Two problems in Black Hole Theory

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In this lecture I will describe two recent pieces of work on Black Hole physics.

- Gravitating Opposites Attract

- Birkhoff's Invariant and Thornes's Hoop Conjecture

Gravitating Opposites Attract

Problem number one is based on joint, and as yet unpublished, work with Robert Beig and Rick Schoen. It is founded on a beautiful recent paper of theirs * excluding the existence of static (non-rotating) electrically neutral 3+1 dimensional solutions of the Einstein's equations representing two (symmetrical) bodies in equilibrium.

We have succeeded in extending the result to electrically charged and rotating black holes both in 3+1 and $n+1$, $\quad n>3$.

*R. Beig and R. M. Schoen, On Static n-body Configurations in Relativity, arXiv:0811.1727 [gr-qc].

The title of this section is taken from an elegant paper by Coleman et al on solitons and monopoles in flat spacetime * .

The feature that opposites attract also seems to extend to quantum mechanical Casimir forces † .

*Y. Aharonov, A. Casher, S. R. Coleman and S. Nussinov, Why opposites attract, Phys. Rev. D **46** (1992) 1877.

†O. Kenneth and I. Klich, Opposites Attract - A Theorem About The Casimir Force, Phys. Rev. Lett. **97** (2006) 160401 [arXiv:quant-ph/0601011].

Electric Charges. For a static electrovac,

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j \qquad A = \phi dt \tag{1}$$

$$\begin{aligned} R_{ij} &= \frac{1}{V}\nabla_i\nabla_j V - \frac{\kappa}{V^2}(\nabla_i\phi\nabla_j\phi - \frac{1}{2}g_{ij}|\nabla\phi|^2)\,, && (2)\\ V\nabla^2 V &= \frac{\kappa}{2}|\nabla\phi|^2\,, && (3)\\ V\nabla^2\phi &= g^{ij}\nabla_i\phi\nabla_j V\,. && (4) \end{aligned}$$

The standard Hamiltonian constraint

$$R = R_{\hat{1}\hat{1}} + R_{\hat{2}\hat{2}} + R_{\hat{3}\hat{3}} = \frac{\kappa}{V^2}|\nabla\phi|^2\,, \tag{5}$$

follows as expected.

Following Beig and Schoen we assume that we have a totally geodesic surface S with Gauss-curvature K, that separates two, possibly charged, black holes. The totally geodesic surface could arise as the fixed point set of a $\mathbf{Z}_2$ isometry.

$$R_{\hat{1}\hat{1}} + R_{\hat{2}\hat{2}} = 2K + R_{\hat{3}\hat{3}} \tag{6}$$

Taking the trace of (2) with respect to the metric on S gives

$$R_{\hat{1}\hat{1}} + R_{\hat{2}\hat{2}} = \frac{1}{V}\nabla^2_S V + \frac{\kappa}{V^2}|\nabla_{\hat{3}}\phi|^2 \tag{7}$$

Thus we arrive at

$$K = \frac{1}{V}\nabla^2_S V + \frac{\kappa}{2V^2}(|\nabla_{\hat{3}}\phi|^2 - |\nabla_{\hat{1}}\phi|^2 - |\nabla_{\hat{2}}\phi|^2)\,, \tag{8}$$

In the neutral case, $\phi = 0$, the strategy of Beig and Schoen was to note that

$$\int_S K dA = 0 = \int_S |\frac{\nabla_S V}{V}|^2 , \qquad (9)$$

Thus S is an equipotential of V. Then the boundary conditions implies that the harmonic function $V = 1$ and hence $R_{ij} = 0$ and the solution is flat.

The argument extends to the electrostatic case straight forwardly if we can assume that $\nabla_S \phi = 0$, that is if S is an equipotential of the electrostatic potential ϕ.

Our assumptions would hold if we had a static system of charged bodies invariant under an isometric action of $\mathbf{Z}_2$ which stabilizes S pointwise and under which the electric field is *odd*, i.e. for unlike

$$\mathbf{Z}_2 : \nabla\phi \rightarrow -\nabla\phi\,. \tag{10}$$

This is the analogue of the situation considered by Coleman et al. for solitons and in Kenneth et al. for Casimir forces.

These results are consistent with previous work on axisymmetric Weyl metrics in which the black holes are represented as rods * The argument also goes through for Einstein-Maxwell-Dilaton theory but that is most easily seen using a slightly streamlined formalism described later.

*G. W. Gibbons,Non-Existence Of Equilibrium Configurations Of Charged Black Holes, Proc. Roy. Soc. Lond. A **372** (1980) 535.

Rotating black holes We re-write the metric as

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\gamma_{ij}dx^i dx^j \,. \tag{11}$$

More generally for stationary metrics we write

$$ds^2 = -e^{2U}(dt + \omega_i dx^i)^2 + e^{-2U}\gamma_{ij}dx^i dx^j \,. \tag{12}$$

We refer to U as the Newtonian potential and define the curvature of the Sagnac connection by

$$F = d\omega \,, \qquad F_{ij} = \partial_i\omega_j - \partial_j\omega_i \,. \tag{13}$$

We use Hodge duality and one of the Einstein equations and in dimension three to introduce a twist potential

$$\star_\gamma F = e^{-4U}d\psi \,. \tag{14}$$

The vacuum Einstein equations may then be cast in as Euler Lagrange equations of the form (modulo boundary terms)

$$\int \sqrt{\gamma} d^3 x (\tilde{R} - 2\gamma^{ij} (\partial_i U \partial_j U + \frac{1}{2} e^{-4U} \partial_i \psi \partial_j \psi))\,, \tag{15}$$

where $\tilde{R}$ is the scalar curvature of the metric γ_{ij}. Thus

$$\tilde{R}_{ij} = 2\partial_i U \partial_j U + \frac{1}{2} e^{-4U} \partial_i \psi \partial_j \psi \tag{16}$$

It follows that Now we evaluate the Gauss-curvature of the totally geodesic surface S *assuming that it is totally geodesic wrt the metric* γ.

$$
\begin{aligned}
2\tilde{K} &= -2G_{ij}n^i n^j\,, && (17)\\
&= 2\partial_{\tilde{1}}U\partial_{\tilde{1}} + 2\partial_{\tilde{2}}U\partial_{\tilde{2}} - 2\partial_{\tilde{3}}U\partial_{\tilde{3}} && (18)\\
&+ \frac{1}{2}e^{-4U}(\partial_{\tilde{1}}\psi\partial_{\tilde{1}}\psi + \partial_{\tilde{2}}\psi\partial_{\tilde{2}}\psi - \partial_{\tilde{3}}\psi\partial_{\tilde{3}}\psi)\,, && (19)
\end{aligned}
$$

where G_{ij} is the Einstein tensor of the metric γ_{ij} , n^i is the unit normal to S wrt to metric γ_{ij} and the $\tilde{2}$ indicates a component in a frame which is orthonormal wrt to the metric γ_{ij}, .

No integration by parts is required to establish the result. However we do need a $\mathbf{Z}_2$ symmetry of both γ_{ij} and U, in the latter case to ensure that $\partial_{\tilde{3}}U = 0$. We also need $\partial_{\tilde{3}}\psi = 0$.

This condition may be better understood by introducing a *gravito-magnetic field* B_i by

$$B = \star_\gamma F = e^{-4U} d\psi \,, \tag{20}$$

so that

$$\begin{aligned} 2\tilde{K} &= -2G_{ij}n^i n^j \,, && (21)\\ &= 2\partial_{\tilde{1}}U\partial_{\tilde{1}} + 2\partial_{\tilde{2}}U\partial_{\tilde{2}} - 2\partial_{\tilde{3}}U\partial_{\tilde{3}} && (22)\\ &+ \frac{1}{2}e^{4U}(B_{\tilde{1}}B_{\tilde{1}} + B_{\tilde{2}}B_{\tilde{2}} - B_{\tilde{3}}B_{\tilde{3}}) \,. && (23) \end{aligned}$$

The condition for an attractive, i,e positive, contribution to the Gauss curvature from the gravito-magnetic field is that $B_{\tilde{3}} = B_i n^i = 0$. This the field lines must lie in the surface S. Now a spinning object generates a gravito-magnetic dipole field and and two symmetrically placed such dipoles with spins pointing along the direction joining them, will give vanishing field in the surface if they are anti-aligned. If the spins are orthogonal, they should also be anti-aligned. Thus the natural assumption on the gravito-magnetic field is that it is odd under the $\mathbf{Z}_2$ reflection

In linear theory, at large separation $r = |\mathbf{r}|$, the mutual potential energy of two spinning bodies with angular momentum $\mathbf{J}_1$ and $\mathbf{J}_2$ is given by Wald *

$$\frac{G}{c^2 r^5}(3(\mathbf{r}.\mathbf{J}_1)(\mathbf{r}.\mathbf{J}_2) - r^2(\mathbf{J}_1.\mathbf{J}_2)) \,. \tag{24}$$

This gives an attractive force if $\mathbf{J}_1 = -\mathbf{J}_2$ and repulsive force if $\mathbf{J}_2 = +\mathbf{J}_1$. Our results are consistent with this.

*R. Wald, Gravitational spin interaction, Phys. Rev. D **6** (1972) 406.

Our results are also consistent with the behaviour of explicit exact *double Kerr* solutions obtained using solution-generating techniques or general existence theorems of Weinstein * . These exhibit conical singularities along the axis between the two sources which may be interpreted as a strut or rod in tension which holds the two black holes apart. For a recent detailed discussion see a recent papers of Herdeiro et al. †

*G. Weinstein, N-black hole stationary and axially symmetric solutions of the Einstein/Maxwell equations. *Comm. Partial Differential Equations* **21** (1996), 1389-143,On the force between rotating co-axial black holes. *Trans. Amer. Math. Soc.* **343** (1994), 899–906, On rotating black holes in equilibrium in general relativity. *Comm. Pure Appl. Math.* **43** (1990) 903–948.

†C. A. R. Herdeiro and C. Rebelo, On the interaction between two Kerr black holes, JHEP **0810** (2008) 017 [arXiv:0808.3941 [gr-qc]] C. A. R. Herdeiro, C. Rebelo, M. Zilhao and M. S. Costa, A Double Myers-Perry Black Hole in Five Dimensions,' JHEP **0807** (2008) 009 [arXiv:0805.1206 [hep-th]]

Higher dimensions. In $n+1$, $N > 3$ spacetime dimensions, we can't use the Gauss-Bonnet theorem.

However the $n-1$ dimensional surface S is asymptotically flat and has zero ADM mass. Thus we can use the Positive Mass Theorem as long as the scalar curvature of S is non-negative.

As Leibniz would have said: Ladies and Gentlemen, let us calculate.

The metric is of the form

$$ds^2 = -e^{2U}(dt + \psi_i dx^i)^2 + e^{-\frac{2U}{n-2}} h_{ij} dx^i dx^j \tag{25}$$

We define $\omega_{ij} = \partial_{[i}\psi_{j]}$. There are the following identities:

$$\mathcal{R}_{ij} = R_{ij} + \frac{n-1}{n-2}(D_i U)(D_j U) - e^{-2U\frac{n-3}{n-2}} \left(2\,\omega_{ik}\omega_j{}^k - \frac{1}{n-2}\, h_{ij}\, \omega_{kl}\omega^{kl}\right) \tag{26}$$

$$e^{2U\frac{n-1}{n-2}}\, \Delta_h U = R_{00} - e^{8U\frac{1}{n-2}}\, \omega_{kl}\omega^{kl} \tag{27}$$

$$e^{-U} D^i \left(e^{2U\frac{n-1}{n-2}}\, \omega_{ij}\right) = R_{jo} \tag{28}$$

Assume that the normal derivative of U across the $(n-1)$ - surface S be zero and also that the pull-back of ω_{ij} to S be zero.
When $R_{ij} = 0$, Eq.(7) implies that

$$\mathcal{R}_{nn} = -2\, e^{-2U\frac{n-3}{n-2}}\, \frac{n-3}{n-2}\, \omega_{n\alpha}\omega^{n\alpha} \tag{29}$$

and

$$\mathcal{R}_{\alpha}{}^{\alpha} = \frac{n-1}{n-2}\,(D_\alpha U)(D^\alpha U) + 2\, e^{-2U\frac{n-3}{n-2}} \frac{1}{n-2}\, \omega_{n\alpha}\omega^{n\alpha} \tag{30}$$

Now the Gauss equation implies that

$$\mathcal{R}_{\alpha}{}^{\alpha} = \mathfrak{R} + \mathcal{R}_{nn} - (\mathrm{tr}\, k)^2 + \mathrm{tr}(k^2)\,, \tag{31}$$

where $\mathfrak{R}$ is the Ricci scalar of S. Thus, if S is totally geodesic, $\mathfrak{R}$ is given by

$$\mathfrak{R} = \frac{n-1}{n-2}\,(D_\alpha U)(D^\alpha U) + 2\, e^{-2U\frac{n-3}{n-2}}\, \omega_{n\alpha}\omega^{n\alpha} \tag{32}$$

In particular the scalar curvature of S, $\mathcal{R}$, is non-negative.

The rest of the argument is similar to the 3+1 dimensional case.

We also derived the analogue of Wald's formula for the force between spinning objects in higher dimensions and checked that our results are consistent with it.

To calculate the sign of the gravitational spin-spin force (SSF), following Wald, use Mathisson-Fock-Papapetrou, i.e.

$$F_i = -\frac{1}{2} S'^{jk} R_{jki0} \qquad \text{large } r \tag{33}$$

in asymptotically flat coordinates. Here S'^{ij} is the spin tensor of a test particle. Now from the Killing identity

$$\nabla_\mu \nabla_\nu \xi_\lambda = R_{\nu\lambda\mu\rho} \xi^\rho \tag{34}$$

applied to the Killing vector $\xi^\mu \partial_\mu = \partial_t$. Using the form

$$ds^2 = -e^{2U}(dt + \psi_i \, dx^i)^2 + e^{-\frac{2U}{n-2}} h_{ij} \, dx^i dx^j \tag{35}$$

and $\omega_{ij} = \partial_{[i} \psi_{j]}$, we find that

$$\partial_i \omega_{jk} = R_{jki0} \tag{36}$$

to leading order. For a stationary rotating spacetime in $n+1$ dimensions (se e.g. Myers-Perry) we have that

$$\psi_i \sim -\frac{S_{ij}x^j}{r^n} \tag{37}$$

where we have omitted a positive N - dependent factor. It follows that the SSF is attractive (resp. repulsive) whenever the quantity $\omega_{ij}S'^{ij}$ is negative (resp. positive), which in turn is the same as the quantity

$$S_{ij}S'^{ij} - nS_{ij}S'^{i}{}_{k}n^j n^k \tag{38}$$

being negative (resp. positive), where $n^i = \frac{x^i}{r}$. Let S_n be the $(N-1)$ - plane in $\mathbf{R}^n$ through the origin with normal n^i and Φ_n be the reflection at that plane, i.e. the involution $x^i \mapsto \bar{x}^i = x^i - 2(x,n)\,n^i$. The map

Φ_n acts by pull-back on the covector field $\psi_i(x)$ as

$$(\Phi_n^\star \psi)_i = \bar{\psi}_i \sim -\frac{\bar{S}_{ij}\bar{x}^j}{\bar{r}^n} \,, \tag{39}$$

where $\bar{S}_{ij} = S_{ij} + 4n^k S_{k[i}n_{j]}$. We call the spin tensor S'_{ij} aligned with S_{ij} when $S'_{ij} = -\bar{S}_{ij}$ (and anti-aligned when $S'_{ij} = \bar{S}_{ij}$). For $n = 3$ note that alignment means that $L'_i = L_i - 2(n, L)\, n_i$, where $S_{ij} = \epsilon_{ijk}L^k$. When S' is aligned with S, the sign of (11) is opposite to that of

$$S_{ij}S^{ij} + (n-4)S_{ij}S^i{}_k n^j n^k \tag{40}$$

The expression (13) is positive definite for $n \geq 3$ (for $n = 3$ use L_i to see this). We conclude that aligned spins are attractive.

Now let ψ_i be the Sagnac connection corresponding to a linear superposition of spins S and S', located at points, say a and $-a$, which are reflections under Φ_n of each other and where $S' = -\bar{S}$. It follows that $\Phi_n^\star \psi = -\psi$ and thus $\Phi_n^\star \omega = -\omega$. So, finally, we have that $\Phi_n^\star \omega|_{S_n} = 0$. Thus a superposition of two aligned spins leads exactly to the situation covered by our theorem

Birkhoff's invariant and Thorne's Hoop Conjecture * Thorne's original Hoop Conjecture † was that

> Horizons form when and only when a mass m get compacted onto a region whose circumference in EVERY direction is $C \leq 4\pi M$.

*G. W. Gibbons

†K S Thorne, *Nonspherical Gravitational Collapse: A Short Review* in *Magic without Magic* ed. J Klauder (San Francisco: Freeman) (1972)

The capitalization “EVERY ” was intended to emphasis the fact that while the collapse of oblate shaped bodies the circumferences are all roughly equal, in the prolate case, a the collapse of a long almost cylindrically shaped body whose girth was never the less small would not necessarily produce a horizon. However, as proposed, the statement is so imprecise as to render either proof or disproof impossible. Presumably for the mass we could take the ADM mass, M_{ADM}, but what about the circumference of the hoop?

We can assume that the horizon * or apparent horizon † is topologically spherical ‡

Suppose that $S = \{S^2, g\}$ is a sphere with arbitrary metric g and $f : S \to \mathcal{R}$ a function on S with just two critical points, a maximum and a minimum. Each level set $f^{-1}(c)$, $c \in \mathcal{R}$ has a length $l(c)$ and for any given function f we define

$$\beta(f) = \mathrm{max}_c \, l(c)\,. \tag{41}$$

*S. W. Hawking, Black holes in general relativity. *Comm. Math. Phys.* **25** (1972), 152-166.

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‡G. W. Gibbons, The time symmetric initial value problem for black holes, *Commun. Math. Phys.* **27** (1972) 87, *Some Aspects of Gravitational Radiation and Gravitational Collapse* Ph.D. Thesis, University of Cambridge (1972), S. W. Hawking, *The event horizon* , in *Black holes (Les astres occlus)* C. M. and B. De Witt(1973) 1-55

We now define the Birkhoff invariant * $\beta(S, g)$ by minimizing $\beta(f)$ over all such functions

$$\beta = \inf_f \beta(f)\,. \tag{42}$$

*G. D. Birkhoff, Dynamical systems with two degrees of freedom *Trans. Amer. Math. Soc.* **18** (1918)

The intuitive meaning of β is the least length of a length of a closed (elastic) string or rubber band which may be slipped over over the surface S. To understand why, note that each function f gives a foliation of S by a one parameter family of curves $f = c$ which we may think of as the string or rubber band at each "moment of time "c. $\beta(f)$ is the longest length of the band during that process. If we change the foliation we can hope to reduce this longest length and the infinum is the best that we can do. The phrase "moment of time "is in quotation marks because we are not regarding f as a physical time function, merely a convenient way of thinking about the geometry of S.

Birkhoff's Theorem then assures us that there exist a closed geodesic γ on S with length $l(\gamma) = \beta(g)$. Clearly, if $l(g)$ is the length of the smallest non-trivial closed (i.e periodic) geodesic then

$$l(g) \leq \beta(g)\,. \tag{43}$$

It seems therefore that the Birkhoff invariant $\beta(g)$ should be taken as a precise formulation of Thorne's rather vague notion of circumference. Thus we make the following

> *Conjecture:* For an outermost marginally trapped surface S lying in a Cauchy hypersurface surface Σ with ADM mass M_{ADM} on which the Dominant Energy condition holds, then
>
> $$\beta(g) \leq 4\pi M_{\text{ADM}}\,. \tag{44}$$

In other words, (44) is conjectured to be a necessary condition for a marginally outermost trapped surface. Bearing in mind Thorne's comments about very prolate shaped surfaces for which $\beta(g)$ can be extremely small, it is not claimed that (44) is a sufficient condition for a closed surface S to be trapped.

Clearly, from (43), this form of the hoop conjecture implies

$$l(g) \leq 4\pi M_{\mathrm{ADM}}\,, \tag{45}$$

and therefore a counter example to (45) would be a counter example to (44).

The Kerr-Newman Horizon We first test the conjecture on the general charged rotating black hole. In standard notation, the metric on the horizon is

$$g = ds^2 = (r_+^2 + a^2)((1 - x^2 \sin^2\theta) d\theta^2 + \frac{\sin^2\theta d\phi^2}{1 - x^2\sin^2\theta}) , \tag{46}$$

with

$$x^2 = \frac{a^2}{r_+^2 + a^2} . \tag{47}$$

This is clearly foliated by the orbits of the group of rotations generated by by $\frac{\partial}{\partial\phi}$ and we take $f = \cos\theta$.

That is, we are thinking of the coordinate θ as a function on S. In this case the greatest length of the small circles, i.e. of the orbits, is l_e, the length of the of the equatorial geodesic at $\theta = \frac{\pi}{2}$ and we have

$$\beta(\cos\theta) = l_e = 2\pi(r_+ + \frac{a^2}{r_+}) = 2\pi(2M - \frac{Q^2}{r_+}) \leq 4\pi M \,. \tag{48}$$

The right hand side of (48) is certainly an upper bound for the Birkhoff invariant and so the conjecture certainly holds in this case.

However the horizon is prolate in character, in the sense that the polar circumference l_p which is the length of a meridional geodesic l_p (i.e. one with $\phi =$ constant and $\phi =$ constant $+ \pi$), is

$$l_p = \sqrt{r_+^2 + a^2} \int_0^\pi \sqrt{1 - x^2 \sin^2\theta}\, d\theta\,. \tag{49}$$

In fact taking $f = \sin\theta\cos\phi$, we have

$$\beta(\sin\theta\cos\phi) = l_p\,, \tag{50}$$

and since

$$\sqrt{r_+^2 + a^2} \leq r_+ + \frac{a^2}{r_+}\,, \tag{51}$$

we have

$$\beta(g) \leq l_p \leq l_e \leq 4\pi M\,. \tag{52}$$

Despite being prolate, the Gaussian curvature K of the surface is given by

$$K = \frac{({r_+}^2 + a^2)(r_+^2 - 3a^2 \cos^2\theta)}{({r_+}^2 + a^2 \cos^2\theta)^3}\,, \tag{53}$$

and can become negative at the poles $\theta = 0, \pi$ as discovered by Smarr * .

*L. Smarr, Surface Geometry of Charged Rotating Black Holes, *Phys Rev* **D 7** (1973) 289

The Kerr-Newman metrics have been generalized to include up to four different charges associated with four different abelian vector fields. In the subclass for which only two charges are non-vanishing we can use these metrics to examine the conjecture. The energy momentum tensor of the system satisfies the Dominant Energy The horizon geometry is given by

$$ds^2 = W d\theta^2 + \frac{(r_{+1} r_{+2} + a^2)^2}{W} \sin^2\theta \, d\phi^2 \,, \tag{54}$$

with

$$W = r_{+1} r_{+2} + a^2 \cos^2\theta \,. \tag{55}$$

and

$$r_{+1} = r_+ + 2m \sinh^2\delta_1 \,, \qquad r_{+2} = r_+ + 2m \sinh^2\delta_2 \tag{56}$$

with r_+ the larger root of $r^2 - 2mr + a^2 = 0$ and δ_1 and δ_2 two parameters specifying the two charges. If $\delta_1 = \delta_2$ we obtain the Kerr-Newman case.

Just as the horizon geometry of the Kerr-Newman solution is isometric to that of the neutral Kerr, so in this more general case, we find an isometric horizon geometry. Of course the interpretation of the parameters occurring in the metric is different, but the geometry is the same. Thus

$$\beta(g) \le l_p \le l_e = 2\pi(\sqrt{r_{+1}r_{+2}} + \frac{a^2}{\sqrt{r_{+1}r_{+2}}})\,. \tag{57}$$

Now for positive x, y, z,

$$xy \le \frac{1}{4}(x+y)^2\,, \quad \Longrightarrow \quad \sqrt{(z+x)(z+y)} \le z + \frac{1}{2}(x+y)\,. \tag{58}$$

Thus,

$$\sqrt{r_{+1}r_{+2}} \le r_+ + m(\sinh^2\delta_1 + \sinh^2\delta_2) \tag{59}$$

and

$$\frac{a^2}{\sqrt{r_{+1}r_{+2}}} \le \frac{a^2}{r_+}\,, \tag{60}$$

Thus

$$(\sqrt{r_{+1}r_{+2}} + \frac{a^2}{\sqrt{r_{+1}r_{+2}}}) \le r_+ + m(\sinh^2\delta_1 + \sinh^2\delta_2) + \frac{a^2}{r_+}. \tag{61}$$

But

$$2m = r_+ + \frac{a^2}{r_+}\,, \tag{62}$$

and the ADM mass is given by

$$M_{ADM} = 2m + 2m(\sinh^2\delta_1 + \sinh^2\delta_2) \tag{63}$$

Thus

$$\beta(g) \leq 4\pi M_{ADM}\,, \tag{64}$$

and the conjecture holds in this case. It would be interesting to check it in the four charge case, but the algebra appears to be rather more complicated.

Collapsing Shells and Convex Bodies This is a class of examples * in which a shell of null matter collapses at the speed of light in which the apparent horizon S may be thought of as a convex body isometrically embedded in Euclidean space $\mathcal{E}^3$. In this case one has

$$8\pi M_{\text{ADM}} \geq \int_S H dA\,, \tag{65}$$

where $H = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2})$ is the mean curvature and R_1 and R_2 the principal radii of curvature of S and dA is the area element on S. The right hand side is called the total mean curvature and it was shown by Álvarez Paiva † that in this case that

$$\beta(g) \leq \frac{1}{2}\int_S H dA\,. \tag{66}$$

*G. W. Gibbons, Collapsing Shells and the Isoperimetric Inequality for Black Holes, *Class. Quant. Grav.* **14** (1997) 2905 [arXiv:hep-th/9701049].

†J .C. Álvarez Paiva, Total mean curvature and closed geodesics. *Bull. Belg. Math. Soc. Simon Stevin* **4** (1997) 373–377.

Combining Álvarez Paiva's (66) with (65) establishes the conjecture (44) in this case.

In fact the proof is close to the ideas of Tod. If $\mathbf{n}$ is a unit vector we define the height function on $S \subset \mathcal{E}^3$ by

$$h = \mathbf{n}.\mathbf{x}\,, \qquad \mathbf{x} \in \mathbf{S}\,. \tag{67}$$

Let $S_{\mathbf{n}}$ be the orthogonal projection of the body S onto a plane with unit normal $\mathbf{n}$ and let $C(\mathbf{n}) = l(\partial S_{\mathbf{n}})$ be the perimeter of $S_{\mathbf{n}}$. Then

$$\beta(g) \le \beta(h) \le C(\mathbf{n})\,. \tag{68}$$

Now

$$\int_S H dA = \frac{1}{2\pi} \int_{S^2} C(\mathbf{n}) d\omega\,, \tag{69}$$

where $d\omega$ is the standard volume element on the round two-sphere S^2 of unit radius. Thus averaging (68) over S^2 and using (69) gives (66).

Quasi-Normal Mass

Some recent constructions of Yau et al. have made us of isometric embeddings of apparent horizons into euclidean space $\mathbb{E}^3$. The ideas described here may be relevant.

The Penrose inequality It is now well established (e.g. Huisken and Ilmanen, Bray) that the area $A(g)$ of the outermost marginally trapped surface should satisfy Penrose's isoperimetric type conjecture that

$$\sqrt{\pi A(g)} \leq 4\pi M_{ADM} \,. \tag{70}$$

Evidently, if we could bound $\beta(g)$ above by by $\sqrt{\pi A(g)}$ we would have a proof of my version (44) of the Hoop conjecture. On the other hand, if we can bound $\sqrt{\pi A(g)}$ above by $\beta(g)$, then the Hoop conjecture would imply the Penrose conjecture.

This raises the question of what is known about bounds for $A(g)$, $\beta(g)$, $l(g)$ and other invariants, either for a surface in general, or one with some additional restrictions.

We begin by noting that the Riemannian metric g on S allows us to define a distance $d(x, y) = d(y, x), x, y \in S$ which is the infinum of the length of all curves from x to y. Then

$$b(x) = \max_y d(x, y) \tag{71}$$

is the furthest we can get from x. The then define

$$\begin{aligned} e(g) &= \min_x b(x) = \min_x \max_y d(x, y) &(72)\\ E(g) &= \max_x b(x) = \max_x \max_y d(x, y) &(73) \end{aligned}$$

Hebda provides a lower bound for A:

$$\sqrt{A(g)} \geq \frac{1}{\sqrt{2}}(2e(s) - E(g)) \,. \tag{74}$$

Using (70) we get

$$4\pi M_{ADM} \geq \sqrt{\frac{\pi}{2}}(2e(s) - E(g))\,, \tag{75}$$

For the sphere the right hand side of (74) is $\sqrt{2\pi^3}M_{ADM}$ which is satisfied but not sharp. There seems therefore no reason to choose $C(g) = \sqrt{\frac{\pi}{2}}(2e(s) - E(g))$, in order to sharpen Thorne's conjecture.

Another lower bound for the area has been given by Croke [**?**]. If, as above, $l(g)$ is the length of the shortest non-trivial geodesic on S, then Croke proves that

$$\sqrt{A(g)} \geq \frac{1}{31} l(g)\,. \tag{76}$$

This is again, far from the best possible result, which Croke conjectures to be

$$\sqrt{A(g)} \geq \frac{1}{3^{\frac{1}{4}} 2^{\frac{1}{2}}} l(g)\,, \tag{77}$$

which is attained for two flat equilateral triangles glued back to back.

If we use (70) and (77 we obtain

$$(\frac{\pi^2}{12})^{\frac{1}{4}} l(g) \leq 4\pi M_{ADM}\,. \tag{78}$$

If one takes $C(g) = l(g)$, then (78) is weaker than Thorne's suggestion and taking $C(g) = (\frac{\pi^2}{12})^{\frac{1}{4}} l(g)$ looks rather perverse, and in any case there is a problem about when it is attained. Moreover, since $\beta(g) \geq l(g)$, we cannot easily relate (78) to my form of the conjecture (44). Curiously however, for a special class of surfaces, we can improve considerably on (74) or (78).

Horizons admitting an anti-podal map Many results for general surfaces rely on on the existence of non-null homotopic closed curves. For a surface with spherical topology no such curves exist. However it is possible to restrict attention to the special class of surfaces for which $\mathbf{Z}_2$ acts freely and isometrically such that $x \to Ix$. The quotient $S^2/I \equiv \mathbf{RP}^2$ and Pu provides a lower bound for$A$ in terms of the the systole sys(S/I), i.e. the length of the shortest non-null homotopic curve:

$$\sqrt{A(S/I)} \geq \sqrt{\frac{2}{\pi}} \mathrm{sys}(S/I)\,. \tag{79}$$

Now the shortest non-null homotopic curve on S/I is a closed geodesic which lifts to a closed geodesic of twice the length on S, thus

$$\mathrm{sys}(S/I) = \min_x d(x, Ix) \leq b(x) \leq e(g)\,, \tag{80}$$

where $b(x)$ and $e(g)$ are taken on the spherical double cover.

If, as before, $l(g)$ is the length of the shortest non-trivial geodesic on S, then for this class of metrics

$$\sqrt{A(g)} \geq \frac{2}{\sqrt{\pi}} \mathrm{sys}(S/I) \geq \frac{l(g)}{\sqrt{\pi}} \tag{81}$$

and hence, using (70) we obtain for this class of metrics,

$$l(g) \leq 4\pi M_{ADM} \, , \tag{82}$$

i.e. the inequality (45) which is a *consequence* of my version of the hoop conjecture (44). Thus *no counter example to to my conjecture can be constructed within the class of horizons admitting an antipodal isometry*.

Of course (45) is of the form of Thorne's suggestion, if we take the circumference $C = l(g)$. However $l(g)$ does not carry with it the idea of the least circumference in all directions. I have argued above that it is $\beta(g)$ which better captures that notion, and so I prefer to think of (45) as a consequence of the more basic inequality (44) and the fact that (45) holds in this special case as a confirmation of the general plausibility of this line of argument.

Injectivity and Convexity Radii

The literature on area, the lengths of geodesics etc is often couched in terms of the injectivity radius $i(g)$ and the convexity radius $c(g)$. In the sequel we mainly follow papers of Berger. The definitions are valid for any dimension.

The injectivity radius $i(x)$ of a point $x \in S$ is the supremum of the distances out to which which the exponential map is a diffeomorphism onto its image. The injectivity radius $i(g)$ of the manifold is the infinum over all points in S of $i(x)$. In the case of an axisymmetric body for which the metric may be written as

$$ds^2 = R^2\{d\theta^2 + a^2(\theta)d\phi^2\} \tag{83}$$

with R an overall constant setting the scale, $0 \leq \theta \leq \pi$, the injectivity radius of the north ($\theta = 0$) or south ($\theta = \pi$pole is $\frac{1}{2}C_p = \pi R$ and

$$i(g) \leq \frac{1}{2}C_p \,. \tag{84}$$

Now local extrema of $a(\theta)$ correspond to azimuthal geodesics. If the Gaussian curvature is positive, there will only be one, and define C_e as its length. Otherwise C_e as the smallest such length.

$$l(g) \leq C_p \,, \qquad l(g) \leq C_e \tag{85}$$

The convexity radius $c(x)$ of a point x is the largest radius for which the geodesics ball $B_c(x)$ centred on x is geodesically convex, that is every point in $B_c(x)$ is connected by a unique geodesic interval lying entirely within $B(x)$. The convexity radius $c(g)$ of the manifold is the infinum over all points in S of $c(x)$. On the round unit-sphere ($R = 1$) we have $i = \pi$ and $c = \frac{\pi}{2}$.

Now Berger proves that

$$l(g) \geq 2c(g)\,. \tag{86}$$

and hence

$$\beta(g) \geq 2c(g) \tag{87}$$

Thus in the case of horizons admitting an antipodal map, we can combine Pu's result and (70 to obtain

$$2c(g) \leq 4\pi M_{ADM}\,, \tag{88}$$

in the case that my form of the hoop conjecture (44) holds we obtain from (87) the same result. In fact Klingenberg has shown that either

$$l(g) = 2i(g) \tag{89}$$

or there is a geodesic segment of length $l(g)$ whose end points are conjugate. Finally for metrics on S^2 we have the so-called *isembolic inequality*

$$\sqrt{\pi A} \geq 2i(g) \geq 4c(g) \tag{90}$$

and hence by (70)

$$4\pi M_{ADM} \geq 2i(g) \geq 4c(g)\,. \tag{91}$$