Twistor Theory and Differential Equations

Maciej Dunajski

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

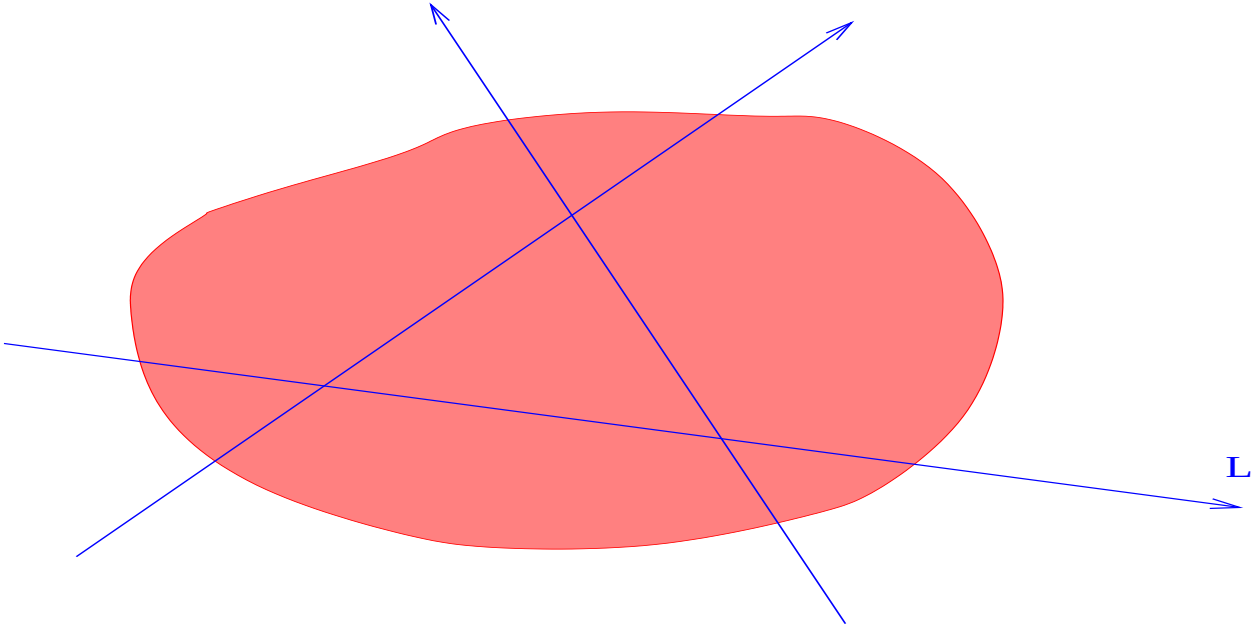
- Work of many: Penrose, Ward, Atiyah, Hitchin, Mason , Sparling, Tod, Woodhouse, ...
- MD, J. Phys. A (2009) `arXiv:math/0902.0274.`
- MD, Prim Plansangkate, Comm. Math. Phys. (2009) `arXiv:math/0809.3015`
- Robert Bryant, MD, Mike Eastwood `arXiv:math/0801.0300`, J. Diff. Geom. (2010).

Motivation: Integral Geometry

- 1917 Radon. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with decay condition ot ∞, $L \subset \mathbb{R}^2$ oriented line.

$$\phi(L) := \int_L f.$$

There exist an inversion formula $\phi \longrightarrow f$.



Differential Equations

- 1938 Fritz John. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, oriented line $L \subset \mathbb{R}^3$.
  Define $\phi(L) = \int_L f$, or

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds.$$

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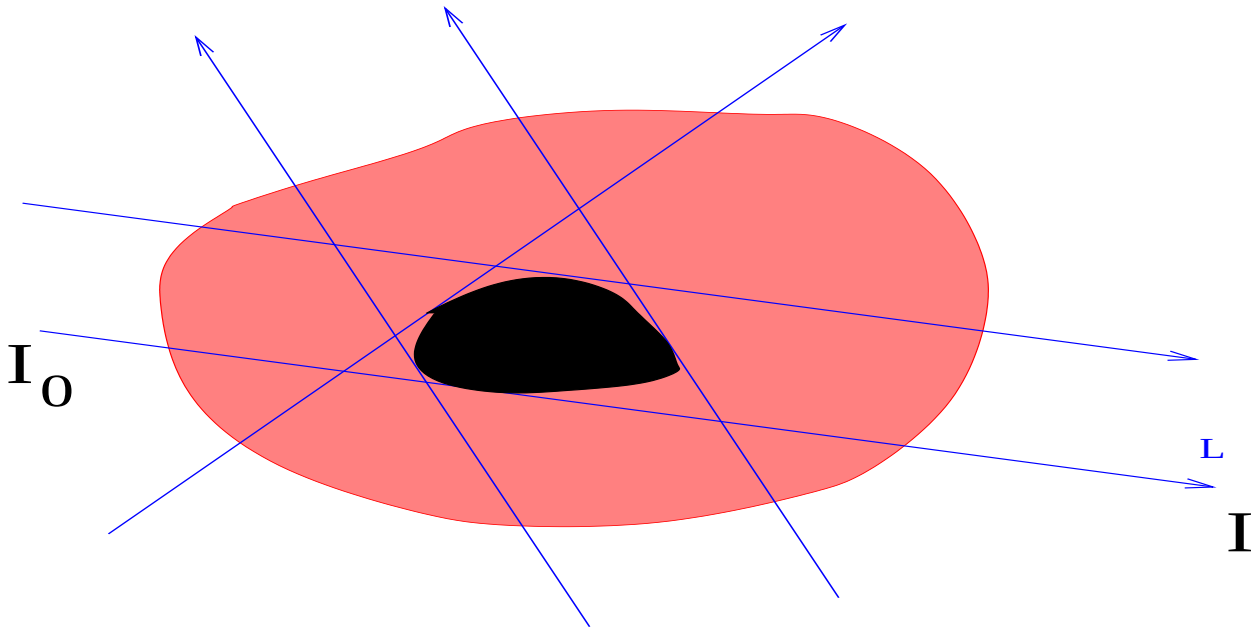
- Change coordinates $\alpha_1 = x + y, \alpha_2 = t + z, \beta_1 = t - z, \beta_2 = x - y$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = 0.$$

Relevant to physics with two times!

Nobel Prize for Mathematics

- 1963 Cormack. Hole theorem.



$$\phi(L) = \int_L \frac{dI}{I} = \log I - \log I_0 = -\int_L f$$

1979 Nobel Prize (in medicine) for image reconstruction.

Penrose's integral formulae

- 1967 Penrose (Twistor theory). Wave equation in Minkowski space.

$$\phi(x,y,z,t) = \oint_{\Gamma\subset\mathbb{CP}^1} f((z+t)+(x+iy)\lambda, (x-iy)-(z-t)\lambda, \lambda)d\lambda$$

$$\text{verify} \qquad \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = 0.$$

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- Mathematically sophisticated: Could modify a contour and add a holomorphic function inside the contour to f. Needs sheaf cohomology (Atiyah).

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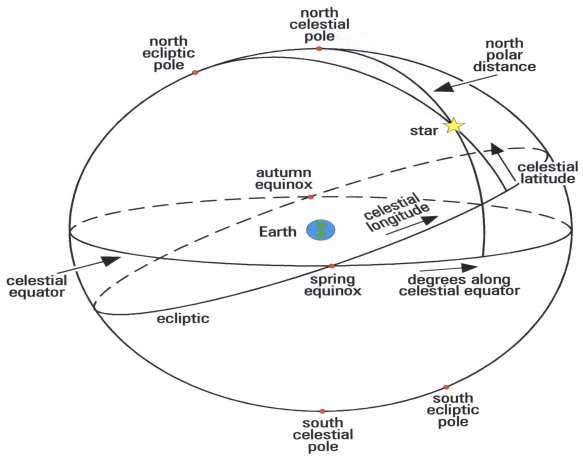
Stereographic projection onto a plane



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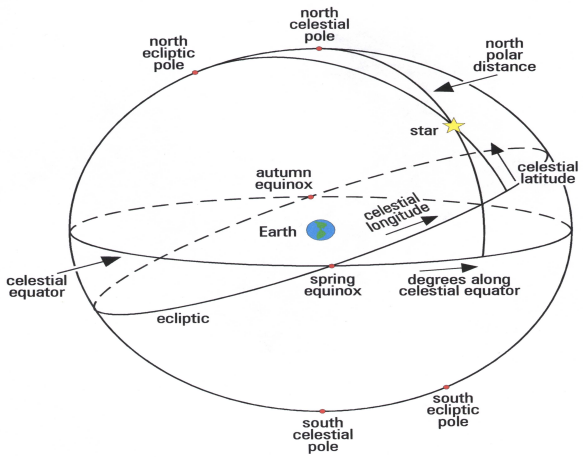

- From north pole $(0,0,1)$, $\quad \lambda = \frac{u_1 + i u_2}{1 - u_3}$.
  From south pole $(0,0,-1)$, $\quad \tilde{\lambda} = \frac{u_1 - i u_2}{1 + u_3}$.

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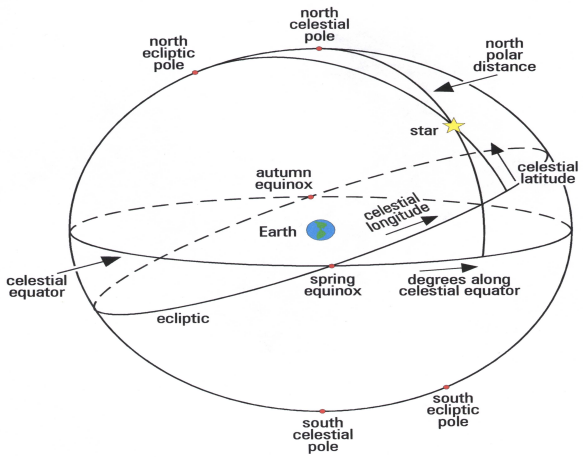

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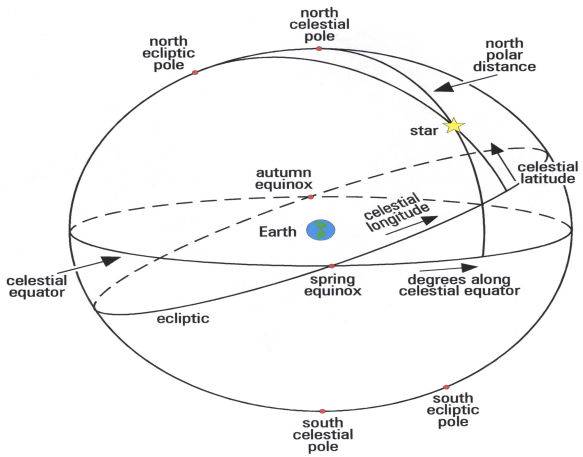

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- Möbius transformations $\xrightarrow{2:1}$ Lorentz transformations.

Twistor Programme

- Twistor correspondence

Space time	$\longleftrightarrow$	Twistor space
Point	$\longleftrightarrow$	Complex line $\mathbb{CP}^1$
Light ray	$\longleftrightarrow$	Point.

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- Space-time points are derived objects in twistor theory. They become 'fuzzy' after quantisation. Attractive framework for quantum gravity.
- 40 years of research: No major impact on physics (so far). Surprisingly major impact on pure mathematics: representation theory, differential geometry, solitons, instantons, integrable systems .

Differential equations and complex numbers

- Harmonic functions on $\mathbb{R}^2$. Complex numbers $\mathbb{R}^2 = \mathbb{C}$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \phi = \mathsf{Re}(f(\zeta)), \quad \zeta = x + iy.$$

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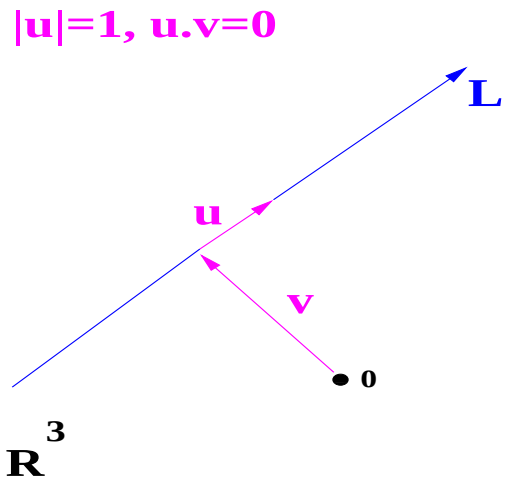
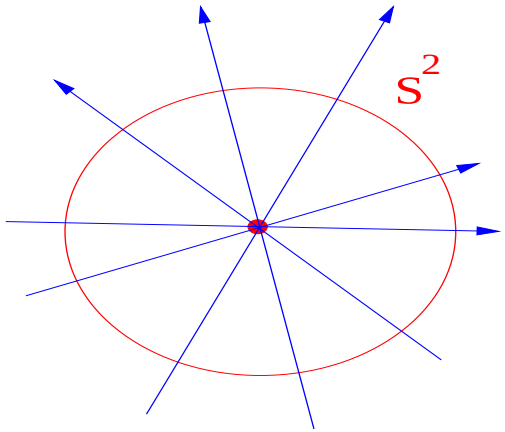
- Twistor space $\mathbb{T}$ = space of oriented lines in $\mathbb{R}^3$. Line $\mathbf{v} + t\mathbf{u}, \quad t \in \mathbb{R}$.





Dimension of $\mathbb{T}$ is four (even!).

Oriented Lines in $\mathbb{R}^3$

- $\mathbb{T} = \{(\mathbf{u}, \mathbf{v}) \in S^2 \times \mathbb{R}^3,\ \mathbf{u}.\mathbf{v} = 0\}$. For each fixed $\mathbf{u}$ this space restricts to a tangent plane to S^2. The twistor space is the union of all tangent planes – the tangent bundle TS^2.

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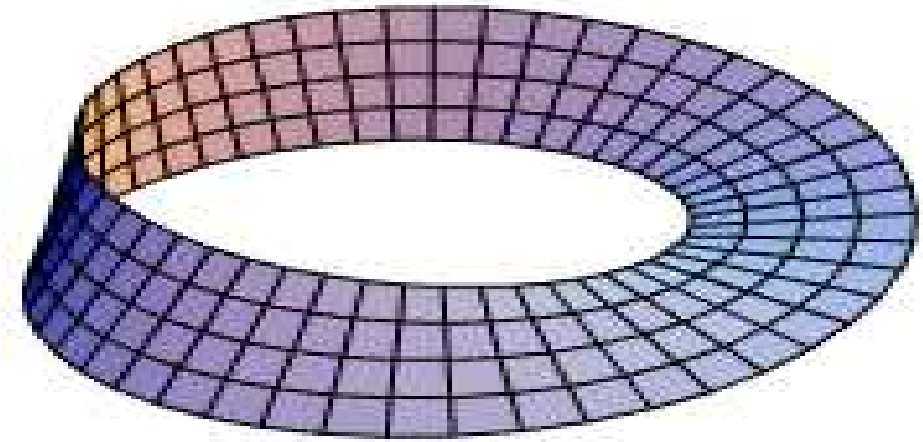
- Topologically nontrivial: Locally $S^2 \times \mathbb{R}^2$ but globally twisted



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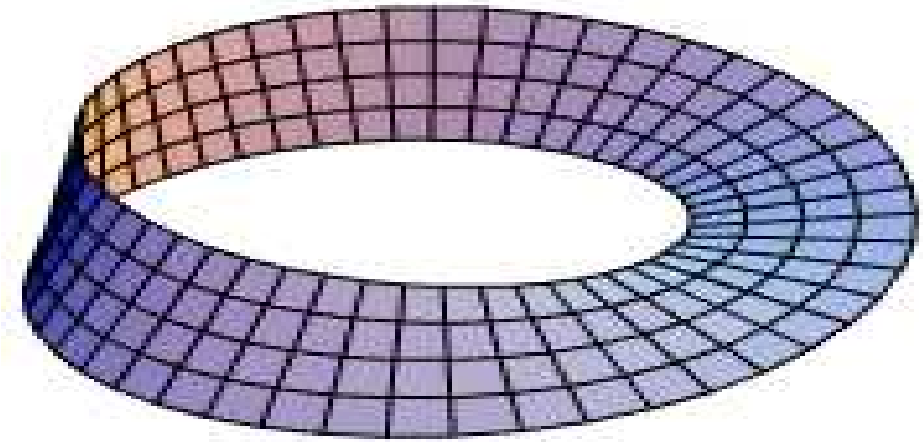

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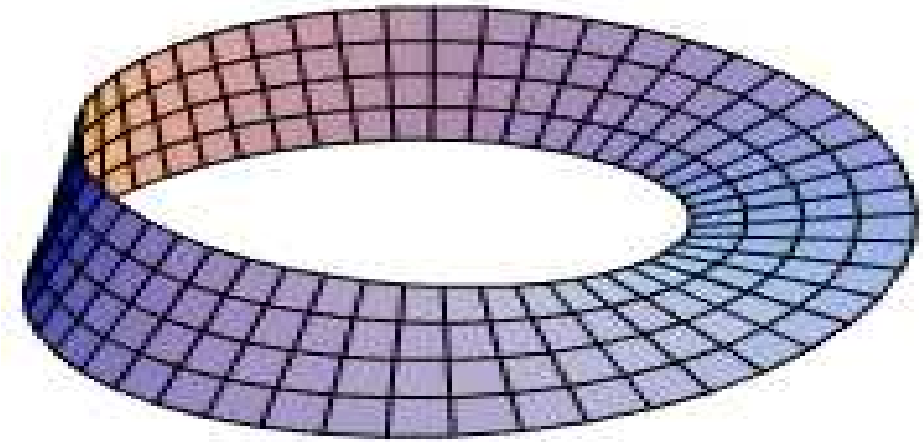

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- Points $\mathbf{p} = (x, y, z)$ in $\mathbb{R}^3$ two–spheres in $\mathbb{T}$; τ–invariant maps

$$\mathbf{u} \longrightarrow (\mathbf{u}, \mathbf{v}(\mathbf{u}) = \mathbf{p} - (\mathbf{p}.\mathbf{u})\mathbf{u}) \in \mathbb{T}.$$

Twistor space as a complex manifold

- Holomorphic coordinates

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{CP}^1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3.$$

Need another coordinate patch $(\tilde{\lambda}, \tilde{\eta})$ containing $\mathbf{u} = (0, 0, 1)$. On the overlap $\tilde{\lambda} = 1/\lambda, \tilde{\eta} = -\eta/\lambda^2$.

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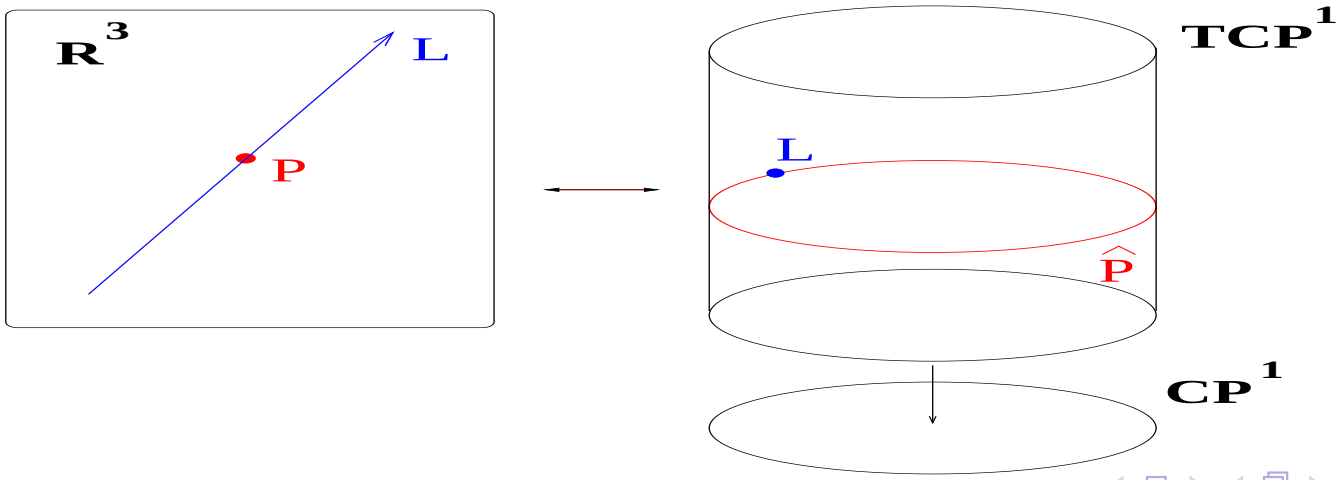
- Points in $\mathbb{R}^3$ are τ-invariant holomorphic maps $\mathbb{CP}^1 \to T\mathbb{CP}^1$

$$\lambda \to (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2 (x - iy)).$$



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To find a harmonic function at $P = (x, y, z)$:

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- Differentiate under the integral to verify

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

(Whittaker, 1903).

Magnetic monopoles and SYZ conjecture

- $(A_j(\mathbf{x}), \Phi(\mathbf{x}))$ Lie algebra valued on $\mathbb{R}^3$,

$$F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.$$

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- Impose a symmmetry: Hitchin system on a surface.

$$F_A - \Phi \wedge \Phi^* - \Phi^* \wedge \Phi = 0, \qquad A = A_z dz + (A_z)^* d\bar{z}, \quad \Phi = Q d\bar{z}.$$

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- SYZ conjecture in the semi–flat limit. $\psi = \psi(z, \bar{z}), U = U(z, \bar{z})$.

$$\psi_{z\bar{z}} + \frac{1}{2} e^{\psi} + |U|^2 e^{-2\psi} = 0, \quad U_{\bar{z}} = 0, \qquad \text{(Loftin, Yau, Zaslov, 2004).}$$

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  LYZ equation is an $SU(2,1)$ Hitchin system

$$A_z = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} & 0 \\ 0 & -\frac{1}{2}\psi_z & -U e^{-\psi} \\ 0 & 0 & \frac{1}{2}\psi_z \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Twistor solution to the monopole equation

- Given $(A_j(\mathbf{x}), \Phi(\mathbf{x}))$ solve a matrix ODE along each oriented line $\mathbf{x}(t) = \mathbf{v} + t\mathbf{u}$

$$\frac{dV}{dt} + (u^j A_j + i\Phi)V = 0.$$

Space of solutions at $L \in \mathbb{T}$ is a complex vector space $\mathbb{C}^2$.

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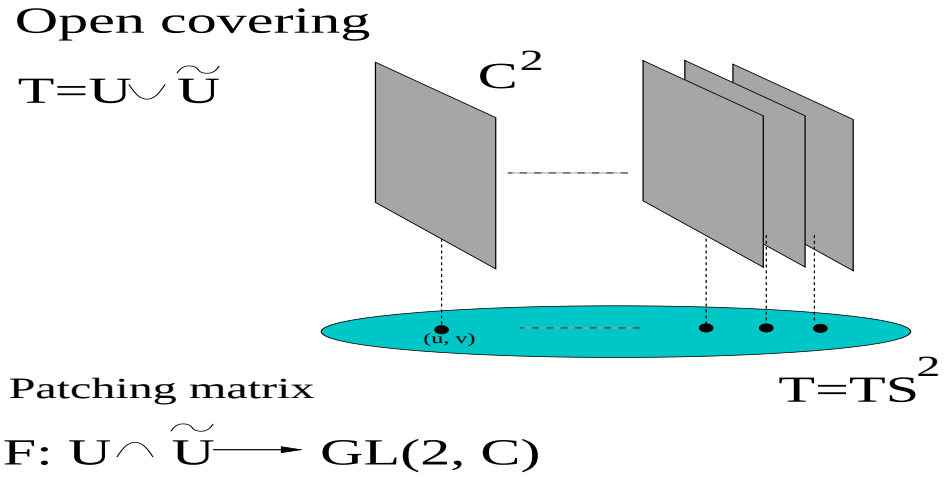

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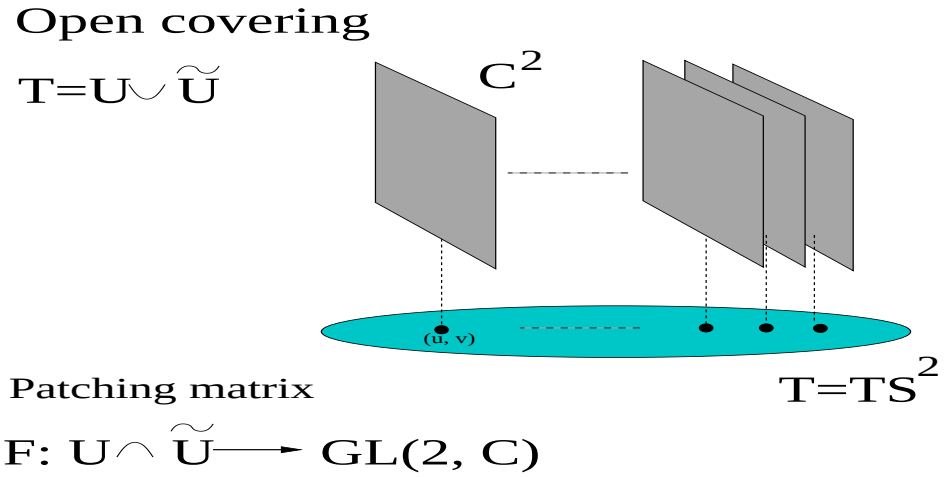

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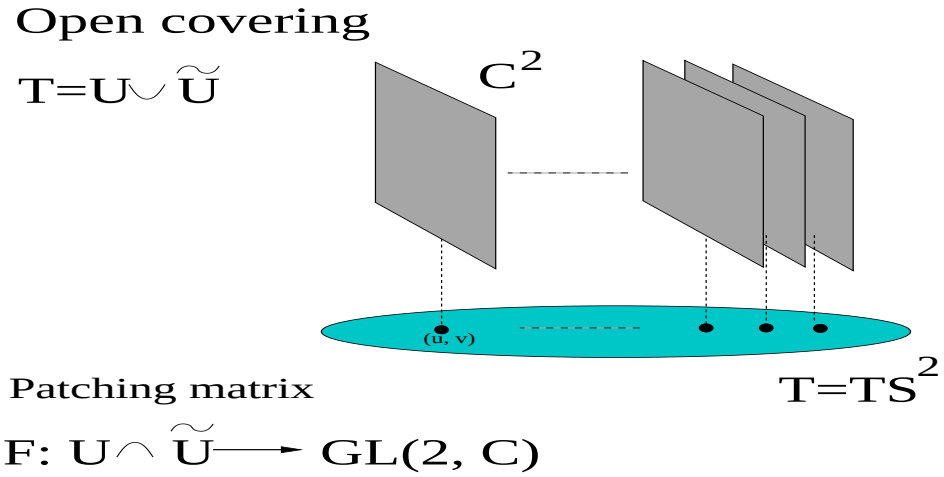

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- Holomorphic vector bundles over $T\mathbb{CP}^1$ - well understood. Take one and work backwards to construct a monopole. (Hitchin, 1982.)

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- Path geometry: $y'' = F(x, y, y')$. Cartan (1922), T. Y. Thomas (1925), Douglas (1936).

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- A projective structure on an open set $M \subset \mathbb{R}^2$: an equivalence class of torsion free connections $[\Gamma]$. Two connections Γ and $\hat{\Gamma}$ are equivalent if they share the same unparametrised geodesics.

$$\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \delta_a{}^c\omega_b + \delta_b{}^c\omega_a, \qquad a, b, c = 1, 2.$$

for some one–form $\omega = \omega_a dx^a$.

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- Look for conditions on a connection Γ^c_{ab} for the existence of a one form ω_a and a symmetric non–degenerate tensor g_{ab} such that the projectively equivalent connection

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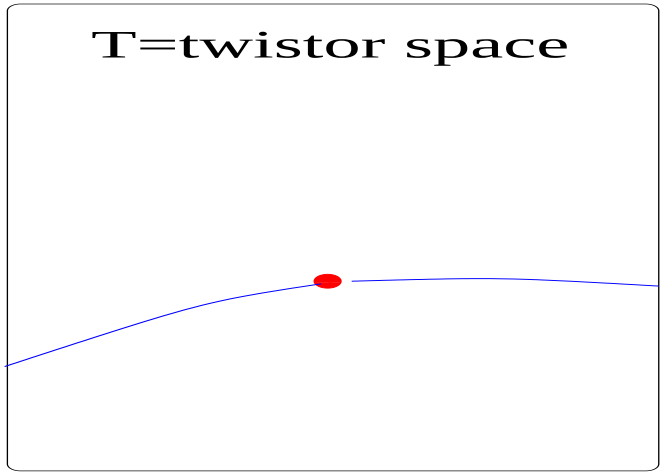
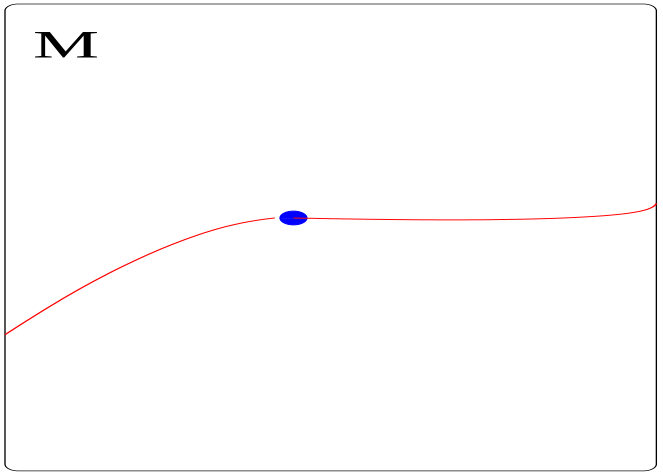
- One-to-one correspondence between holomorphic projective structures $(M, [\Gamma])$ and complex surfaces $\mathbb{T}$ with a family of rational curves.





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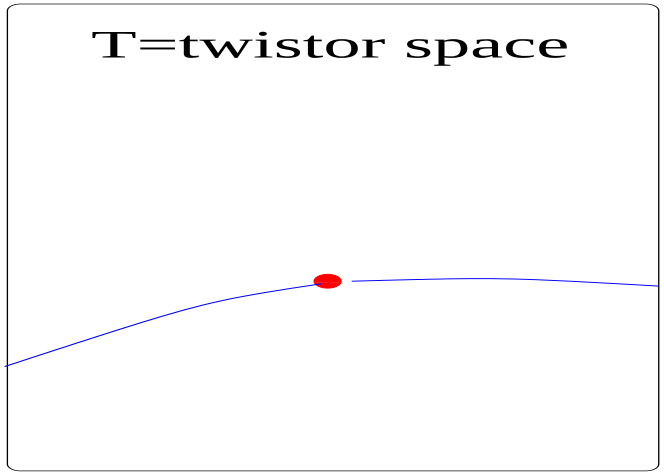

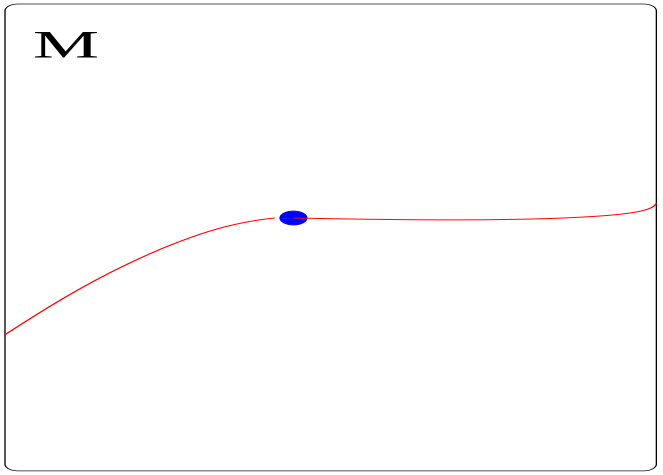

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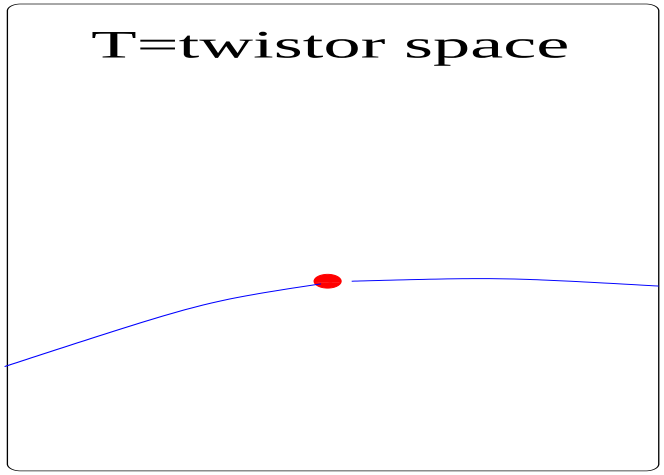

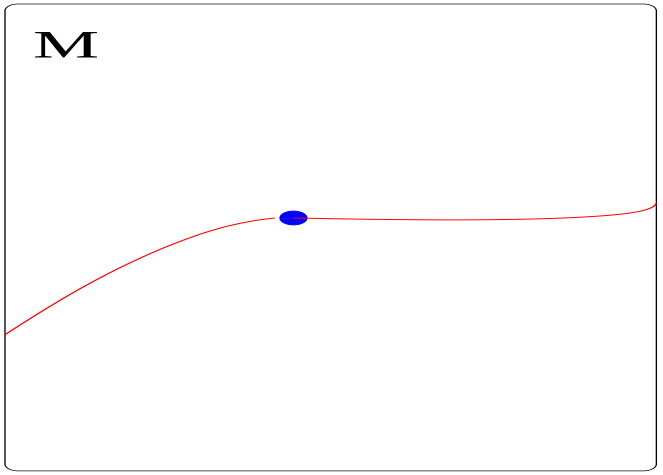

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- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

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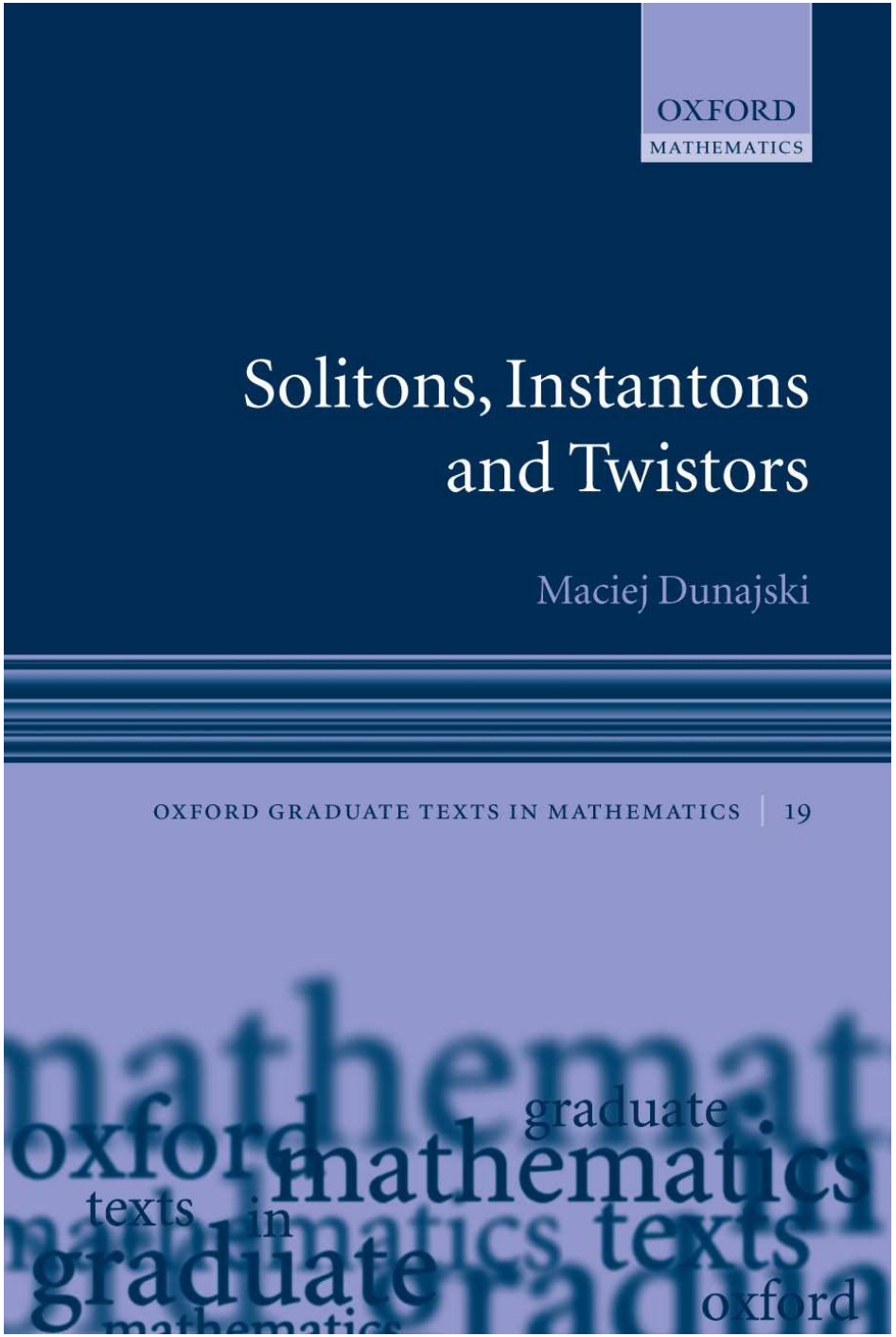
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