Noncommutative Gauge Theory Anomalies

- String Theory in the presence of Antisymmetric Field B:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \left(g_{ij}\partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j \right)$$

- Cannonical Quantiztion then forces Noncommutativity of space:

$$[x^i, x^j] = i\theta^{ij}$$

- Open String Amplitudes give a vertex leading to a Noncommutative Gauge Theory

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]_\star .$$

Wher

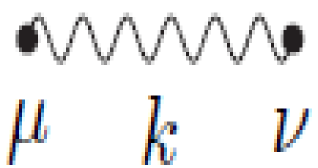
$$f(x) \star g(x) \equiv e^{\frac{i\theta_{\mu\nu}}{2} \frac{\partial}{\partial \xi_\mu} \frac{\partial}{\partial \zeta_\nu}} f(x+\xi)\, g(x+\zeta)\Big|_{\xi=\zeta=0}$$

* Product is associative but not commutative.

The NC QED Lagrangian:

$$\mathcal{L} = \bar{\psi} \star (i\partial\!\!\!/ - gA\!\!\!/) \star \psi - \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu};$$

Same propagators

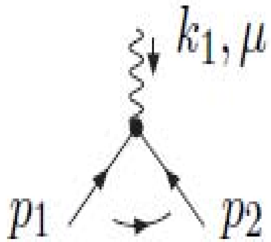


$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2}.$$

as

$$\int \mathrm{d}x^d f \star_M g = \int \mathrm{d}x^d f g$$

But, the vertices have a crucial extra phase factor:



$$V_\mu (p_1, p_2; k_1) = ig (2\pi)^4 \delta^4 (p_1 + p_2 + k_1) \gamma_\mu \exp\left(\frac{-i\theta_{\eta\sigma}}{2} p_1^\eta p_2^\sigma\right).$$

Thus better UV behavior.

- Planar loops diagrams involve no phases.
- Nonplanar diagrams involve phases. Thus finite as long as with on shell external lines
- But, if the external lines are part of a larger diagram and thus off shell, they give an IR divergent contribution .
- Nonrenormalizable.
- Serious: UV/IR problem

- Explicit Example: One loop in scalar theory.

$$S = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} g^2 \phi \star \phi \star \phi \star \phi\right)$$

Two point functions

$$\Gamma^{(2)}_{1\,planar} = \frac{g^2}{3(2\pi)^4} \int \frac{d^4k}{k^2 + m^2}$$

$$\Gamma^{(2)}_{1\,nonplanar} = \frac{g^2}{6(2\pi)^4} \int \frac{d^4k}{k^2 + m^2} e^{ik \times p}$$

Result of calculation:

$$\Gamma^{(2)}_{1\ planar} = \frac{g^2}{48\pi^2}\left(\Lambda^2 - m^2 \ln(\frac{\Lambda^2}{m^2}) + O(1)\right)$$

$$\Gamma^{(2)}_{1\ nonplanar} = \frac{g^2}{96\pi^2}\left(\Lambda^2_{eff} - m^2 \ln(\frac{\Lambda^2_{eff}}{m^2}) + O(1)\right),$$

where

$$\Lambda^2_{eff} = \frac{1}{1/\Lambda^2 + p \circ p}.$$

Two limits:

$$p \circ p \ll \frac{1}{\Lambda^2}$$

Then the commutative theory result

$$p \circ p \gg \frac{1}{\Lambda^2}$$

Then a finite result.

Note that the two limits do not " commute"

This is famous "UV/IR Mixing".

- Anomalies. Breaking of classical symmetries due to quantum corrections.
- Two currents for Global Axial symmetry:
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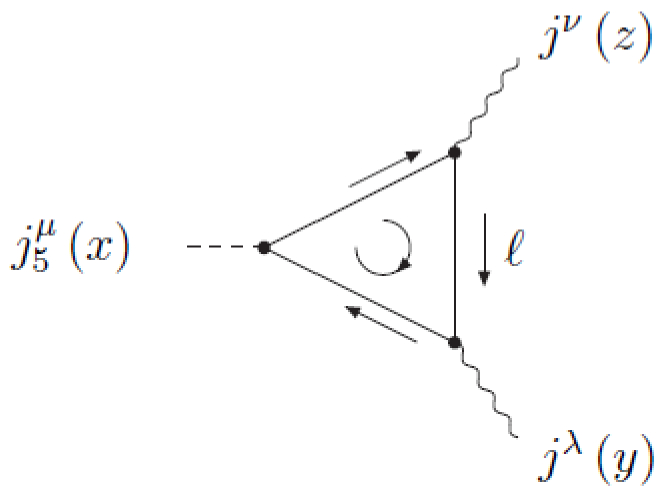
$$J^5_\mu \equiv \psi_\beta \star \bar{\psi}_\alpha (\gamma_\mu \gamma_5)^{\alpha\beta}, \qquad D_\mu j^\mu_5 = 0$$

$$j^5_\mu \equiv \bar{\psi}_\alpha \star \psi_\beta (\gamma_\mu \gamma_5)^{\alpha\beta}. \qquad \partial_\mu j^{\mu,5}_a = 0$$

The Triangle Diagram for the anomaly is



$j^{\nu}(z)$
$j_5^{\mu}(x)$
ℓ
$j^{\lambda}(y)$

For the “ Covariant “ Current,

$$D_\mu J_5^\mu = -\frac{g^2}{16\pi^2} F_{\mu\nu} \star \tilde{F}^{\mu\nu},$$

it is the usual * generalization of “ Adler Anomaly”.

But, anomaly for the invariant current is unusual:

$$\partial_\mu j^{\mu,5} = \begin{cases} 0 & (\Theta p)^2 \gg \frac{1}{\Lambda^2} \qquad \text{UV limit} \\ -\frac{g^2}{16\pi^2} F_{\mu\nu} \star' \tilde{F}^{\mu\nu} + \cdots & (\Theta p)^2 \ll \frac{1}{\Lambda^2} \qquad \text{IR limit} \end{cases}$$

Where,

$$f(x) \star' g(x) \equiv f(x+\xi) \frac{\sin\left(\frac{\Theta_{\mu\nu}}{2}\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\zeta_\nu}\right)}{\left(\frac{\Theta_{\mu\nu}}{2}\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\zeta_\nu}\right)} g(x+\zeta)\Big|_{\xi=\zeta=0}$$

To make rigorous mathematical sense out of this anomaly one needs to regularise the expressions by compactification of the spacial directions

Generalization of the Star Product

- Various generalizations.
- A Translation Invariant Generalization :

$$f \star g = \frac{1}{(2\pi)^{\frac{d}{2}}} \int \mathrm{d}p^d \mathrm{d}q^d \mathrm{d}k^d \mathrm{e}^{\mathrm{i}p\cdot x} \tilde{f}(q) \tilde{g}(k) K(p,q,k)$$

Translation invariance ,

$$\mathcal{T}_a(f) \star \mathcal{T}_a(g) = \mathcal{T}_a(f \star g) \qquad \mathcal{T}_a(f)(x) = f(x+a)$$

- It implies,

$$K(p,q,k) = \mathrm{e}^{\alpha(p,q)}\delta(k-p+q)$$

$$f \star g = \frac{1}{(2\pi)^{\frac{d}{2}}} \int \mathrm{d}p^d \mathrm{d}q^d \mathrm{e}^{\mathrm{i}p\cdot x} \tilde{f}(q)\tilde{g}(p-q)\mathrm{e}^{\alpha(p,q)}$$

Associativity:

$$\alpha(p,q) + \alpha(q,r) = \alpha(p,r) + \alpha(p-r, q-r)$$

Generally,

$$\alpha(p,q) = \frac{1}{2}[\eta(q) - \eta(p) + \eta(q-p)] - \frac{i}{2}\omega(p,q) + i\xi(-p,q)$$

$$\eta(p) \equiv \alpha(0,p)$$

$\omega(p,q)$ is antisymmetric under $p \leftrightarrow q$

$\xi(p,q)$ is symmetric under $p \leftrightarrow q$

$$\omega(p,q) \equiv -i[\alpha(p+q,p) - \alpha(p+q,q)]$$

$$\eta(p) = \eta(-p)$$
$$\eta(0) = 0$$

$$
\begin{aligned}
\omega(p,p) &= 0\\
\omega(p,0) &= 0\\
\omega(0,p) &= 0 \qquad \omega\\
\omega(p,q) &= -\omega(q,p)\\
\omega(-p,-q) &= \omega(p,-q)\\
\omega(p,q) &= \omega(p-q,p) = \omega(-q,p).
\end{aligned}
$$

- The most general solution in 2 directions is:

$$\omega(p,q) = \sum_{k=0}^{\infty} b_{2k+1}(p_2 q_1 - p_1 q_2)^{2k+1} = \sum_{k=0}^{\infty} b_{2k+1}(\mathbf{p} \wedge \mathbf{q})^{2k+1},$$

New NC Gauge Theory

- Diagram are now more complicated.

$$S_{\alpha\beta}(p) = \left(\frac{i}{\not{p} - m}\right)_{\alpha\beta} e^{-\eta(p)}$$

$$D_{\mu\nu}(k, \xi = 1) = -\frac{ig^{\mu\nu}}{k^2} e^{-\eta(k)}$$

$$V_\mu(p, q; k) = \ ig(2\pi)^4 \delta(p - k - q)(\gamma_\mu)_{\alpha\beta}\ e^{i\omega(p,q) - i\xi(p,-q) + \frac{1}{2}[\eta(p) + \eta(q) + \eta(k)]}$$