Twistor Theory and Differential Equations

Maciej Dunajski

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

- Work of many: Penrose, Ward, Atiyah, Hitchin, Mason, Sparling, Tod, Woodhouse, ...
1917 Radon. \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) with decay condition at \( \infty \), \( L \subset \mathbb{R}^2 \) oriented line.

\[
\phi(L) := \int_L f.
\]

There exist an inversion formula \( \phi \rightarrow f \).
1938 Fritz John. \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \), oriented line \( L \subset \mathbb{R}^3 \).

Define \( \phi(L) = \int_L f \), or

\[
\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds.
\]
1938 Fritz John. $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, oriented line $L \subset \mathbb{R}^3$. Define $\phi(L) = \int_L f$, or

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds.$$

The space of oriented lines is 4 dimensional, and $4 > 3$ so expect one condition on $\phi$. 

Differentiate under the integral: ultrahyperbolic wave equation

$$\partial^2 \phi / \partial \alpha_1 \partial \beta_2 - \partial^2 \phi / \partial \alpha_2 \partial \beta_1 = 0.$$

Change coordinates $\alpha_1 = x + y, \alpha_2 = t + z, \beta_1 = t - z, \beta_2 = x - y$.

$$\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial z^2 - \partial^2 \phi / \partial y^2 - \partial^2 \phi / \partial t^2 = 0.$$

Relevant to physics with two times!
1938 Fritz John. $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, oriented line $L \subset \mathbb{R}^3$.
Define $\phi(L) = \int_L f$, or

$$
\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds.
$$

The space of oriented lines is 4 dimensional, and $4 > 3$ so expect one condition on $\phi$.

Differentiate under the integral: ultrahyperbolic wave equation

$$
\frac{\partial^2 \phi}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \phi}{\partial \alpha_2 \partial \beta_1} = 0.
$$
1938 Fritz John. \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \), oriented line \( L \subset \mathbb{R}^3 \).

Define \( \phi(L) = \int_L f \), or

\[
\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds.
\]

The space of oriented lines is 4 dimensional, and \( 4 > 3 \) so expect one condition on \( \phi \).

Differentiate under the integral: ultrahyperbolic wave equation

\[
\frac{\partial^2 \phi}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \phi}{\partial \alpha_2 \partial \beta_1} = 0.
\]

Change coordinates \( \alpha_1 = x + y, \alpha_2 = t + z, \beta_1 = t - z, \beta_2 = x - y \).

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = 0.
\]

Relevant to physics with two times!

\[ \phi(L) = \int_L \frac{dI}{I} = \log I - \log I_0 = -\int_L f \]

1979 Nobel Prize (in medicine) for image reconstruction.
1967 Penrose (Twistor theory). Wave equation in Minkowski space.

\[ \phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{C}P^1} f((z + t) + (x + iy)\lambda, (x - iy) - (z - t)\lambda, \lambda) d\lambda \]

verify

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0. \]
1967 Penrose (Twistor theory). Wave equation in Minkowski space.

\[ \phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} f((z + t) + (x + iy)\lambda, (x - iy) - (z - t)\lambda, \lambda) d\lambda \]

verify

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0. \]

Mathematically sophisticated: Could modify a contour and add a holomorphic function inside the contour to \( f \). Needs sheaf cohomology (Atiyah).
Quantum Physics. Complex wave function, Hilbert spaces, ...
Complex Numbers in Physics

- Quantum Physics. Complex wave function, Hilbert spaces, ...
- Classical Physics. Complex numbers in the sky! Celestial sphere

\[(u_1)^2 + (u_2)^2 + (u_3)^2 = 1\]

Stereographic projection onto a plane
Complex Numbers in Physics

- Quantum Physics. Complex wave function, Hilbert spaces, ...
- Classical Physics. Complex numbers in the sky! Celestial sphere

\[(u_1)^2 + (u_2)^2 + (u_3)^2 = 1\]

Stereographic projection onto a plane

From north pole \((0, 0, 1)\), \(\lambda = \frac{u_1 + iu_2}{1 - u_3}\).
From south pole \((0, 0, -1)\), \(\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}\).
Complex Numbers in Physics

- Quantum Physics. Complex wave function, Hilbert spaces, ...
- Classical Physics. Complex numbers in the sky! Celestial sphere

\[(u_1)^2 + (u_2)^2 + (u_3)^2 = 1\]

Stereographic projection onto a plane

- From north pole \((0, 0, 1)\), \(\lambda = \frac{u_1 + iu_2}{1 - u_3}\).
- From south pole \((0, 0, -1)\), \(\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}\).
- On the overlap \(\tilde{\lambda} = 1/\lambda\). This makes \(S^2\) into a complex manifold \(\mathbb{CP}^1\) (Riemann sphere).
**Complex Numbers in Physics**

- Quantum Physics. Complex wave function, Hilbert spaces, ...
- Classical Physics. Complex numbers in the sky! Celestial sphere

\[(u_1)^2 + (u_2)^2 + (u_3)^2 = 1\]

Stereographic projection onto a plane

- From north pole \((0, 0, 1)\), \(\lambda = \frac{u_1 + iu_2}{1 - u_3}\).
- From south pole \((0, 0, -1)\), \(\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}\).

On the overlap \(\tilde{\lambda} = 1/\lambda\). This makes \(S^2\) into a complex manifold \(\mathbb{CP}^1\) (Riemann sphere).

- Möbius transformations \(\xrightarrow{2:1}\) Lorentz transformations.
**Twistor Programme**

- **Twistor correspondence**

  Space time $\leftrightarrow$ Twistor space

  Point $\leftrightarrow$ Complex line $\mathbb{CP}^1$

  Light ray $\leftrightarrow$ Point.
Twistor Programme

- Twistor correspondence

  \[
  \text{Space time} \leftrightarrow \text{Twistor space} \\
  \text{Point} \leftrightarrow \text{Complex line } \mathbb{CP}^1 \\
  \text{Light ray} \leftrightarrow \text{Point.}
  \]

- Space-time points are derived objects in twistor theory. They become ‘fuzzy’ after quantisation. Attractive framework for quantum gravity.
Twistor Programme

- Twistor correspondence

  Space time $\leftrightarrow$ Twistor space
  Point $\leftrightarrow$ Complex line $\mathbb{CP}^1$
  Light ray $\leftrightarrow$ Point.

- Space-time points are derived objects in twistor theory. They become ‘fuzzy’ after quantisation. Attractive framework for quantum gravity.

- 40 years of research: No major impact on physics (so far).
  Surprisingly major impact on pure mathematics: representation theory, differential geometry, solitons, instantons, integrable systems.
Differential equations and complex numbers

- Harmonic functions on $\mathbb{R}^2$. Complex numbers $\mathbb{R}^2 = \mathbb{C}$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \phi = \text{Re}(f(\zeta)), \quad \zeta = x + iy.$$
Harmonic functions on $\mathbb{R}^2$. Complex numbers $\mathbb{R}^2 = \mathbb{C}$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \phi = \text{Re}(f(\zeta)), \quad \zeta = x + iy.$$

Harmonic functions on $\mathbb{R}^3$? Problem: 3 is an odd number, $\mathbb{R}^3 \neq \mathbb{C}^n$. 

Twistor space $T = \text{space of oriented lines in } \mathbb{R}^3$. Line $v + tu$, $t \in \mathbb{R}$.

Dimension of $T$ is four (even!).

Dunajski (DAMTP, Cambridge)  
Twistor Theory  
April 2010 8 / 1
Harmonic functions on $\mathbb{R}^2$. Complex numbers $\mathbb{R}^2 = \mathbb{C}$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \phi = \text{Re}(f(\zeta)), \quad \zeta = x + iy.$$

Harmonic functions on $\mathbb{R}^3$? Problem: 3 is an odd number, $\mathbb{R}^3 \neq \mathbb{C}^n$.

Twistor space $\mathbb{T} = \text{space of oriented lines in } \mathbb{R}^3$. Line $\mathbf{v} + t \mathbf{u}$, $t \in \mathbb{R}$.

$|\mathbf{u}|=1$, $\mathbf{u} \cdot \mathbf{v} = 0$

Dimension of $\mathbb{T}$ is four (even!).
Oriented Lines in $\mathbb{R}^3$

- $T = \{(u, v) \in S^2 \times \mathbb{R}^3, \ u \cdot v = 0\}$. For each fixed $u$ this space restricts to a tangent plane to $S^2$. The twistor space is the union of all tangent planes – the tangent bundle $TS^2$. 

Dunajski (DAMTP, Cambridge)  
Twistor Theory  
April 2010  9 / 1
Oriented Lines in $\mathbb{R}^3$

- $T = \{(u, v) \in S^2 \times \mathbb{R}^3, \; u \cdot v = 0\}$. For each fixed $u$ this space restricts to a tangent plane to $S^2$. The twistor space is the union of all tangent planes – the tangent bundle $TS^2$.

- Topologically nontrivial: Locally $S^2 \times \mathbb{R}^2$ but globally twisted.
Oriented Lines in $\mathbb{R}^3$

- $\mathbb{T} = \{(u, v) \in S^2 \times \mathbb{R}^3, \ u.v = 0\}$. For each fixed $u$ this space restricts to a tangent plane to $S^2$. The twistor space is the union of all tangent planes – the tangent bundle $TS^2$.
- Topologically nontrivial: Locally $S^2 \times \mathbb{R}^2$ but globally twisted

- Reversing the orientation of lines $\tau : \mathbb{T} \longrightarrow \mathbb{T}, \ \tau(u, v) = (-u, v)$. 

Dunajski (DAMTP, Cambridge)
**Oriented Lines in $\mathbb{R}^3$**

- $\mathbb{T} = \{(u, v) \in S^2 \times \mathbb{R}^3, \ u \cdot v = 0\}$. For each fixed $u$ this space restricts to a tangent plane to $S^2$. The twistor space is the union of all tangent planes – the tangent bundle $TS^2$.

- Topologically nontrivial: Locally $S^2 \times \mathbb{R}^2$ but globally twisted

- Reversing the orientation of lines $\tau : \mathbb{T} \longrightarrow \mathbb{T}$, $\tau(u, v) = (-u, v)$.

- Points $p = (x, y, z)$ in $\mathbb{R}^3$ two–spheres in $\mathbb{T}$; $\tau$–invariant maps

  $$u \longrightarrow (u, v(u) = p - (p \cdot u)u) \in \mathbb{T}.$$

Twistor space as a complex manifold

- Holomorphic coordinates

\[ \lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{CP}^1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3. \]

Need another coordinate patch \((\tilde{\lambda}, \tilde{\eta})\) containing \(u = (0, 0, 1)\). On the overlap \(\tilde{\lambda} = 1/\lambda, \tilde{\eta} = -\eta/\lambda^2\).
Twistor space as a complex manifold

- Holomorphic coordinates

\[ \lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{CP}^1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2}v_3. \]

Need another coordinate patch \((\tilde{\lambda}, \tilde{\eta})\) containing \(u = (0, 0, 1)\). On the overlap \(\tilde{\lambda} = 1/\lambda, \tilde{\eta} = -\eta/\lambda^2\).

- Points in \(\mathbb{R}^3\) are \(\tau\)-invariant holomorphic maps \(\mathbb{CP}^1 \to T\mathbb{CP}^1\)

\[ \lambda \to (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2(x - iy)). \]
Harmonic functions on $\mathbb{R}^3$

To find a harmonic function at $P = (x, y, z)$:

- Restrict a twistor function $f(\lambda, \eta)$ to $\hat{P} = \mathbb{CP}^1 = S^2$.
To find a harmonic function at $P = (x, y, z)$:

- Restrict a twistor function $f(\lambda, \eta)$ to $\hat{P} = \mathbb{CP}^1 = S^2$.
- Integrate along a closed contour

$$\phi(x, y, z) = \oint_{\Gamma \subset \hat{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2 (x - iy))d\lambda,$$

(Whittaker, 1903).
To find a harmonic function at \( P = (x, y, z) \):

- Restrict a twistor function \( f(\lambda, \eta) \) to \( \hat{P} = \mathbb{CP}^1 = S^2 \).
- Integrate along a closed contour

\[
\phi(x, y, z) = \oint_{\Gamma \subset \hat{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2(x - iy)) d\lambda,
\]

- Differentiate under the integral to verify

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.
\]

(Whittaker, 1903).
(\(A_j(x), \Phi(x)\)) Lie algebra valued on \(\mathbb{R}^3\),

\[
F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.
\]
Magnetic monopoles and SYZ conjecture

- \((A_j(x), \Phi(x))\) Lie algebra valued on \(\mathbb{R}^3\),

\[
F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.
\]

- Monopole equation \(\partial_j \Phi + [A_j, \Phi] = (1/2)\varepsilon_{jkl}F_{kl}\).
Magnetic monopoles and SYZ conjecture

- \((A_j(x), \Phi(x))\) Lie algebra valued on \(\mathbb{R}^3\),

\[
F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.
\]

- Monopole equation \(\partial_j \Phi + [A_j, \Phi] = (1/2)\varepsilon_{jkl}F_{kl}\).

- Impose a symmetry: Hitchin system on a surface.

\[
F_A - \Phi \wedge \Phi^* - \Phi^* \wedge \Phi = 0, \quad A = A_z dz + (A_z)^* d\bar{z}, \quad \Phi = Q d\bar{z}.
\]
Magnetic monopoles and SYZ conjecture

- \((A_j(x), \Phi(x))\) Lie algebra valued on \(\mathbb{R}^3\),
  \[
  F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.
  \]

- Monopole equation \(\partial_j \Phi + [A_j, \Phi] = (1/2)\epsilon_{jkl}F_{kl}\).

- Impose a symmetry: Hitchin system on a surface.
  \[
  F_A - \Phi \wedge \Phi^* - \Phi^* \wedge \Phi = 0, \quad A = A_z dz + (A_z)^* d\bar{z}, \quad \Phi = Qd\bar{z}.
  \]

- SYZ conjecture in the semi–flat limit. \(\psi = \psi(z, \bar{z}), U = U(z, \bar{z})\).
  \[
  \psi_{z\bar{z}} + \frac{1}{2}e^\psi + |U|^2e^{-2\psi} = 0, \quad U_{\bar{z}} = 0, \quad \text{(Loftin, Yau, Zaslov, 2004).}
  \]
Magnetic monopoles and SYZ conjecture

\((A_j(x), \Phi(x))\) Lie algebra valued on \(\mathbb{R}^3\),

\[F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.\]

Monopole equation \(\partial_j \Phi + [A_j, \Phi] = (1/2)\varepsilon_{jkl}F_{kl}\).

Impose a symmetry: Hitchin system on a surface.

\[F_A - \Phi \wedge \Phi^* - \Phi^* \wedge \Phi = 0, \quad A = A_z dz + (A_z)^* d\bar{z}, \quad \Phi = Q d\bar{z}.\]

SYZ conjecture in the semi–flat limit. \(\psi = \psi(z, \bar{z}), U = U(z, \bar{z}).\)

\[\psi_{z\bar{z}} + \frac{1}{2} e^{\psi} + |U|^2 e^{-2\psi} = 0, \quad U_{\bar{z}} = 0, \quad (\text{Loftin, Yau, Zaslov, 2004}).\]

LYZ equation is an \(SU(2, 1)\) Hitchin system

\[A_z = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} & 0 \\ 0 & -\frac{1}{2} \psi_z & -U e^{-\psi} \\ 0 & 0 & \frac{1}{2} \psi_z \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\]
Given \((A_j(x), \Phi(x))\) solve a matrix ODE along each oriented line
\[ x(t) = v + tu \]
\[ \frac{dV}{dt} + (u^j A_j + i\Phi)V = 0. \]

Space of solutions at \(L \in \mathbb{T}\) is a complex vector space \(\mathbb{C}^2\).
Twistor solution to the monopole equation

- Given \((A_j(x), \Phi(x))\) solve a matrix ODE along each oriented line \(x(t) = v + tu\)

\[
\frac{dV}{dt} + (u^j A_j + i\Phi) V = 0.
\]

Space of solutions at \(L \in \mathbb{T}\) is a complex vector space \(\mathbb{C}^2\).  
- Complex vector bundle over \(\mathbb{T}\) with patching matrix \(F(\lambda, \bar{\lambda}, \eta, \bar{\eta})\).  

Open covering \(C^2\)

Patching matrix \(F: U \cup \bar{U} \rightarrow GL(2, \mathbb{C})\)
Twistor solution to the monopole equation

- Given \((A_j(x), \Phi(x))\) solve a matrix ODE along each oriented line 
  \[ x(t) = v + tu \]
  
  \[ \frac{dV}{dt} + (u^j A_j + i\Phi)V = 0. \]

  Space of solutions at \(L \in \mathbb{T}\) is a complex vector space \(\mathbb{C}^2\).
- Complex vector bundle over \(\mathbb{T}\) with patching matrix \(F(\lambda, \bar{\lambda}, \eta, \bar{\eta})\).

Monopole equation \(\leftrightarrow\) Cauchy–Riemann eq. \(\frac{\partial F}{\partial \lambda} = 0, \frac{\partial F}{\partial \bar{\eta}} = 0\).
Twistor solution to the monopole equation

- Given \((A_j(x), \Phi(x))\) solve a matrix ODE along each oriented line \(x(t) = v + tu\)

\[
\frac{dV}{dt} + (u^j A_j + i\Phi) V = 0.
\]

Space of solutions at \(L \in \mathbb{T}\) is a complex vector space \(\mathbb{C}^2\).

- Complex vector bundle over \(\mathbb{T}\) with patching matrix \(F(\lambda, \bar{\lambda}, \eta, \bar{\eta})\).

- Monopole equation \(\leftrightarrow\) Cauchy–Riemann eq. \(\frac{\partial F}{\partial \lambda} = 0, \frac{\partial F}{\partial \bar{\eta}} = 0\).

- Holomorphic vector bundles over \(T\mathbb{CP}^1\) - well understood. Take one and work backwards to construct a monopole. (Hitchin, 1982.)
A problem of R. Liouville (1889)

Cover a plane with curves, one curve through each point in each direction. How can you tell whether these curves are geodesics of some metric?
A problem of R. Liouville (1889)

Cover a plane with curves, one curve through each point in each direction. How can you tell whether these curves are geodesics of some metric?

- Path geometry: \( y'' = F(x, y, y') \). Cartan (1922), T. Y. Thomas (1925), Douglas (1936).
A problem of R. Liouville (1889)

Cover a plane with curves, one curve through each point in each direction. How can you tell whether these curves are geodesics of some metric?

- Path geometry: $y'' = F(x, y, y')$. Cartan (1922), T. Y. Thomas (1925), Douglas (1936).

- When are the paths unparametrised geodesics of some connection $\Gamma$ on $M \subset \mathbb{R}^2$? Eliminate the parameter in $\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c \sim \dot{x}^a$.

$$y'' = A_0(x, y) + A_1(x, y)y' + A_2(x, y)(y')^2 + A_3(x, y)(y')^3, \quad x^a = (x, y).$$
A problem of R. Liouville (1889)

Cover a plane with curves, one curve through each point in each direction. How can you tell whether these curves are geodesics of some metric?

- Path geometry: \( y'' = F(x, y, y') \). Cartan (1922), T. Y. Thomas (1925), Douglas (1936).

- When are the paths unparametrised geodesics of some connection \( \Gamma \) on \( M \subset \mathbb{R}^2 \)? Eliminate the parameter in \( \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c \sim \dot{x}^a \).

\[
y'' = A_0(x, y) + A_1(x, y)y' + A_2(x, y)(y')^2 + A_3(x, y)(y')^3, \quad x^a = (x, y).
\]

- A projective structure on an open set \( M \subset \mathbb{R}^2 \): an equivalence class of torsion free connections \([\Gamma]\). Two connections \( \Gamma \) and \( \hat{\Gamma} \) are equivalent if they share the same unparametrised geodesics.

\[
\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \delta_a^c \omega_b + \delta_b^c \omega_a, \quad a, b, c = 1, 2.
\]

for some one–form \( \omega = \omega_a dx^a \).
Summary of the results

- Look for conditions on a connection $\Gamma^c_{ab}$ for the existence of a one form $\omega_a$ and a symmetric non-degenerate tensor $g_{ab}$ such that the projectively equivalent connection

$$\Gamma^c_{ab} + \delta^c_a \omega_b + \delta^c_b \omega_a$$

is the Levi-Civita connection for $g_{ab}$.
Look for conditions on a connection $\Gamma_{ab}^c$ for the existence of a one form $\omega_a$ and a symmetric non-degenerate tensor $g_{ab}$ such that the projectively equivalent connection

$$\Gamma_{ab}^c + \delta_a^c \omega_b + \delta_b^c \omega_a$$

is the Levi-Civita connection for $g_{ab}$.

Overdetermined system: 4 equations for 3 unknowns.
Summary of the Results

- Look for conditions on a connection $\Gamma_{ab}^c$ for the existence of a one form $\omega_a$ and a symmetric non-degenerate tensor $g_{ab}$ such that the projectively equivalent connection

$$\Gamma_{ab}^c + \delta_a^c \omega_b + \delta_b^c \omega_a$$

is the Levi-Civita connection for $g_{ab}$.

- Overdetermined system: 4 equations for 3 unknowns.


  Necessary condition: obstruction of order 5 in the components of a connection. Point invariant for a second order ODE whose integral curves are the geodesics of $[\Gamma]$ or a scalar projective invariant of the projective class.

- Sufficient conditions: Vanishing of two invariants of order at most 8.

  In the smooth case even if $M$ is simply connected.
Summary of the results

- Look for conditions on a connection $\Gamma^c_{ab}$ for the existence of a one form $\omega_a$ and a symmetric non–degenerate tensor $g_{ab}$ such that the projectively equivalent connection

$$\Gamma^c_{ab} + \delta^c_a \omega_b + \delta^c_b \omega_a$$

is the Levi-Civita connection for $g_{ab}$.

- Overdetermined system: 4 equations for 3 unknowns.


- Necessary condition: obstruction of order 5 in the components of a connection. Point invariant for a second order ODE whose integral curves are the geodesics of $[\Gamma]$ or a scalar projective invariant of the projective class.

- Sufficient conditions: Vanishing of two invariants of order at most 8. In the smooth case even if $M$ is simply connected.
One-to-one correspondence between holomorphic projective structures \((M, [\Gamma])\) and complex surfaces \(\mathbb{T}\) with a family of rational curves.

\[ T = \text{twistor space} \]

points \(\longleftrightarrow\) geodesics

rational curves with normal bundle \(\mathcal{O}(1)\) \(\longleftrightarrow\) points
Nonabelian Radon Transform

- One-to-one correspondence between holomorphic projective structures $(\mathcal{M}, [\Gamma])$ and complex surfaces $\mathbb{T}$ with a family of rational curves.

- Double fibration $\mathcal{M} \leftarrow \mathbb{P}(TM) \longrightarrow \mathbb{T} = \mathbb{P}(TM)/D_x$, where $D_x = z^a \frac{\partial}{\partial x^a} - \Gamma^c_{ab} z^a z^b \frac{\partial}{\partial z^c}$ is a geodesic spray.
Nonabelian Radon Transform

- One-to-one correspondence between holomorphic projective structures $(M, [\Gamma])$ and complex surfaces $\mathbb{T}$ with a family of rational curves.

\[ T = \text{twistor space} \]

\[ \text{points} \quad \longleftrightarrow \quad \text{geodesics} \]

\[ \text{rational curves with normal bundle } \mathcal{O}(1) \quad \longleftrightarrow \quad \text{points} \]

- Double fibration $M \leftarrow \mathbb{P}(TM) \longrightarrow \mathbb{T} = \mathbb{P}(TM)/D_x$, where

\[ D_x = z^a \frac{\partial}{\partial x^a} - \Gamma^c_{ab} z^a z^b \frac{\partial}{\partial z^c} \] is a geodesic spray.

- $(M, [\Gamma])$ is metrisable iff $\mathbb{T}$ is equipped with a preferred section of the line bundle $\kappa_{\mathbb{T}}^{-2/3}$, where $\kappa_{\mathbb{T}}$ is the canonical bundle.
Twistor Theory

- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

\[
\begin{align*}
\text{point} & \iff \text{line (complex)} \\
\text{line} & \iff \text{point}.
\end{align*}
\]
Twistor Theory

- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).
  
  \[
  \text{point} \leftrightarrow \text{line (complex)}
  
  \text{line} \leftrightarrow \text{point}.
  \]

- Complex numbers are essential
  
  nonlinear PDEs $\leftrightarrow$ linear Cauchy–Riemann equations.
Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

- point $\leftrightarrow$ line (complex)
- line $\leftrightarrow$ point.

Complex numbers are essential nonlinear PDEs $\leftrightarrow$ linear Cauchy–Riemann equations.

Its status as a physical theory is not clear.
Twistor Theory

- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

  \[
  \text{point} \longleftrightarrow \text{line (complex)}
  \]

  \[
  \text{line} \longleftrightarrow \text{point}.
  \]

- Complex numbers are essential

  nonlinear PDEs \(\longleftrightarrow\) linear Cauchy–Riemann equations.

- Its status as a physical theory is not clear
  - Weakness: effective only in low dimensions.
Twistor Theory

- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

  \[
  \text{point} \leftrightarrow \text{line (complex)}
  \]

  \[
  \text{line} \leftrightarrow \text{point}.
  \]

- Complex numbers are essential

  \[
  \text{nonlinear PDEs} \leftrightarrow \text{linear Cauchy–Riemann equations}.
  \]

- Its status as a physical theory is not clear
  - Weakness: effective only in low dimensions.
  - Strength: effective only in low dimensions.
Solitons, Instantons and Twistors

Maciej Dunajski