Extremal Black Hole Entropy

One of the successes of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric black holes in terms of microscopic quantum states.

$$S_{BH}(\vec{Q}) = \ln d_{micro}(\vec{Q})$$

Strominger, Vafa

$d_{micro}(\vec{Q})$: degeneracy of microstates carrying a given set of charges $\vec{Q}$

$$S_{BH}(\vec{Q}) = A/4G_N$$

$A=$ Area of event horizon of a black hole of charge $\vec{Q}$

This formula is quite remarkable since it relates a geometric quantity in space-time to a counting problem.

However the Bekenstein-Hawking formula for the entropy receives α' and g_s corrections.

Our goal is to search for an exact relation of the form

$$d_{macro}(\vec{Q}) = d_{micro}(\vec{Q})$$

$d_{macro}(\vec{Q})$: Some generalization of the Bekenstein-Hawking formula taking into account α' and g_s corrections.

We shall focus on extremal, BPS black holes.

Extremality: essential for the separation between the horizon degrees of freedom and those living outside the horizon by an infinite throat

Supersymmetry: (probably) needed for ensuring stability of extremal black holes.

Also we shall work in some fixed duality frame so that we can clearly distinguish between classical and quantum effects.

Plan

1. A precise proposal for $d_{macro}(\vec{Q})$

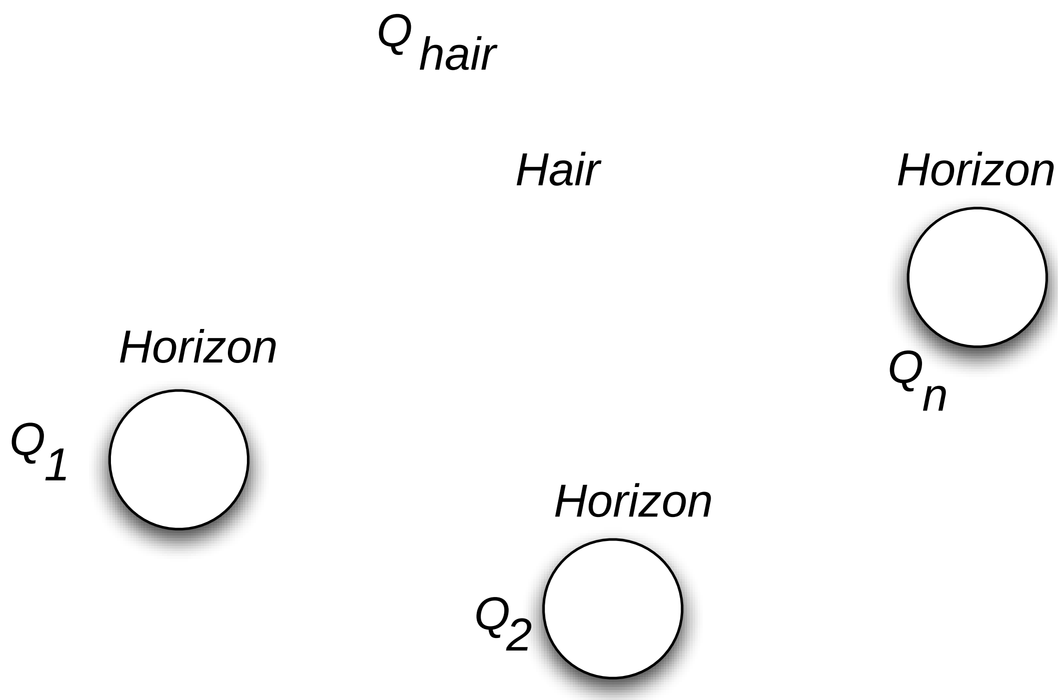
2. Some exact results for $d_{micro}(\vec{Q})$ in type IIB string theory on $K3 \times T^2$.

3. Comparison of $d_{macro}(\vec{Q})$ with $d_{micro}(\vec{Q})$.

Take a macroscopic configuration of charge $\vec{Q}$

In general such a configuration could involve an n centered black hole with charges $\vec{Q}_1, \cdots \vec{Q}_n$ and hair with charge $\vec{Q}_{hair}$.

Hair: smooth normalizable deformations of the black hole solution with support outside the horizon(s).



Q_{hair}
Hair
Horizon
Horizon
Q_n
Q_1
Horizon
Q_2

Proposal for $d_{macro}(\vec{Q})$:

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} d_{hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

$d_{hor}(\vec{Q}_{hor})$: contribution from the horizon with charge $\vec{Q}_{hor}$

d_{hair}: contribution from the hair of the n-centered black hole, with the horizons carrying charges $\vec{Q}_1, \cdots \vec{Q}_n$, and the hair carrying charge $\vec{Q}_{hair}$.

Our main focus on this talk will be on $d_{hor}(\vec{Q})$.

Our goal: Find a macroscopic prescription for computing $d_{hor}(\vec{Q})$

To leading order in g_s but all orders in α', $d_{hor}(\vec{Q})$ is given by the exponential of the Wald entropy

– can be computed using the entropy function formalism

We shall begin with a lightening review of the results of the entropy function formalism.

How do we define an extremal black hole in a general higher derivative theory of gravity?

Reissner-Nordstrom solution in $D=4$:

$$\begin{aligned} ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\ &\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\ &\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Define

$$2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}$$

and take $\lambda \to 0$ limit.

$$ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$ds^2 = \rho_+^2 \left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right] + \rho_+^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

This describes an $AdS_2 \times S^2$ space and has $SO(2,1) \times SO(3)$ isometry.

The electromagnetic fields at the horizon also have $SO(2,1) \times SO(3)$ isometry.

More generally, for all known extremal black holes in all dimensions, the time translation symmetry gets enhanced to $SO(2,1)$ in the near horizon limit.

Postulate: An extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry.

Regarding all other directions (including angular coordinates) as compact we can regard the near horizon geometry of an extremal black hole as

$$AdS_2 \times \text{a compact space (fibered over } AdS_2\text{)}$$

Note: Magnetic charges are encoded in the fluxes through the compact space.

Consider string theory in such a background containing two dimensional metric $g_{\mu\nu}$ and $U(1)$ gauge fields $A_\mu^{(i)}$ among other fields.

The most general field configuration consistent with $SO(2,1)$ isometry:

$$ds^2 \equiv g_{\mu\nu}^{(2)} dx^\mu dx^\nu = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$

$$F_{rt}^{(i)} = e_i, \quad \cdots\cdots\cdots$$

$\mathcal{L}^{(2)}(v, \vec{e}, \cdots)$: The Lagrangian density evaluated in this background.

For black hole with electric charge $\vec{q}$, define

$$\mathcal{E}(\vec{q}, v, \vec{e}, \cdots) \equiv 2\pi \left(e_i \, q_i - v \, \mathcal{L}^{(2)} \right)$$

One finds that

1. All the near horizon parameters are obtained by extremizing $\mathcal{E}$ with respect to v, e_i and the other near horizon parameters.

2. $S_{wald}(\vec{q}) = \mathcal{E}$ at this extremum.

Thus in the classical limit

$$d_{hor}(\vec{q}) = e^{S_{wald}(\vec{q})} = e^{\mathcal{E}}$$

We shall propose an expression for $d_{hor}(\vec{q})$ in the full quantum theory as a path integral over the Euclidean continuation of the near horizon geometry.

$\longrightarrow$ Quantum entropy function

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$

$$F^{(i)}_{rt} = e_i$$

Euclidean continuation:

$$t = -i\theta, \quad r = \cosh\eta, \quad 0 \le \eta < \infty$$

This gives

$$\begin{aligned} ds^2 &= v\left(d\eta^2 + \sinh^2\eta\, d\theta^2\right), \quad \to \theta \equiv \theta + 2\pi, \\ F^{(i)}_{\theta\eta} &= i e_i \sinh\eta \end{aligned}$$

$$\to \quad A^{(i)}_\theta = -i\, e_i\, (\cosh\eta - 1) = -i\, e_i\, (r-1)\,.$$

Proposal for the quantum entropy function $d_{hor}(\vec{q})$

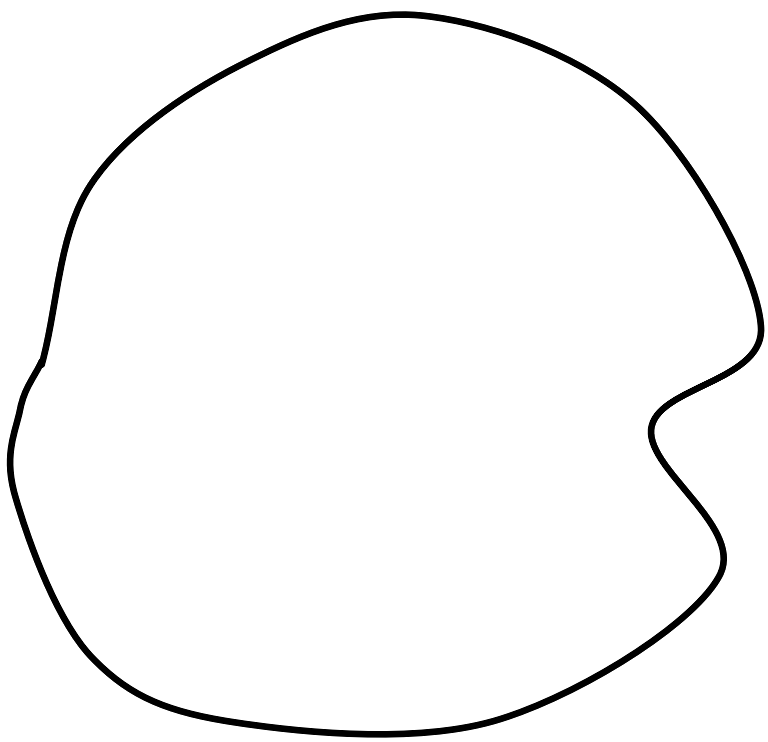
$$d_{hor}(\vec{q}) = \left\langle \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

$\langle\ \rangle_{AdS_2}$: unnormalized path integral over various fields of string theory on euclidean global AdS_2.

$\oint$: a closed contour at the boundary of AdS_2.

'finite': Infrared finite part of the amplitude.

We need to regularize the infinite volume of AdS_2 by putting a cut-off $r \leq r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.



Cut-off: $r \le r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

The superscript '$finite$' refers to the finite part of the amplitude defined by expressing it as

$$e^{CL} \times \text{finite part}$$

L: length of the boundary of AdS_2.

C: A constant

The definition can be shown to be independent of the choice of $f(\theta)$.

We shall work with $f(\theta) = 1$.

The role of

$$\exp[-iq_i \oint d\theta\, A_\theta^{(i)}]$$

We could absorb this into the boundary terms in the action.

However we have displayed it explicitly since it plays a special role.

It is the only term in the boundary action that involves the gauge field and not its field strength.

Why do we need this term?

In AdS_d the Maxwell's equation has two solutions in the asymptotic region:

$A_\theta^{(i)} \sim r^{-d+3}$: electric field mode

$A_\theta^{(i)} \sim$ constant: constant mode

Thus for $d \geq 4$ the constant mode of the gauge field is dominant at infinity.

We fix the constant mode by a boundary condition and integrate over the electric field mode.

However for $d = 2$,

Electric field mode: $A_\theta^{(i)} \sim r$

Constant mode: $A_\theta^{(i)} \sim$ constant

Thus the electric field mode is dominant

$\longrightarrow$ we must work in a sector with fixed asymptotic electric field i.e. fixed charge, and allow the constant mode to fluctuate.

However now the extremization of the action no longer gives the classical equations of motion.

The variation of the action contains boundary terms proportional to $\delta A_\theta^{(i)}$ which are no longer constrained to vanish by boundary condition.

$\rightarrow$ we need to add new boundary term in the action to cancel the boundary terms proportional to $\delta A_\theta^{(i)}$.

The $\exp[-iq_i \oint d\theta\, A_\theta^{(i)}]$ precisely achieves this task.

$$d_{hor}(\vec{q}) = \left\langle \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

We shall try to justify this proposal by showing that

1. In the classical limit

$$\ln d_{hor}(\vec{q}) \to S_{wald}(\vec{q})$$

2. This fits in with the usual rules of AdS/CFT correspondence.

Classical limit:

$$\left\langle \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}$$

In the classical limit this reduces to

$$e^{-\mathcal{S}}\, \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)cl}]$$

$$A_\theta^{(i)cl} = -i\, e_i\, (r_0 - 1)$$

$\mathcal{S} =$ Euclidean action $= \mathcal{S}_{bulk} + \mathcal{S}_{boundary}$

$$\mathcal{S}_{bulk} = -\int_1^{r_0} dr\sqrt{\det g}\, d\theta\, \mathcal{L}^{(2)} = -(r_0 - 1)\, 2\pi v\, \mathcal{L}^{(2)}$$

$$-iq_i \oint d\theta\, A_\theta^{(i)cl} = -2\pi\, \vec{q}\cdot\vec{e}\,(r_0 - 1)$$

$$\mathcal{S}_{boundary} = -2\pi\, K r_0 + \mathcal{O}(r_0^{-1})$$

K: some constant which depends on the details of the boundary terms.

The length of the boundary is

$$L = 2\pi\sqrt{v}r_0 + \mathcal{O}(r_0^{-1})\,.$$

This gives

$$\left\langle \text{exp}[-iq_i \oint d\theta \, A_\theta^{(i)}] \right\rangle_{AdS_2}$$

$$= \left[e^{L(v\,\mathcal{L}^{(2)} + K - \vec{e}\cdot\vec{q})/\sqrt{v} + 2\pi(\vec{e}\cdot\vec{q} - v\,\mathcal{L}^{(2)}) + \mathcal{O}(r_0^{-1})} \right]$$

Extracting the finite part we get

$$d_{hor}(\vec{q}) \simeq \text{exp}\left[2\pi(\vec{e}\cdot\vec{q} - v\,\mathcal{L}^{(2)})\right] = \text{exp}\left[S_{wald}(\vec{q})\right]$$

Note: A change in the boundary action changes K but the finite part is insensitive to such a change.

AdS_2/CFT_1 **correspondence**

Euclidean AdS_2 is the Poincare disk.

$\to$ its boundary is a circle of circumference L.

Thus AdS/CFT correspondence $\to$

$$\left\langle \exp[-iq_i \oint d\theta \, A_\theta^{(i)}] \right\rangle_{AdS_2} = Z_{CFT_1} = Tr_{\vec{q}} e^{-LH}$$

$Tr_{\vec{q}}$: trace over states of charge $\vec{q}$ in CFT_1

H: Hamiltonian of dual CFT_1

Thus we have, for large L,

$$\begin{aligned}\left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2} &= Tr_{\vec{q}}\, e^{-LH} \\ &= d_{CFT}(\vec{q})e^{-E_0 L}\,.\end{aligned}$$

E_0, $d_{CFT}(\vec{q})$: ground state energy, degeneracy

Taking the finite part we get

$$d_{hor}(\vec{q}) = d_{CFT}(\vec{q})$$

Note: In the more conventional units we take the length of the boundary to be finite, but scale energies by L.

Only the ground states of the CFT survive.

What can we say about CFT_1?

It should be identified as the infrared limit of the quantum mechanics associated with the microscopic description of the black hole, after stripping off the hair contribution.

Thus d_{CFT} together with the hair contribution should give us the microscopic degeneracies.

– agrees with our proposal.

Degeneracy or Index?

Often in the microscopic theory we compute the index rather than degeneracy.

– protected against quantum corrections.

e.g. in $D = 4$ we calculate the helicity trace index

$$B_{2n} = (-1)^n \, Tr_{\vec{Q}} \left[(-1)^{2h} \, (2h)^{2n} \right]$$

$4n$: Number of broken SUSY generators

Thus on the black hole side also we should compute the index.

The $(2h)^{2n}$ factor is needed to absorb the fermion zero modes associated with broken SUSY.

For a black hole solution these zero modes form part of hair degrees of freedom.

Thus B_{2n} for the black hole takes the form

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i+\vec{Q}_{hair}=\vec{Q}}} \left\{\prod_{i=1}^n I_{hor}(\vec{Q}_i)\right\} B_{2n;hair}(\vec{Q}_{hair};\{\vec{Q}_i\})$$

I_{hor}: Witten index associated with the horizon

Since in $D=4$ the black hole horizons always have $h=0$ we get

$$I_{hor}(\vec{Q}_{hor}) = d_{hor}(\vec{Q}_{hor})$$

This gives the following formula for the index on the macroscopic side

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} B_{2n;hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

– can be computed using quantum entropy function.

Summary

We have a concrete proposal for relating the extremal black hole entropy to the microscopic degeneracy

$$d_{hor}(\vec{q}) = \left\langle \mathrm{exp}[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

should agree with the microscopic degeneracies after removing the hair contribution.

1. It reduces to the relation between wald entropy and statistical entropy in the classical limit.

2. It is in the spirit of AdS/CFT correspondence.

Results for $d_{micro}(\vec{Q})$ in type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1$

We shall focus on a special class of states in this theory consisting of

D5/D3/D1 branes wrapped on 4/2/0 cycles of $K3 \times (S^1$ or $\widetilde{S}^1)$

Q: D-branes charges wrapped on 4/2/0 cycles of $K3 \times \widetilde{S}^1$

P: D-branes charges wrapped on 4/2/0 cycles of $K3 \times S^1$

Q and P are each 24 dimensional vectors.

Note: The $\vec{Q}$ used earlier now stands for (Q, P)

$B_6(Q, P)$: microscopic 6th helicity trace index of quarter BPS states carrying charges (Q, P).

Besides depending on the charges, $B_6(Q, P)$ also depends on the asymptotic values of the moduli fields as the degeneracy can jump as we cross walls of marginal stability.

In order to facilitate comparison with the macroscopic results it will be convenient to choose the asymptotic moduli such that only single centered black holes contribute to $B_6(Q, P)$.

We shall proceed with this choice.

Duality symmetries

The duality symmetries which take D-branes to D-branes is given by

$$O(4,20;\mathbb{Z})_T \times SL(2,\mathbb{Z})_S$$

$O(4,20;\mathbb{Z})_T$: global diffeomorphism + mirror symmetry of K3

$SL(2,\mathbb{Z})_S$: global diffeomorphism of $S^1 \times \widetilde{S}^1$

We can use duality transformations to simplify the dependence of B_6 on (Q, P).

An invariant of $O(4, 20; \mathbb{Z})_T \times SL(2, \mathbb{Z})_S$:

$$\ell \equiv \mathrm{gcd}\{Q_i P_j - Q_j P_i\}$$

Dabholkar, Gaiotto, Nampuri

With the help of $SL(2, \mathbb{Z})_S$ transformation any charge vector can be brought to the form

$$(Q, P) = (\ell Q_0, P_0), \quad \mathrm{gcd}\{Q_{0i} P_{0j} - P_{0i} Q_{0j}\} = 1$$

Banerjee, A.S.

We shall proceed with this choice.

Intersection form of 4/2/0 forms on $K3$ defines $O(4,20;\mathbb{Z})$ invariant inner products

$$Q^2, \qquad P^2, \qquad Q\cdot P$$

One finds that for $(Q,P) = (\ell Q_0, P_0)$ the microscopic result for $B_6(Q,P)$ takes the form

$$\sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s)$$

Banerjee, A.S., Srivastava; Dabholkar, Gomes, Murthy

$$
\begin{aligned}
& f(2m, 2n, p) \\
= \; & (-1)^{p+1} \int_{iM_1-1/2}^{iM_1+1/2} d\rho \int_{iM_2-1/2}^{iM_2+1/2} d\sigma \int_{iM_3-1/2}^{iM_3+1/2} dv \\
& e^{-2i\pi(\sigma m+\rho n+vq)} \, \Phi_{10}(\rho, \sigma, v)^{-1}
\end{aligned}
$$

$$
M_1 = 2\Lambda \frac{Q^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}, \quad M_2 = 2\Lambda \frac{P^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}, \quad M_3 = -2\Lambda \frac{Q\cdot P}{\sqrt{Q^2P^2-(Q\cdot P)^2}}.
$$

Φ_{10}: Igusa cusp form

Λ: a large positive number

$$
\begin{aligned}
& f(2m, 2n, p) \\
= \;& (-1)^{p+1} \int_{iM_1-1/2}^{iM_1+1/2} d\rho \int_{iM_2-1/2}^{iM_2+1/2} d\sigma \\
& \int_{iM_3-1/2}^{iM_3+1/2} dv \, e^{-2i\pi(\sigma m+\rho n+vq)} \, \Phi_{10}(\rho, \sigma, v)^{-1}
\end{aligned}
$$

For large charges

$$
\begin{aligned}
& f(Q^2/s^2, P^2, Q\cdot P/s) \\
= \;& \exp\left[2\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/s\right] \\
& \times \text{ Series expansion in inverse powers of charges} \\
& + \text{Exponentially suppressed corrections}
\end{aligned}
$$

Our goal: understand this formula from macroscopic viewpoint.

Macroscopic computation of $B_6(Q,P)$

For single centered black holes

$$B_6(\vec{Q}) = \sum_{\substack{\vec{Q}_{hor},\vec{Q}_{hair} \\ \vec{Q}_{hor}+\vec{Q}_{hair}=\vec{Q}}} d_{hor}(\vec{Q}_{hor})\, B_{6;hair}(\vec{Q}_{hair};\vec{Q}_{hor})$$

For black holes carrying D-brane charges only hair degrees of freedom are the fermion zero modes associated with broken supersymmetry.

(Closed string excitations do not carry D-brane charges)

$$\rightarrow \qquad \vec{Q}_{hair} = 0, \qquad B_{6;hair} = 1$$

In this case we have

$$B_6(\vec{Q}) = d_{hor}(\vec{Q})$$

$d_{hor}(\vec{Q})$: Finite part of the partition function on AdS_2 with the boundary condition that the asymptotic values of the electric fields are fixed.

– can be expressed as sum over contributions from different saddle points.

The leading saddle point

$$K3 \times S^1 \times \widetilde{S}^1 \times AdS_2 \times S^2$$

$$\begin{aligned} ds^2 &= v\left(\frac{dr^2}{r^2-1} + (r^2-1)\,d\theta^2\right) + \frac{R^2}{\tau_2}\left|dx^4 + \tau dx^5\right|^2 \\ &\quad + w(d\psi^2 + \sin^2\psi d\phi^2) + \widehat{g}_{mn}(\vec{u})du^m du^n \end{aligned}$$

v, w, R: real constants

$\tau = \tau_1 + i\tau_2$: a complex constant $\in$ UHP

$\widehat{g}_{mn}(\vec{u})$: metric on K3

x^4: coordinate along $\widetilde{S}^1$

x^5: coordinate along S^1

There are also background RR fluxes.

Q: represent RR fluxes through the cycles of $K3$ times the 3-cycle spanned by (x^5, ψ, ϕ).

P: represent RR fluxes through the cycles of $K3$ times the 3-cycle spanned by (x^4, ψ, ϕ).

The classical contribution to $d_{hor}(Q,P)$ from this saddle point is exponential of the Wald entropy:

$$\exp\left[2\pi\sqrt{Q^2P^2-(Q\cdot P)^2}\right]$$

IIB coupling constant at the horizon $\sim$ charge^{-2}

Quantum corrections computed via path integral over AdS_2

$\to$ a multiplicative factor containing a series expansion in inverse powers of charges

This has the structure of asymptotic expansion of $f(Q^2, P^2, (Q\cdot P)^2)$.

In principle one should be able to calculate the subleading corrections systematically by evaluating the path integral around the saddle point

This can then be compared with the microscopic result for $f(Q^2, P^2, Q \cdot P)$.

Part of the corrections to f has been understood as coming from some specific quatum corrections to the path integral, but a completely systematic analysis has not yet been done.

Work in progress.

Can we understand the origin of the $f(Q^2/s^2, P^2, Q\cdot P/s)$ term for $s > 1$ in the macroscopic formula?

We shall now argue that these arise from new saddle points obtained by taking freely acting $\mathbb{Z}_s$ orbifold of the original saddle point.

Consistency checks:

1. Classical contribution from this saddle point must go as

$$\exp\left[2\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s\right]$$

2. This saddle point should exist iff $\ell/s \in \mathbb{Z}$

Consider an orbifold of the leading saddle point by the transformation

$$\theta \to \theta + 2\pi/s, \quad \phi \to \phi + 2\pi/s, \quad x^5 \to x^5 + 2\pi/s$$

At $r = 1$ $(\eta = 0)$ the shift in θ is irrelevant.

$\to$ the identification is $(\phi, x^5) \equiv (\phi + 2\pi/s, x^5 + 2\pi/s)$.

Thus the RR flux Q through the cycle at $r = 1$, spanned by (x^5, ψ, ϕ) times a cycle of K3 gets divided by s.

Flux quantization $\to$ the orbifold is well defined only if Q is divisible by s, i.e. if

$$l/s \in \mathbb{Z}$$

Denoting the (r, θ, ϕ, x^5) coordinates of the orbifold by $(\widetilde{r}, \widetilde{\theta}, \widetilde{\phi}, \widetilde{x}^5)$ we get the new metric

$$\begin{aligned} ds^2 &= v\left(\frac{d\widetilde{r}^2}{\widetilde{r}^2 - 1} + (\widetilde{r}^2 - 1)\, d\widetilde{\theta}^2\right) \\ &\quad + \frac{R^2}{\tau_2}\left|dx^4 + \tau d\widetilde{x}^5\right|^2 \\ &\quad + w(d\psi^2 + \sin^2\psi d\widetilde{\phi}^2) + \widehat{g}_{mn}(\vec{u})du^m du^n \end{aligned}$$

$$(\widetilde{\theta} + 2\pi/s, \widetilde{\phi} + 2\pi/s, \widetilde{x}^5 + 2\pi/s) \equiv (\widetilde{\theta}, \widetilde{\phi}, \widetilde{x}^5)$$

Define

$$\theta = s\widetilde{\theta}, \quad r = \widetilde{r}/s, \quad \phi = \widetilde{\phi} - \widetilde{\theta}, \quad x^5 = \widetilde{x}^5 - \widetilde{\theta}$$

Then

$$\begin{aligned} ds^2 &= v\left(\frac{dr^2}{r^2 - s^{-2}} + (r^2 - s^{-2})\, d\theta^2\right) \\ &\quad + \frac{R^2}{\tau_2}\left|dx^4 + \tau(dx^5 + s^{-1}d\theta)\right|^2 \\ &\quad + w(d\psi^2 + \sin^2\psi(d\phi + s^{-1}d\theta)^2) \\ &\quad + \widehat{g}_{mn}(\vec{u})du^m du^n \end{aligned}$$

$$(\theta + 2\pi, \phi, x^5) \equiv (\theta, \phi, x^5)$$

This has the same asymptotic behaviour as the original saddle point and hence is an admissible saddle point.

Its contribution to $d_{hor}(Q,P)$ in the classical limit is given by

$$\exp[S_{wald}/s] = \exp\left[2\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/s\right]$$

This is the same behaviour as of $f(Q^2/s^2, P^2, Q\cdot P/s)$.

Thus this saddle point is the ideal candidate for the contribution $f(Q^2/s^2, P^2, Q\cdot P/s)$ in the microscopic formula.

Furthermore it exists iff $s|\ell$, as the case for the term $f(Q^2/s^2, P^2, Q\cdot P/s)$ in the microscopic formula.

Black hole hair removal:

– a consistency check for the formula

$$B_{2n;macro}(Q) = \sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} B_{2n;hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

Suppose two black holes have identical near horizon geometry but different asymptotic geometries.

Suppose further that we know the appropriate index for both these black holes from microscopic analysis, and can compute the hair contribution for both the black holes.

Then by stripping off the hair contribution we can get the 'microscopic result' for $d_{hor}(\vec{Q})$ for both the black holes.

They must agree.

An example: Banerjee, Mandal. A.S.

System 1: BMPV black hole

– A five dimensional rotating black hole in type IIB on $K3 \times S^1$.

System 2: A four dimensional black hole in type IIB on $K3 \times T^2$ obtained by placing the BMPV black hole in Taub-NUT

They have identical near horizon geometries but different index and different $B_{2n;hair}$

But d_{hor} computed by stripping off $B_{2n;hair}$ from the index gives the same result for both.