Iran Lectures

Part 1: Gravity duals of Fluid dynamics and a local 2nd law of thermodynamics

Part 2: Pure states vs black holes in D1-D5

~~Lectures 1+2~~ : AdS_5, [hydrodynamics, Navier-Stokes vs Einstein equations, increase of local entropy from local area increase, white hole, B.H. formation]

~~Lecture 3~~ : D1-D5 system [scattering off 2-charge system, pure states — BH & non-BH, Kibble Zurek mechanism]

Refs

Part 1 : Bhatta, Hubeny, Minwalla, Rangamani 0712

w/ " " " "
Reall, ~~Fa~~ Morita, Loganayagam 0803

w/ Morita, Wadia (in progress) 09..

Part 2 : w/ Dhr ~~090~~ 0812

w/ Dm (in progress)

Motivation: ~~(a) The arrow of time in thermodynamics is in apparent violation of the fundamental laws and is understood as a coarse grained notion. The black hole area theorem is apparently exact.~~ Can we relate these in AdS/CFT

Plan:

(0): ~~Gravity - hydrodynamics connection & the time arr~~

Lectures 1 & 2

(i) Thermodynamics
(ii) Hydrodynamics
(iii) Navier Stokes from radial ADM
(iv) Explicit connection between solutions of Einstein's equat & solutions of Navier Stokes

(v) Arrow of time: 2nd law of thermodynamics from Raychaudhuri equation

[(vi) White hole and violation of 2nd law - optional]

Lecture 3 Emergence of "~~black holes" from pure states~~ in D1-D5 system

(i) review of 2-charge systems - CFT
(ii) Fuzzball - classical solutions
(iii) CWZ counting
(iv) Scattering
(v) Equilibration - dynamics
(vi) Black hole formation (Kibble-Zurek)
(vii) Naked singularity - Gregory Laflamme - Matrix model

(i) A very brief review of AdS/CFT at finite T.

T=0

- $N=4$ SYM on $\mathbb{R}^{3,1}$ $\Longleftrightarrow$ IIB string theory on $AdS_5^P \times S^5$

where AdS_5^P refers to the Poincare patch of AdS_5, described by

$$ds^2_{AdS_5^P} = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2}dr^2, \qquad r \in (0,\infty), \quad t, \vec{x} \in \mathbb{R}^{3,1}$$

$$ds^2_{S^5} = R^2 d\Omega_5^2$$

- $\mathcal{N}=4$ SYM on $S^3 \times$ time $\Longleftrightarrow$ IIB string on $AdS_5 \times S^5$

$$ds^2_{AdS_5} = \cancel{R^2\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\right)} \qquad \cancel{\rho \in (0,\infty)}$$

$$= \left(-\left(1+\frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{1+r^2/R^2} + r^2 d\Omega_3^2\right) \qquad \Omega_3 \in S^3, \quad r = \cancel{R\sinh\rho} \in (0,\infty)$$

$$\boxed{\frac{R^4}{g_s l_s^4} = \frac{R^4}{l_p^4} = N \ (\text{rank}) \qquad g_s = g_{YM}^2}$$

$T \neq 0$

- $\mathcal{N}=4$ SYM on $\mathbb{R}^{3,1}$ $\Longleftrightarrow$ IIB string in black 3-brane geometry in $AdS_5 \times S^5$

$$ds^2_{BB} = \frac{r^2}{R^2}\left(-f\,dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2 f}dr^2$$

$$f = 1 - \frac{r_0^4}{r^4}$$

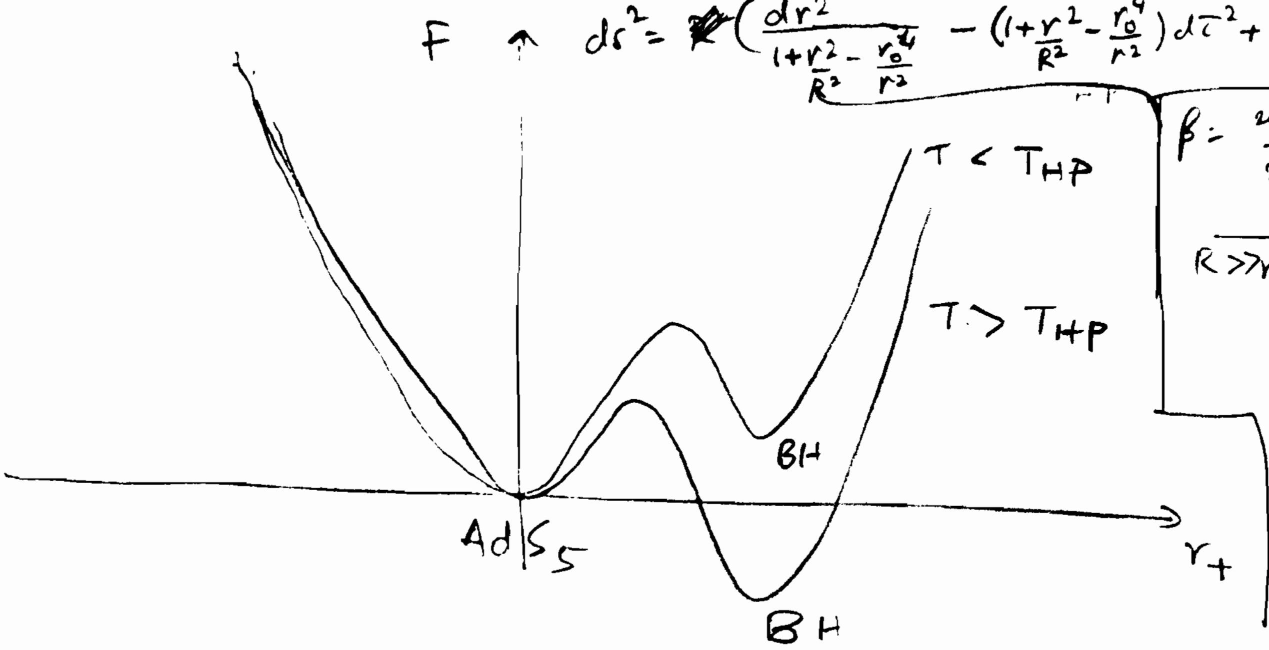
$$\boxed{\frac{r_0}{\pi R^2} = T}$$

Note: $S_{BB} = V.\ \frac{\pi^2}{2} N^2 T^3 \quad (\lambda >> 1) \quad \lambda \equiv g_{YM}^2 N$

$S_{YM} = V.\ \frac{2\pi^2}{3} N^2 T^3 = \frac{4}{3} S_{BB} \quad (\lambda << 1)$

- $\mathcal{N}=4$ SYM on $S^3 \times$ time $\Longleftrightarrow$ IIB theory in AdS_5 $(T < T_{HP})$
or AdS_5-schwarzchild $(T > T_{HP})$

$$ds^2 = \underbrace{\left(\frac{dr^2}{1+\frac{r^2}{R^2}-\frac{r_0^4}{r^2}} - \left(1+\frac{r^2}{R^2}-\frac{r_0^4}{r^2}\right)d\tau^2 + r^2 d\Omega_3^2\right)}_{\beta = \frac{2\pi R^2 r_+}{2r_+^2+R^2} \xrightarrow{R >> r_+} 2\pi r_+}$$



Understanding the correspondence at finite T

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2\left(1-\frac{r_0^4}{r^4}\right)+d\vec{x}^2\right)+\frac{R^2}{r^2}dr^2\underbrace{\left(1-\frac{r_0^4}{r^4}\right)^{-1}}_{1+\frac{r_0^4}{r^4}+O\left(\frac{r_0}{r}\right)^8}$$

$$= ds^2_{AdS} + \frac{r_0^4}{r^2R^2}dt^2 + R^2$$

$$g_{MN} = \bar{g}_{MN} + h_{MN} \quad \text{—— (A)}$$

where h_{MN} = normalizable mode $\sim \frac{r_0^4}{r^4}\bar{g}_{MN}$

$\Rightarrow$ excited state of the YM theory $|\psi\rangle$

$h_{00} = \frac{r_0^4}{r^4}\bar{g}_{00} \Rightarrow h_0^0 = r_0^4/r^4$

s.t.

$\langle\psi|T_0^0|\psi\rangle = r^4 h_0^0 \approx T^4$

$|\psi\rangle$ is meant to be a pure state defined by a series of expectation values

$h_{\mu\nu}$ can be chosen $=0$

$C^{(4)} = ?$

obtd. for near ext. D3 brane at near horizon limit

$$\langle\psi|\mathcal{O}|\psi\rangle = \delta\phi$$

where $\delta\phi$ is the (normalizable) excitation over AdS value for the SUGRA mode ϕ. operators $\mathcal{O}$

Given sufficient number of these, we can determine $|\psi\rangle$, but otherwise we can reproduce $h_{\mu\nu}$ by

$$\mathrm{Tr}\left(T_{\mu\nu}e^{-\beta H}\right) = \sum_{\text{states}}\langle\psi_E|T_{\mu\nu}|\psi_E\rangle e^{-\beta E} = h_{\mu\nu}$$

If we insist on the black brane soln as

$h_{\mu\nu}$ = as in (A) [and $C^{(4)}$ = as from black 3-brane]

and $\delta\phi = 0$ for all other string modes then we must demand $\langle\mathcal{O}\rangle = 0 \;\forall\, \mathcal{O}$ other than $T_{\mu\nu}$

This is impossible to solve with the mixed state $e^{-\beta H}$ but possible with some pure state $|\psi\rangle$

1.36

Task: Construct a static geometry which will give ~~static with~~

$\langle T_{00} \rangle = T^4$

Ans: ~~create a static geometry with~~

$\lim_{r\to\infty} h_{00}(x,r) = \frac{T^4}{r^4}$ (unique?)

Task: Construct a fluid:

$\langle T_{00}(x) \rangle = aT^4(x)\left(\frac{4}{1-\vec{\beta}^2(x)}-1\right) +$ derivatives

$\langle T_{ii}(x) \rangle = \frac{a}{3} T^4(x) \frac{\beta_i^2}{1-\vec{\beta}^2}$ + derivatives

$\langle T_{ij}(x) \rangle = -\underbrace{6T^3(x)}_{\eta}(\partial_i\beta_j + \partial_j\beta_i)$ $(i \neq j)$ + derivatives

$T_{0i}(x) = -\frac{aT^4\beta_i}{1-\vec{\beta}^2}$

Ans. create a time-dependent geometry with boundary data $h_{00}(r\to\infty, x) = \frac{T_{00}(x)}{r^4}$ etc.

Q. can't we specify $T_{00}(x)$ etc arbitrarily?

There are 2 kinds of restrictions

① initial value constraints (does not restrict a, b, since $\partial_\mu T^{\mu\nu}=0$ can be satisfied with any a, b.)

② (h_{00}, h_{ii}, h_{ij} and Θ_{00}, Θ_{ii}, Θ_{ij}) must evolve to black brane with $T(x)$, $\beta_i(x)$

Fick's law ~~[illegible]~~ $A_i = \frac{J_i}{\epsilon}$ $F_{i0} = J_i$ $T_{\mu\nu}$ ~~[illegible]~~ $K_{\mu i}$

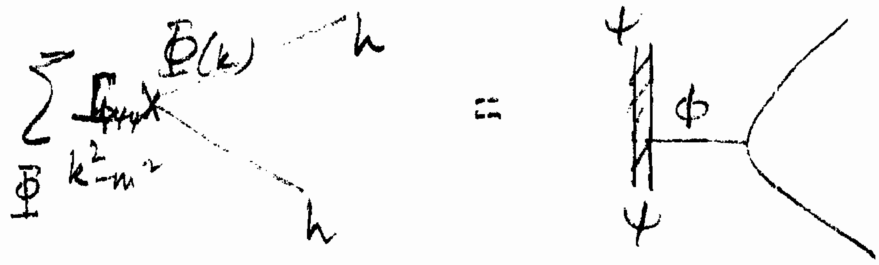
Puzzle: If $T_{00}^0(x) = r^4 h_{00}^0$? or $r^4(H_{00}^0 - 1)$ ← more uniform

Could we relax this? Could there be some other end-point geometry

$\phi = \frac{\langle\mathcal{O}\rangle}{r^4}$ / $\frac{\partial\mathcal{I}}{\partial\mathcal{J}} = \langle\phi\rangle =$

[illegible]

$$(m^2+\nabla^2)\Phi = g_{k\ell}\,\Gamma\,\bar{\Psi}^{*}\Psi \qquad \Gamma \propto C_{\psi^*\psi\Phi}$$



<u>Here</u>

$\langle \Psi \,|\, T_{00}^{(k)} \,|\, \Psi \rangle \Leftrightarrow h_{00}$

$\langle \Psi \,|\,$ irrelevant op $\,|\, \Psi \rangle \Leftrightarrow$ massive mode

(ii) Hydrodynamics

The systems considered so far were static. They were in equilibrium.

Suppose we want to take the system out of equilibrium:

In the bulk, this can be done by exciting a time-dependent nonnormalizable mode. In the boundary, it corresponds to coupling to a time-dependent source term in the field theory.

If the perturbation applied exists for a finite time interval (or vanishes suitably as $t \to \infty$) one expects the system to eventually settle back to a state which closely approximates a thermal state at a new temp $T + \Delta T$ where ΔT depends on the energy pumped into the system.

ing

[[cf Kibble-Zurek mechanism:

$$H = -J \sum_i \sigma_i^3 \sigma_{i+1}^3 + h(t) \sum_i \sigma_i^1$$

Ferro. gr. state $\to$ thermal state]]

[[We will briefly mention recent work by Swm? et al on b.h. formation]]

LOCAL THERMODYNAMICS : Consider a strongly coupled CFT. At high enough densities such a system should equilibrate locally defined by $(\vec{x},t)$-dependent thermodynamic quantities eg. $\rho(\vec{x},t)$ (no. density), $\varepsilon(\vec{x},t)$ (energy density), $p(\vec{x},t)$ (pressure), $T(\vec{x},t)$ (temp.) etc.

CONFORMAL HYDRODYNAMICS

We will, in fact, assume that the system of our interest is described as a fluid characterized entirely by a local velocity $u^\mu(x)$ and local temp. $T(x)$. These are the independently specifiable dynamical variables. $\longrightarrow$ 4 variables

Such data can be uniquely evolved from just the conservation of stress tensor

$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow 4 \text{ equations}$$

provided we know the constitutive eqn of the stress tensor of the fluid. For a perfect fluid the eqn is

$$T^\mu{}_\nu = \begin{pmatrix} \varepsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} = (\varepsilon + p)\, u^\mu u_\nu + p\,\delta^\mu{}_\nu$$

where $\varepsilon = \varepsilon(T)$, $p = p(T)$

for a conformal fluid $\varepsilon - 3p = 0$

and $\varepsilon(T) = 3aT^4$

$$\therefore\ T_{\mu\nu} = \frac{4}{3}\varepsilon\, u_\mu u_\nu + \frac{1}{3}\varepsilon\,\eta_{\mu\nu} = \frac{\varepsilon}{3}\left(\eta_{\mu\nu} + 4u_\mu u_\nu\right)$$

$$= aT^4(\eta_{\mu\nu} + 4u_\mu u_\nu)$$

Imperfect fluid: ~~insert~~ Velocity gradient terms, e.g. viscosity etc. are important

General expansion in velocity & temp. gradients:

$$T_{\mu\nu} = \underbrace{aT^4}_{\varepsilon/4}(\eta_{\mu\nu} + u_\mu u_\nu) \overbrace{+ bT^3}^{-2\eta} \sigma_{\mu\nu}$$
$$+ T^2 \left(c\, T^{2,a}_{\mu\nu} + d\, T^{2,b}_{\mu\nu} + e\, T^{2,c}_{\mu\nu} + f\, T^{2,d}_{\mu\nu} + g\, T^{2,e}_{\mu\nu} \right.$$
$$+ \ldots$$

$$\sigma_{\mu\nu} = P_{\mu\alpha} P_{\nu\beta}\, \partial^{(\alpha} u^{\beta)} - \tfrac{1}{3} P_{\mu\nu} \partial_\alpha u^\alpha \sim O(\partial u)$$

insert (p. 7)

$$T^{2,a}_{\mu\nu} = \epsilon_{\alpha\beta\gamma(\mu}\, \sigma_{\nu)}{}^{\gamma} u^\alpha \ell^\beta \qquad \ell_\mu = \epsilon_{\alpha\beta\gamma\mu} u^\alpha \partial^\beta u^\gamma$$
$$\sim O(\partial u)^2$$

$$T^{2,b}_{\mu\nu}, T^{2,c}_{\mu\nu}, T^{2,d}_{\mu\nu} \sim O(\partial u)^2, \quad T^{2,e}_{\mu\nu} \sim O(\partial^2 u) \;\&$$

$$P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$$

We will determine a, b (equiv. η), c, d, e, f, g from gravity!

Detailed expressions: Loganayagam et al (p.T.O.)

Get
$$\left.\begin{aligned} a &= \pi^4 \\ b &= -2\pi^3 \\ c &= \pi^2 \ln 2 \\ d &= \pi^2 \cdot 2 \\ e &= \pi^2 (2 - \ln 2) \end{aligned}\right\} R_{AdS}=1 \text{ units}$$

Putting in R_{AdS} ($R^4_{AdS} \propto N^2$)

$$a = \frac{\pi^2}{32} N^2$$
$$\Rightarrow \varepsilon = \frac{\pi^2}{8} N^2 T^4$$
$$\Rightarrow s = \frac{\partial \varepsilon}{\partial T} = \frac{\pi^2}{2} N^2 T^3$$

insert (**) p. 1.3b

$$T_{2b}^{\mu\nu} = \sigma^{\mu\alpha}\sigma^{\nu}{}_{\alpha} - \frac{1}{3} P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta}$$

$$T_{2c}^{\mu\nu} = \partial_\alpha u^\alpha \sigma^{\mu\nu}$$

$$T_{2d}^{\mu\nu} = Du^\mu Du^\nu - \frac{1}{3} P^{\mu\nu} Du^\alpha Du_\alpha \qquad D \equiv u^\alpha \partial_\alpha$$

$$T_{2e}^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta} D(\partial_\alpha u_{\beta)}) - \frac{1}{3} P^{\mu\nu} P^{\alpha\beta} D(\partial_\alpha u_\beta)$$

iii) Navier Stokes from radial ADM:

Recall usual Navier Stokes eq^n:

Non-rel $\Rightarrow$ $\varepsilon(x^\mu) \to \rho(x^\mu)$
since energy $\to$ rest mass

Non-rel $\Downarrow$ $\boxed{u_0 \simeq 1 \quad u_i \simeq v_i}$ $\varepsilon \simeq \rho$

Ideal fluid: $T_{00} = \rho$

$T_{0i} = \rho v_i$, $T_{ik} = p\delta_{ik} + \rho v_i v_k$

$$\partial_\mu T^{0\mu} = 0 \Rightarrow \partial_t \rho + \partial_i(\rho v^i) = 0 \Rightarrow \text{continuity eq}$$

$$\partial_\mu T^{i\mu} = 0 \Rightarrow \partial_t(\rho v^i) + \partial_j(p\delta^{ij} + \rho v^i v^j) = 0$$

$$\Rightarrow \cancel{(\partial_t \rho) v^i} + \underbrace{\rho \partial_t v^i} + \partial^i p$$
$$+ \cancel{v^i \partial_j(\rho v^j)} + \rho \underbrace{v^j \partial_j v^i} = 0$$

$$\Rightarrow \rho \frac{dv^i}{dt} + \partial^i p = 0$$

$$\Rightarrow \rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p \Rightarrow \text{Force law}$$

for imperfect fluid

$$T_{ik} = p\delta_{ik} + \rho v_i v_k - \tilde{\sigma}_{ik}$$

$$\tilde{\sigma}_{ik} = \eta\left(v_{(i,k)} - \frac{2}{3}\delta_{ik} v_{l,l}\right) + \zeta \delta_{ik} v_{l,l}$$

$$v_{(i,k)} = v_{i,k} + v_{k,i}$$

$\zeta = 0$ for CFT

$$\partial_\mu T^{i\mu} = 0 \Rightarrow \rho\frac{d\vec{v}}{dt} = -\vec{\nabla} p + \eta \nabla^2 \vec{v} \Longrightarrow \text{Navier Stokes}$$

Gravity dual to hydrodynamics

· Linear analysis (KSS)

QFT : $S = S_0 + \int_x J_a(x)\, O_a(x)$ $\quad \langle O_a(x)\rangle_{J=0} = 0$

Linear resp. $\langle O_a(x)\rangle_{J\neq 0} = -\int_y \underbrace{G^R_{ab}(x-y)}\, J_b(y)$

$\frac{1}{i}\,\theta(x^0-y^0)\, \langle [O_a(x), O_b(y)]\rangle$

Similarly $S = S_0 + \int_x \underbrace{T_{\mu\nu}(x)\, h^{\mu\nu}(x)}$

$T_{\mu\nu} = T^{\text{perfect}}_{\mu\nu} + \sigma_{\mu\nu}\big|_h$ + higher deriv. $\Rightarrow T_{xy} = \sigma_{xy}$ + higher deriv

$\sigma_{\mu\nu} = P_{\mu\alpha} P_{\nu\beta}\, [\eta\, \nabla_\alpha u_\beta, + (\zeta - \frac{2\eta}{3})\, g_{\alpha\beta}\, \nabla\cdot u]$

$\Rightarrow \sigma_{xy} = \eta\, (\nabla_x u_y + \nabla_y u_x)/2$ $\qquad P_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$

$= \eta\, (\mp \Gamma^0_{xy}) = \eta\, \partial_0 h_{xy}$ $\quad \because u_\mu = (1, \vec{0})$

However by lin. response

$\langle \sigma^{(\xi)}_{xy}\rangle = -\int_{\xi'} G^R_{xy,xy}(\xi-\xi')\, h_{xy}(\xi') = \eta\, \partial_0 h_{xy}$

$\Rightarrow G^R_{xy,xy}(\omega, \vec{0}) = -i\eta\omega + O(\omega^2)$ since higher derivatives have been ignored

$\Rightarrow \eta = \lim_{\omega\to 0}\left[-\frac{1}{i\omega}\, G^R_{xy,xy}(\omega, \vec{0})\right]$

$G^R_{xy,xy}(\omega, \vec{0}) = \int d^3\vec{x}\, dt\, e^{-i\omega t} \langle [\sigma_{xy}(t, \vec{x}), \sigma_{xy}(0, \vec{0})]\rangle$

$\shortparallel$

$\langle [T_{xy}(t, \vec{x}), T_{xy}(0, \vec{0})]\rangle$

since at this order $T_{xy} = \sigma_{xy}$

$\Rightarrow \frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$

· Non linear analysis

Geometries dual to fluids

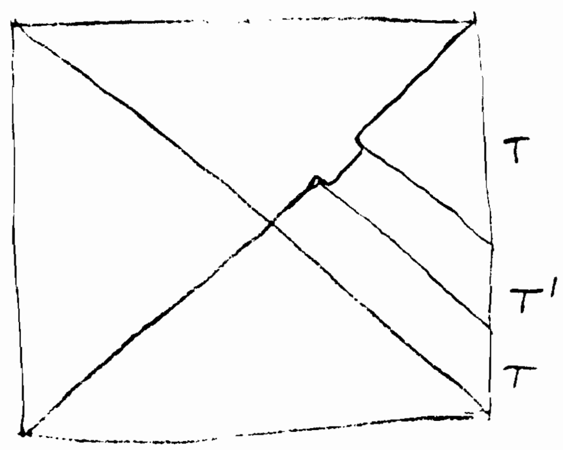
Hint:
$$r_0 = T \pi R^2$$

For $T = T(x)$, perhaps the dual is a fluctuating black brane with horizon at

$$r = r_0(x) = T(x)\, \pi R^2 \quad ?$$

The answer is yes

Consider a geometry that looks locally like a black brane with horizon at $r_0 = r_0(x)$



One way to obtain it is to consider the static black brane as a soliton and excite its collective coordinates...

Kink

$$\int d^2x \left[\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \;-\; \lambda(\phi^2-a^2)^2 \right]$$

Symmetry : Poincaré (1,1)

$\phi_s = a \tanh(\sqrt{2\lambda}\,(x - x_0))$ is a static classical solution

~~$\phi(x,t) = \tanh(A(x\,\mathrm{ch}\beta + t\,\mathrm{sh}\beta - x_0))$ is also a classical solution~~

~~β, x_0 are 2 zero modes~~

Collective coordinate $x_0 \longrightarrow x_0(t)$ $\quad \phi_s(x,t) = a\tanh(\sqrt{2\lambda}(x - x_0(t)))$

$$S_{eff} = \tfrac{1}{2} M \int \dot{x}_0^2\,dt + S(x_0,\Phi), \quad M = \int \left(\frac{\partial \phi_s}{\partial x_0}\right)^2 dx$$

This is not a classical solution unless $\frac{\delta S_{eff}}{\delta x_0} = \ddot{x}_0 + \ldots$

Boosted black brane :

$+dt$, $t + r_* = v$

$-dt^2 F(r) + \frac{dr}{F(r)}$

$= F(r)\,(-(v - r_*)^2 + dr_*^2)$

$= F(r)\,(-dv^2 + 2\,dv\,dr_*)$

$$ds^2 = -r^2 f(r)\,dv^2 + 2\,dv\,dr + r^2 d\vec{x}^2$$

$$= -r^2 f(r)\,(u_\mu dx^\mu)^2 + 2\,u_\mu dx^\mu dr + r^2 P_{\mu\nu}\,dx^\mu dx^\nu$$

$P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu \rightarrow$ projection operator

4 zero modes

- $u_\mu = \left(\sqrt{1-\beta^2}, \frac{\vec\beta}{\sqrt{1-\vec\beta^2}}\right) \longrightarrow$ (3 zero modes)
- $r_0 \longrightarrow$ (1 zero mode)

Consider promoting $u_\mu \to u_\mu(x)$, $r_0 \to r_0(x)$

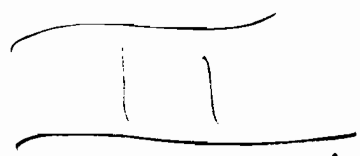
The resulting metric is not a classical solution of Einstein eqn unless $u_\mu(x)$, $r_0(x)$ satisfy the Navier-Stokes equation !!

We will provide 2 proofs

Proof 1: using radial ~~Hamiltonian~~ ADM:

Recall usual ADM (Hamiltonian treatment of GR)

f. E.M. field



$A_i,\ \pi^i \equiv E^i$

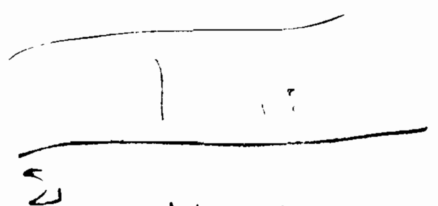
initial value data is constrained

$\partial_i \pi^i = 0$

$\vec{\nabla}\cdot\vec{E} = 0$ Gauss law

$(\partial_i F^{i0} = 0$

$\nu = 0$ component of $\partial_\mu F^{\mu\nu} = 0)$



Σ $\quad \gamma_{ij},\ \pi^{ij}$

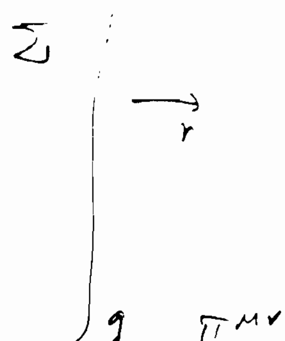
$\pi^{ij} = \pm\sqrt{\gamma}\left(K^{ij} - K h^{ij}\right)$

$D_i\left(\frac{1}{\sqrt{\gamma}}\pi^{ij}\right) = 0$ $\quad \cancel{D_i \pi^{ij} = 0}$ $\quad D_i$ uses $g^{(3)}$

Constraint equation (A component of Einstein's equation)

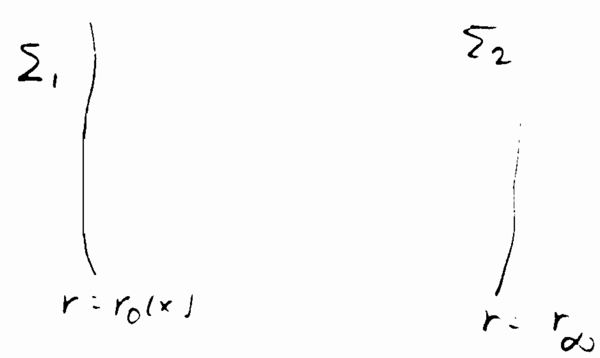
see p. D2

Suppose we set up the ~~same~~ Hamiltonian problem in the radial direction.



$$\cancel{D_\mu(\sqrt{\gamma}\,\pi^{\mu\nu}) = 0}$$

$$D_\mu\left(\gamma^{-1/2}\pi^{\mu\nu}\right) = 0$$



Suppose we apply the radial ADM constraint to the surface Σ_2 then we get

$$D_\mu\left(\gamma^{-1/2}\pi^{\mu\nu}\right)\Big|_{\Sigma_2} = 0$$

However, notice that, at $r = r_\infty$, $\gamma_{\mu\nu} = r_\infty^2\,\eta_{\mu\nu}$ $\Rightarrow \partial_\mu \pi^{\mu\nu} = 0$ — (A) *

$$T^{\mu\nu} = \frac{2\sqrt{-g}}{\pi}\,\pi^{\mu\nu} = 2\pi^{\mu\nu} \rightarrow \boxed{\partial_\mu T^{\mu\nu} = 0}$$

$\therefore$ We get $\partial_\mu T^{\mu\nu} = 0 \iff$ Relativistic Navier-Stokes

If we consider the bulk geometry as ~~~~ evolved from an initial data provided at the boundary and demand that the ~~~~ bulk geometry satisfies Einstein equation, then we ... the boundary matt must satisfy Navier-Stokes equation.

Qualifier: $T^{\mu\nu}$ defined in (A) is the bare $T^{\mu\nu}$

$$\left(T^{\mu\nu}\right)_{ren} = \left(T^{\mu\nu}_{bare} - T^{\mu\nu}_{c.t}\right)$$

next page

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\Big|_{boundary}$$

$$\therefore \pi^{\mu\nu} = \sqrt{-g}\,T^{\mu\nu} = r_\infty^4\,T^{\mu\nu}$$

On the other hand,

$$\delta S = \int_{\partial M} d^4x\,\pi^{\mu\nu}\,\delta g_{\mu\nu}$$

$$\left(S = \int dt\, L(q,\dot q)\right) \quad (\partial L/\partial \dot q)$$

$$\Rightarrow \delta S = \int dt\left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q\right) = \Big[\frac{\partial L}{\partial \dot q}\delta q$$

$$\delta S = \left[p\,\delta q\right]_{t_i}^{t_f}$$

indeed the above formula (cf. Balasubramanian & Kraus 1999) follows from

$$S_{ren} = S_{bare} + S_{c.t}[g] \quad \text{where} \quad T_{c.t}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{c.t.}}{\delta g_{\mu\nu}}$$

so that

$$\Pi^{\mu\nu}_{ren} = \Pi^{\mu\nu}_{bare} + \frac{\delta S_{c.t.}}{\delta g_{\mu\nu}}$$

$$\Rightarrow \quad \Pi^{\mu\nu}_{ren} = r_\infty^4 \, T^{\mu\nu}_{ren}$$

The ADM constraint, applied to S_{ren} gives

$$\partial_\mu \Pi^{\mu\nu}_{ren} = 0$$

$$\Longrightarrow \quad \boxed{\partial_\mu T^{\mu\nu}_{ren} = 0} \qquad \underline{\text{Navier Stokes}} \ **$$

[[Alternatively,

$$T^{\mu\nu}_{c.t.} = \text{of the form } A g^{\mu\nu} + B G^{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu}[g] - \tfrac{1}{2} R[g]\, g_{\mu\nu}$$

$\partial_\mu T^{\mu\nu}_{c.t.} = 0$ is automatically true

Hence $\partial_\mu T^{\mu\nu}_{bare} = 0 \Rightarrow \partial_\mu T^{\mu\nu}_{ren} = 0$]]

<u>Explicit expression</u>

$$T^{\mu\nu}_{ren} = \frac{1}{8\pi G} \left[\Theta^{\mu\nu} - \Theta g^{\mu\nu} - \frac{3}{R} g^{\mu\nu} \right]$$

$\Theta^{\mu\nu} = \nabla^{(\mu} n^{\nu)}$ $\quad n^\mu$ = outward normal to ∂M

The counterterms (Balasubramanian + Kraus)

$$T^{\mu\nu}_{BY}(x) = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}(x)} \quad \text{(factor of 2)}$$

$$= \frac{1}{8\pi G} \left[\Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} \right]$$

$$\Theta^{\mu\nu} = -\frac{1}{2} \left(\nabla^\mu \hat{n}^\nu + \nabla^\nu \hat{n}^\mu \right)$$

"Pf":

$$S = -\frac{1}{16\pi G} \int_M d^5 y \sqrt{-g^{(5)}} \left(R - \frac{20}{\ell^2} \right)$$

$$- \frac{1}{8\pi G} \int_{\partial M} d^4 x \sqrt{-\gamma^{(4)}}\, \Theta$$

Gibbons Hawking

$$\delta \sqrt{\gamma} = -\frac{\sqrt{\gamma}}{2} \gamma_{\mu\nu} \delta \gamma^{\mu\nu}$$

$$\delta \left(\sqrt{\gamma}\, \Theta_{\mu\nu} \gamma^{\mu\nu} \right) = -\frac{\sqrt{\gamma}}{2} \gamma_{\mu\nu}\, \Theta\, \delta\gamma^{\mu\nu} + \sqrt{\gamma}\, \Theta_{\mu\nu} \delta \gamma^{\mu\nu}$$

$$= \sqrt{\gamma} \left[\Theta_{\mu\nu} - \frac{1}{2} \Theta \gamma_{\mu\nu} \right]$$

$$\cancel{\delta(\sqrt{g}) = \sqrt{g}\left(-\frac{1}{2}\right) R}$$

Holographic stress-tensor

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{eff}}{\delta\gamma^{\mu\nu}} \qquad \text{S_{eff} of CFT}$$

FT ~~S_{eff}~~ has UV divergences, ~~needs~~ needs counter terms (usual QFT)

Principle: Allow all possible diff-inv. counterterms at the boundary

$$S_{ct} = \int_{\partial M} d^4x \left[\sqrt{-\gamma}\ \cancel{\left(a\, L_{GB} + b(C_{\mu\nu\rho\sigma})^2 + c\, R^2\right)}\right.$$
$$\left(a + b R^{(4)} + \cdots\right)$$

Compute

$$M_{ADM} = \int_\Sigma \cancel{d^3x}\ d^3x \sqrt{\sigma}\, N_\Sigma\, \epsilon \qquad \epsilon = u^\mu u^\nu T_{\mu\nu}$$

where $\gamma_{\mu\nu}dx^\mu dx^\nu =: -N_\Sigma^2 dt^2 + \sigma_{ab}(dx^a + N_\Sigma^a dt)(dx^b + N_\Sigma^b dt)$

(ADM split of bdry)

$\frac{r^2}{\ell^3}$ ~~$+\ldots$~~

[AdS_3: $\epsilon \sim T_{00} \sim \frac{1}{8\pi G}\left(-\frac{r^2}{\ell^3} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}}\right)$

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}(-dt^2 + dx^2)$$

$$-\gamma_{tt} = \gamma_{xx} = \frac{r^2}{\ell^2}$$

$$\hat n^\mu = \frac{r}{\ell}\delta^{\mu r} \qquad \hat n^\mu \hat n^\nu \gamma_{\mu\nu} = \gamma_{rr}\frac{r^2}{\ell^2} = 1\ \checkmark$$

$$M = \int dx\, T_{tt} \sim \underset{\to\infty}{r^2} \quad \text{need}\quad S_{ct} = -\frac{1}{\ell}\int\sqrt{-\gamma}$$]

Does Π get renormalized too?

$$\Pi^{ct}_{\mu\nu} = \frac{\delta S_{ct}}{\delta \dot{\gamma}^{\mu\nu}_{,r}} \overset{?}{=} 0$$

e.g. $\quad S_{ct} = \int_{\partial M_r} \sqrt{-\gamma}$

Q. What happens to Gauss law itself?

Recall, in the bulk

$$\mathcal{L}_G = \sqrt{\gamma}\, N \left[{}^{(4)}R + \Theta_{\mu\nu}\Theta^{\mu\nu} - \Theta^2 \right]$$

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr)$$

$\boxed{D_\mu = \text{cov. deriv. w.r.t. } \gamma_{\mu\nu}}$

$$\Theta_{\mu\nu} = \frac{1}{2N}\left[\gamma'_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu \right]$$

$$\Pi_{\mu\nu} = \sqrt{\gamma}\left(\Theta_{\mu\nu} - \Theta \gamma_{\mu\nu}\right)$$

$$\mathcal{H}_C^{\text{radial}} = \Pi^{\mu\nu}\gamma'_{\mu\nu} - \mathcal{L}_G$$

$$= -\sqrt{\gamma}\, N\, {}^{(4)}R + \frac{N}{\sqrt{\gamma}}\left[\Pi^{\mu\nu}\Pi_{\mu\nu} - \frac{\Pi^2}{2}\right] + 2\Pi^{\mu\nu} D_\mu N_\nu$$

$$= \sqrt{\gamma}\left\{ N\left(-{}^{(4)}R + \gamma^{-1}\Pi^{\mu\nu}\Pi_{\mu\nu} - \frac{\gamma^{-1}}{2}(\Pi^\mu_\mu)^2 \right) \right.$$

$$\underbrace{- 2 N_\mu \left(D_\nu \left(\gamma^{-1/2}\Pi^{\mu\nu} \right) \right)}_{=0} + 2 D_\mu \left(\gamma^{-1/2} N_\nu \Pi^{\mu\nu} \right)$$

cf. $\int A_0 \vec{\nabla}\cdot\vec{E}$

Thus ~~structure~~

$$D_\mu \left(\gamma^{-1/2} \, \pi^{\mu\nu}\right) = 0 \quad \text{— (B)}$$

and $\quad D_\mu \left(T^{\mu\nu}_{ct}\right) = 0 \quad$ — (A)

$$\Rightarrow \quad \boxed{D_\mu \left(T^{\mu\nu}_{ren}\right) = 0}$$

proof of (A)

AdS$_3$ $\quad 8\pi G \, T^{\mu\nu}_{ct} = -\frac{1}{\ell}\gamma^{\mu\nu}$

AdS$_4$ $\quad 8\pi G \, T^{\mu\nu}_{ct} = -\frac{2}{\ell}\gamma^{\mu\nu} - \ell G^{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R \gamma_{\mu\nu} \qquad R = R[\gamma]$$

AdS$_5$ $\quad 8\pi G \, T^{\mu\nu}_{ct} = -\frac{3}{\ell}\gamma^{\mu\nu} - \frac{\ell}{2} G^{\mu\nu}$

recall $D_\mu G^{\mu\nu} = 0$ (Bianchi) ! Any deep reason ? ?

Generalization to L_{GB} :

(B) is still true ||

(A) ?

What about arbitrary higher derivative theories ?

S. More detailed proof (Proof 2, by construction)

insert (###) p. 22 [BHMR paper]

Derivative expansion

We set $\partial_\mu \sim \epsilon$ (formally $x \to \epsilon x$)

$\beta_i(x) = \beta_i^{(0)}(x) + \epsilon\, \beta_i^{(1)}(x) + \cdots$, $T(x) = T_0(x) + \epsilon T_1(x) + \cdots$

$$g_{MN} = g^{(0)}_{MN} + \epsilon\, g^{(1)}_{MN} + \epsilon^2 g^{(2)}_{MN} + \cdots$$

$g^{(0)}_{MN} \Rightarrow ds^2_{(0)} =$ local black brane

ghs, "$\nabla^{(0)2}$" $g^{(1)}_{MN} = S^{(1)}$ (includes $\beta_i^{(1)}(x)$, $r_0^{(1)}(x)$ etc.) — ⊛

refers to $g^{(0)}_{MN}[\beta_i^{(0)}, r_0^{(0)}]$ * see D.1 — dynamical eqn.

constraint eqn:

$b \equiv \frac{1}{r_0}$

$$\partial_\mu T^{\mu\nu,(0)} = 0 \Rightarrow \quad \partial_i b^{(0)} = 2 \partial_v \beta_i^{(0)}$$

$$\partial_v b^{(0)} = \frac{\partial_i \beta_i^{(0)}}{3}$$

⊛ $g^{(1)}_{MN} = g^{(1)}_{MN}$, particular soln + zero modes of "$\nabla^{(0)2}$"

The zero modes correspond to $\frac{\partial \phi_{cl}}{\partial x_0}$; by a gauge choice, we chose
$$\int \delta\phi \, \frac{\partial \phi_{cl}}{\partial x} \, dx = 0$$

Eventually g_{MN} can be built

$\Rightarrow$ GRAVITY DUALS OF FLUID

Details $\qquad b \equiv \frac{1}{r_0}$ $\qquad$ D.1

avity dual of Fluid : ϵ-perturbation

$$H\left[g^{(0)}\left(\beta_i^{(0)}, b^{(0)}\right)\right] g^{(1)}(x) = S^{(1)}$$

15 → 4 Constraint eqns (cf. $\partial_\mu T^{\mu\nu}_{(1)} = 0$)
→ 10 dynamical eqns (1 redundant)

$$g^{(1)}(x) = g_p^{(1)}(x) + f_b(x)\, \underline{g_b} + f_i(x)\, \underline{g_i}$$

where $\quad g_b = \frac{\partial g^{(0)}}{\partial b}, \quad g_i = \frac{\partial g^{(0)}}{\partial \beta_i}$

(*A) $\partial_i \beta_i^{(0)} = 3 \partial$ ... ; $\partial_v \beta_i^{(0)} = \partial_i$ ... ; equiv $\ddot{x}_0$ ...

$$\Rightarrow \quad g(x) = g^{(0)}(x) + \epsilon\left(\underline{g_p^{(1)}(x)} + \left(f_b(x) + b^{(1)}\right) g_b + \left(f_i + \beta_i^{(1)}\right) g_i\right)$$

Choose Landau-Gervais-Sakita gauge $[u^{(0)}_\mu T^{(1)\mu\nu} = 0]$

$$g(x) = g^{(0)}(x) + \epsilon\, g_p^{(1)}(x)$$

~~Specifically~~ $\quad ds^2 = \left(-r^2 f(b^{(0)}(x) r)\left(\cancel{u^{(0)}_\mu dx^\mu}\, dv^2\right) - 2\left(\cancel{u^{(0)}_\mu dx^\mu}\, dv\right) dr + r^2 dx_i dx_i\right)$

$$F(r) = \int_r^\infty \frac{(x^2+x+1)\, dx}{x(x+1)(x^2+1)}$$

$$\sigma_{ij}^{(0)} = \partial_{(i}\beta_{j)}^{(0)} - \frac{1}{3}\delta_{ij}\, \partial_m \beta_m^{(0)}$$

$$+ \Big(-2x^\mu \partial_\mu \beta_i^{(0)} dr\, dx^i - 2x^\mu \partial_\mu \beta_i^{(0)} r^2 (1 - f(b^{(0)} \ldots)) dv\, dx^i - \frac{4 x^\mu \partial_\mu b^{(0)}}{r^2} dv^2$$

$$+ 2r^2 F(r)\, \sigma_{ij}^{(0)} dx^i dx^j + \frac{2}{3} r^2 \partial_i \beta_i^{(0)} dv^2 + 2r\, \partial_v \beta_i^{(0)} dr\, dx^i\Big)$$

This satisfies Einstein eqn provided (*A) is satisfied.

$$T_{\mu\nu} = \frac{1}{b^4}\left(\eta^{\mu\nu} + 4u^\mu u^\nu\right) - \frac{2}{b^3}\sigma_{\mu\nu}$$

$$\text{III} \quad \hat{\cdot} = (-2) \lim_{r\to\infty} r^4 \left(\Theta_{\mu\nu} - \delta_{\mu\nu}\right)$$

$$\Rightarrow \quad \varepsilon = (\pi T)^4 = \pi^4 T^4 \qquad s = \frac{\partial \varepsilon}{\partial T} = 4\pi^4 T^3$$

$$\eta = + (\pi T)^3 = + \pi^3 T^3 \qquad \frac{\eta}{s} = \frac{1}{4\pi} \;!!$$

Lecture 2

§. Existence of horizon (B & paper)

The solutions constructed above (p. 1·14) correspond to fluctuating black branes + else!

Because of "+ else" it is not clear that the net solution has a horizon.

However, ∃ a horizon

$r(x) = r_0(x) + \epsilon r_1(x) + \ldots$ eg

(Eq. 5.4)
$r_1(x) = 0$
$r_2(x) = \frac{1}{r_0 \cdot 4}\left(\frac{1}{3}\sigma_{\mu\nu} \ldots + S_1 \text{ etc.}\right)$

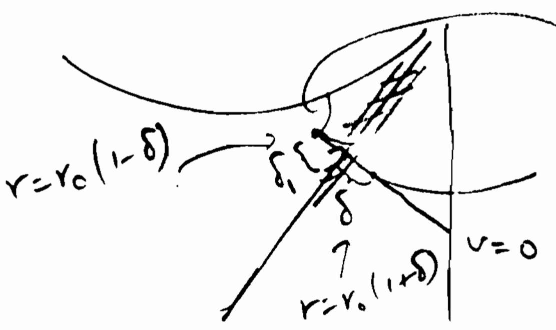
We obtain this by looking for a null surface $f(r, x) = 0$ s.t.

$$g^{MN} \frac{\partial f}{\partial x^M} \frac{\partial f}{\partial x^N} = 0$$

which coincides with $r(x) = r_0$ at infinite future.

Why is this a horizon?

We can show that

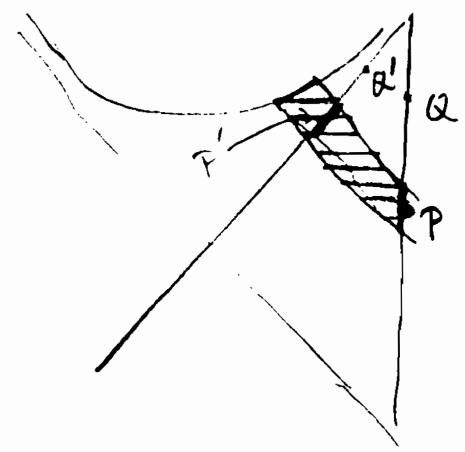


finite proper time to reach the singularity $(\sim -\ln \delta_1)$

finite proper time to reach the boundary $(\sim -\ln \delta) = \int \frac{dr}{r^2 f} <$

Construction of Fluid entropy current from gravity

The tube map :



∴ $L_0 = \frac{1}{T}$ represents the scale of variation of the fields

The geometrical meaning of the ϵ - expansion.

Suppose

L = length scale of variation of $T(x)$

eg $T(x) = T(x_0)\left[1 + \frac{\Delta x}{L}\right]$

$\Rightarrow \frac{\partial_\mu T}{T^2} = \frac{1}{LT}$

$\Rightarrow \partial_\mu \left(\frac{1}{T}\right) = \frac{1}{LT}$

If $LT \gg 1$ $\quad \partial_\mu\left(\frac{1}{T}\right) \ll 1$

∴ Tube of width $\ll \frac{1}{T}$ will approximate uniform black brane.

For a uniform black brane, use E-F coordinates

$ds^2 = 2dvdr - r^2 f(r) dv^2 + r^2 dx^{i2}$

A natural map from ~~...~~ $\mathcal{B}$ to $\mathcal{H}$ is

~~$\mathcal{Q}(r_H, \vec{x}) \to \mathcal{Q}(r_\infty, \vec{x}) \in \mathcal{B}$~~

$$\mathcal{B} \ni \vec{x} \xrightarrow{\ f\ } (r_H(\vec{x}), \vec{x}) \in \mathcal{H}$$

This is an example of a more general set of maps connecting $\mathcal{B}$ to $\mathcal{H}$ thru' null ingoing geodesics (~~4 vector fields~~ parametrized by 4 velocity vectors t^μ at each point x^μ ; $t^\mu(x)$)

~~Fig~~ ~~this~~

$a_{Q'} \geq a_{p'}$ ($\because \theta \geq 0$ on horizon)

hence $(f^* a)_Q \geq (f^* a)_p$

$\therefore$ $f^* a$ is a ~~good~~ choice of the entropy current. [why ???]

~~This fixes parameters~~

$da \geq 0$

$d(f^* a) = f^* da \geq 0$

This fixes parameter:

$$(4\pi\eta)^{-1} J^\mu_S$$
$$= \left[1 + b^2\left(A_1 \sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2\, \omega_{\alpha\beta}\omega^{\alpha\beta} + A_3 \mathcal{R}\right)\right] u^\mu$$
$$+ b^2\left[B_1 D_\lambda \sigma^{\mu\lambda} + B_2 D_\lambda \omega^{\mu\lambda}\right]$$
$$+ C_1 b\, l^\mu + C_2 b^2 u^\lambda D_\lambda l^\mu$$

$D_\lambda Q^{\mu\cdots}_{\nu\cdots}$ $\rightarrow$ cov. deriv

$$\mathcal{A}_\mu = u^\lambda \nabla_\lambda u_\mu - \frac{\nabla_\lambda u^\lambda}{d-1} u_\mu$$

$$\mathcal{R} = R - 6\left(\nabla_\lambda A^\lambda - A_\lambda A^\lambda\right)$$

$A_1 = -\frac{\ln 2}{8}$

$A_2 = -\frac{1}{8}$

$A_3 = \frac{1}{8}$

$B_1 = \frac{1}{4}$

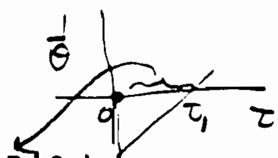
$B_2 = -\frac{1}{2}$

$C_1 = C_2 = \ldots$

$\dot\theta = -\frac{1}{3}\theta^2 - (+ve)$

$\downarrow$

$\frac{d}{d\tau}\left(\frac{1}{\theta}\right) = \frac{1}{3}$



$\tau_1 \leq 3\lambda\theta_0$

$\frac{1}{\theta} - \frac{1}{\theta_0} = \tau - \tau_0$

$\frac{1}{\theta(\tau_1)} = 0 - \varepsilon$

$\Rightarrow \theta(\tau_1) = -\infty$

$\therefore \frac{da}{d\tau}$

$u^a \nabla_a \theta \equiv \dot\theta = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}u^a u^b$

Raychaudhuri eqn.