Kaluza-Klein Reductions and AdS/CFT

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Kaluza-Klein Theory

Pure Einstein gravity in $D = 5$ with an S^1 of radius R.

$$g_{MN} = \begin{pmatrix} e^{\phi} g_{\mu\nu} + e^{-2\phi} A_\mu A_\nu & e^{-2\phi} A_\mu \\ e^{-2\phi} A_\nu & e^{-2\phi} \end{pmatrix}$$

Let $y = y + 2\pi R$ be a coordinate on S^1, x^μ a $D = 4$ coordinate and expand:

$$\begin{aligned} \phi(x,y) &= \sum \phi^n(x) e^{iny/R} \\ A_\mu(x,y) &= \sum A_\mu^n(x) e^{iny/R} \\ g_{\mu\nu}(x,y) &= \sum g_{\mu\nu}^n(x) e^{iny/R} \end{aligned}$$

$D = 5$ Einstein equations $R_{MN} = 0 \Rightarrow$ the $D = 4$ fields have

$$mass^2 \sim n^2/R^2$$

There is a $U(1)$ gauge symmetry arising from $D = 5$ diffeos of the form:

$$x^\mu \to x^\mu, \qquad y \to y + \Lambda(x)$$

which induces

$$A_\mu(x) \equiv A_\mu^{n=0}(x) \;\; \to \;\; A_\mu(x) - \partial_\mu \Lambda$$

Furthermore the fields with $n \neq 0$ have non-zero charge:

$$\begin{aligned} \phi^n &\to e^{in\Lambda/R}\phi^n \\ A_\mu^n &\to e^{in\Lambda/R}A_\mu^n \\ g_{\mu\nu}^n &\to e^{in\Lambda/R}g_{\mu\nu}^n \end{aligned}$$

KK Reduction I: Low-energy effective theory

Consider $E << 1/R$: ignore all modes with $mass \neq 0$ and just keep $n = 0$ massless modes: (write $\phi^0(x) = \phi(x)$ etc)

$$\nabla_\mu(e^{-3\phi}F^{\mu\nu}) = 0$$
$$\nabla^2\phi + \tfrac{1}{4}e^{-3\phi}F^2 = 0$$
$$R_{\mu\nu} - \tfrac{3}{2}\nabla_\mu\phi\nabla_\nu\phi - \tfrac{1}{2}e^{-3\phi}F_{\mu\rho}F_\nu{}^\rho + \tfrac{1}{8}g_{\mu\nu}e^{-3\phi}F^2 = 0$$

Equations of motion can be derived from $D = 4$ low-energy effective action:

$$S = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{4}e^{-3\phi}F^2\right].$$

$D = 4$ gravity, $U(1)$ gauge-field, scalar.

Historical Aside: Idea of extra dimensions for unification is not Kaluza in 1919 (or 1921) or Klein in 1926 but Nordstrom in 1914!

A. Nordstrom gravity I: $\nabla^2\psi = -4\pi G\rho$

B. Nordstrom gravity II: $g_{\mu\nu} = e^{2\psi}\eta_{\mu\nu},\ R = 24\pi GT$

C. Nordstrom in 1914: $D = 5$ Electromagnetism on $\mathbb{R}^{1,3}\times S^1$ with $A_M(x,y) = [A_\mu(x), \psi(x)]$ to unify gravity and electromagnetism.

KK Reduction II: consistent truncation

Break up $D = 4$ fields into a finite number L ,including $g_{\mu\nu}$, and an infinite number H. Schematically:

$$D^2 H \sim \sum_{l\geq 0} a_l L^l + \sum_{l\geq 0} b_l H^l + \sum_{k,l\geq 1} c_{kl} H^k L^l$$
$$D^2 L \sim \sum_{l\geq 0} d_l L^l + \sum_{l\geq 0} e_l H^l + \sum_{k,l\geq 1} f_{kl} H^k L^l$$

Consistent to set $H = 0$ only if $a_l = 0$ and then one obtains equations of $D = 4$ theory of gravity with a finite number of fields *whose solutions give rise to exact solutions of the $D = 5$ theory.*

Dangerous terms are

$$S = \int d^4x \sqrt{g}[R + ... + (DH)^2 + H \sum_k a_k L^k + ...]$$

The standard $D = 5$ KK reduction is consistent
$H \leftrightarrow$ heavy massive modes with $n \neq 0$. Charged
$L \leftrightarrow$ massless modes with $n = 0$. Neutral

Dangerous terms are absent because of the $U(1)$ gauge invariance. Observe that in general one must keep all of the neutral fields. e.g. it is inconsistent to set $\phi = 0$.

Any solution of theory in low dimensions gives a solution of higher dimension theory. e.g. $D = 4$ vacuum solution $\mathbb{R}^{1,3}$, $\phi = A_\mu = 0$, "uplifts" to $\mathbb{R}^{1,3} \times S^1$ solution.

Consistent truncations provide a powerful tool for solving higher dimensional Einstein Equations. Applications to AdS/CFT in string/M-theory.

Generalisations:

1. Start with a more general theory of gravity in any dimension and reduce on T^n: consistent to truncate to $U(1)^n$ invariant fields to get a lower dimensional theory with (at least) $U(1)^n$ symmetry.

2. Reduce on a group manifold G with $G_L \times G_R$ invariant metric: consistent to truncate to G_R invariant fields to get a lower dimensional theory with G_L symmetry.

3. Reductions on spheres, supersymmetry and AdS/CFT...

Maximally supersymmetric solutions with spheres:

$D = 11$ SUGRA

$$AdS_4 \times S^7, \qquad G_4 = Vol(AdS_4)$$

Consistent KK reduction of $D = 11$ SUGRA on S^7 to $D = 4$ $N = 8$ $SO(8)$ gauged SUGRA.

$D = 4$ fields: $g_{\mu\nu}$, $\mathbf{28}$ A_μ, $\mathbf{35_s} + \mathbf{35_c}$ ϕ, fermions.

Comments

1. The truncation has been argued to exist, but it is not explicit. There are further truncations which are consistent eg keeping $U(1)^4$ gauge fields. Very useful for constructing solutions
2. There is no known group theory argument why this is true.
3. The truncation keeps both massless and massive fields.

Similarly:

type IIB SUGRA

$$AdS_5 \times S^5, \qquad F_5 = Vol(AdS_5) + Vol(S^5)$$

Consistent KK reduction of type IIB on S^5 to $D = 5$ $SO(6)$ gauged SUGRA.

$D = 11$ SUGRA

$$AdS_4 \times S^7, \qquad G_4 = Vol(AdS_4)$$

Consistent KK reduction of $D = 11$ on S^4 to $D = 7$ $SO(5)$ gauged SUGRA.

All of these solutions are dual to maximally supersymmetric SCFTs via the AdS/CFT correspondence:

$$\begin{array}{lcl} AdS_5 \times S^5 & \leftrightarrow & \text{N=4 SYM in d=4} \\ AdS_4 \times S^7 & \leftrightarrow & \text{N=8 SCFT in d=3} \\ AdS_7 \times S^4 & \leftrightarrow & \text{(0,2) SCFT in d=6} \end{array}$$

Is there are a dual interpretation of the consistent truncation?

1. Fields kept are those of the superconformal current multiplet

$$\begin{array}{ccc} g_{\mu\nu} & \leftrightarrow & T_{ij} \\ A^I_\mu & \leftrightarrow & J^I_i \\ & \dots & \end{array}$$

2. The consistency of the truncation is roughly that for $N \to \infty$

$$< \mathcal{O} T \dots T >= 0, \qquad \mathcal{O} \neq T$$

Proven for d=2 SCFT David, Sahoo, Sen

Conjecture JPG, O. Varela : for any supersymmetric $AdS \times_w M$ solution of string/M-theory there is a consistent KK truncation on M to a gauged supergravity whose fields $g_{\mu\nu}, A^I_\mu ...$ are dual to the superconformal current multiplet: $T_{ij}, J^I_i, ...$

Would like explicit KK reduction formulae as this allows one to uplift solutions of lower dimensional SUGRA to get explicit solutions in D=10, 11.

How?

1. Precise characterisation of general classes of AdS solutions using G-structure techniques.

2. Using G-structure results to construct explicit ansatz.

Sasaki-Einstein Solutions

D3-branes at apex of CY_3 cones:

$$\begin{aligned} ds^2 &= AdS_5 \times SE_5 \\ F_5 &= Vol(AdS_5) + Vol(SE_5) \end{aligned}$$

M2-branes at apex of CY_4 cones:

$$\begin{aligned} ds^2 &= AdS_4 \times SE_7 \\ G_4 &= Vol(AdS_4) \end{aligned}$$

Every SE space has a "Reeb-vector" ξ which is a Killing vector. This is dual to abelian R-symmetry. Choose local coordinates where $\xi = \partial_\psi$:

Locally, the Sasaki-Einstein metric can be written

$$ds^2(SE_{2n+1}) = ds^2(KE_{2n}) + (d\psi + a)^2$$

where KE is a Kähler-Einstein space, with Kähler form J_{KE} and holomorphic $(n, 0)$ form Ω_{KE}, and $da = 2J_{KE}$

$*$Reduction of $D = 11$ on SE_7 to minimal $N = 2$ $D = 4$ gauged SUGRA JPG O. Varela

$$\begin{aligned} ds^2 &= ds_4^2 + ds^2(KE_6) + (d\psi + a + A)^2 \\ G_4 &= Vol_4 + *F \wedge J_{KE} \end{aligned}$$

Substitute into $D = 11$ equations of motion to get minimal $D = 4$ gauged SUGRA:

$$S = \int d^4x \sqrt{g}[R - F^2 + 6]$$

$*$Reduction of type IIB on SE_5 to minimal $D = 5$ gauge SUGRA. Bosonic fields are $g_{\mu\nu}$, A_μ Buchel, Liu

An application:
Gutowski, Reall black hole solutions of $D = 5$ gauged supergravity can be uplifted on any SE_5 space to obtain solutions of type IIB SUGRA. State counting interpretation applies to general class of SCFTs.

There are other classes of supersymmetric AdS_{d+1} solutions for various d for both string and M-theory where conjecture is also proven JPG, E. O'Colgain, O.Varela

Generalisation to include additional massive "breathing" modes.

JPG, S. Kim, O. Varela, D. Waldram (Maldacena, Martelli, Tachikawa)

Consider $D = 11$ reduced on SE_7. KK ansatz:

$$ds_{11}^2 = ds_4^2 + e^{2v}\left[e^{-2u} ds^2(KE_6) + e^{12u}(d\psi + a + A)^2\right]$$

$$G_4 = f\mathrm{vol}_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J_{KE} + dh \wedge J_{KE} \wedge (\eta + A_1)$$
$$+ 2hJ_{KE} \wedge J_{KE} + \sqrt{3}\left[-\tfrac{i}{4} D\chi \wedge \Omega_{KE} + \chi(\eta + A_1) \wedge \Omega_{KE} + \text{c.c.}\right],$$

This gives a consistent truncation to a $D = 4$ theory with
6 scalars: u, v, h, χ, H_3
2 vectors: dA, H_2

The scalar v is a "breathing mode" and u is a "squashing mode".

The truncation is consistent with $N = 2$ SUGRA. More precisely:

$N = 2$ supergravity multiplet: $g_{\mu\nu}$ + vector
$N = 2$ vector multiplet: vector + 2 scalars
$N = 2$ hypermultiplet: 4 scalars

Aside: can view this as a kind of flux compactification of $D = 11$ on $S^1 \times KE_6$:

If we consider the $AdS_4 \times SE_7$ vacuum solution, we can identify the multiplet of operators in the dual $N = 1$ SCFT in $d = 4$:
massless graviton, massless vector $\leftrightarrow$ conformal current supermultiplet
massive vector plus 5 massive scalars $\leftrightarrow$ "breathing mode supermultiplet"; it's a long vector multiple with $4 \leq \Delta \leq 6$.

Similarly:

$AdS_5 \times SE_5$ solutions of type IIB

There exists a consistent truncation of the SE_5 to $D = 5$ SUGRA that contains the breathing mode multiplet. Maldacena, Martelli, Tachikawa; JPG, S.Kim, O. Varela, D. Waldram

Interestingly the operator dual to the breathing mode is known in the SCFT and corresponds to $\Delta = 8$ operator $\sim TrF^4$.

Application 1. Holographic Superconductivity

Skew Whiffing: We can also consider the skew-whiffed solution $AdS_4 \times SE_7$, $G_4 = -vol(AdS_4)$. This breaks all supersymmetry but is perturbatively stable. Can do a consistent KK reduction to get an $N = 2$ SUGRA theory with an AdS_4 vacuum that spontaneously breaks the supersymmetry.

For this case, the fields include metric, vector field plus charged scalar which have been used to study holgraphic superconductivity Gubser; Denef, Hartnoll

Application 2. Holographic solutions with non-relativistic conformal symmetry:

Invariant under Galilean transformations:

time and spatial translations, spatial rotations, Galilean boosts and a central "mass" operator

PLUS scale transformations: $\mathbf{x} \to \mu \mathbf{x}$, $t \to \mu^z t$, where z is the dynamical exponent

For any SE_5 space one can construct type IIB solutions with such symmetry with three spatial dimensions and $z = 2, 4$ Herzog et al, Maldacean et al; Ross et al

For any SE_7 space one can construct $D = 11$ solutions with such symmetry with two spatial dimensions and $z = 3$ JPG et al

Can be generalised to other $z \geq 4$ and $z \geq 3$, respectively JPG, Donos

IIB case:

$$
\begin{aligned}
ds^2 &= \frac{dr^2}{r^2} + r^2\left[2dx^+dx^- + dx_1^2 + dx_2^2\right] + ds^2(SE_5) + 2r^2 C dx^+ \\
F_5 &= 4r^3 dx^+ \wedge dx^- \wedge dr \wedge dx_1 \wedge dx_2 + 4Vol(SE_5) \\
&- dx^+ \wedge \left[*_{CY_3} dC + d(r^4 C) \wedge dx_1 \wedge dx_2\right]
\end{aligned}
$$

with

$$d *_{CY} dC = 0 \ .$$

is supersymmetric. Choose

$$C = r^\lambda \beta, \qquad \Delta_{SE}\beta = \mu\beta, \qquad d^\dagger\beta = 0, \qquad \mu = \lambda(\lambda+2)$$

then $z = 2 + \lambda$ and since $\mu \geq 8$ we have $z \geq 4$.

e.g. $SE_5 = S^5$ case the spectrum of Δ_{S^5} on one-forms is known:

$$\mu = (s+1)(s+3), \qquad s = 1, 2, 3, ...$$

Leading to solutions with

$$\begin{aligned} z &= 4, \qquad \mathbf{15} \\ z &= 5, \qquad \mathbf{64} \\ z &= 6, \qquad \mathbf{175} \end{aligned}$$

etc.

The original $z = 4$ solution found via consistent truncation is when $C = r^2(d\psi + a)$

Conclusions

1. Conjecture: for any supersymmetric AdS solution of string/M-theory there exists a consistent KK reduction to gravity supermultiplet. This is now a theorem in many cases.

2. For certain classes of AdS solutions, based on Sasaki-Einstein spaces, we know that this can be extended to include additional massive modes, the breathing mode supermultiplet.

3. Helpful in constructing new solutions - non relativistic holography and holographic superconductivity.

4. What is dual AdS/CFT reason for all of these consistent truncations?

5. How general is the truncation with breathing mode supermultiplets? Consider $AdS_4 \times M_7$, $G_4 = Vol(AdS_4)$:
N=1 done; N=2 done; N=3 almost certainly true
Conjecture it is true for N=8 ($M_7 = S^7$)
Gravity supermultiplet :
metric: $g_{\mu\nu}$
vectors: $\mathbf{28}$
scalars: $\mathbf{35_s} + \mathbf{35_c}$
Breathing mode supermultiplet :
scalars: $\mathbf{1} + \mathbf{294_v} + \mathbf{840'_s} + \mathbf{300} + \mathbf{35_s}$
vectors: $\mathbf{567_v} + \mathbf{350} + \mathbf{28}$
massive spin 2: $\mathbf{35_v}$

Would be remarkable since no-one has managed to construct theories with massive spin two interacting with gravity...

6. Other classes of AdS geometries.