scalars: $1 + 294_v + 840_s' + 300 + 35_s$
vectors: $567_v + 350 + 28$
massive spin 2: $35_v$

Would be remarkable since no-one has managed to construct theories with massive spin two interacting with gravity...

6. Other classes of $AdS$ geometries.