Kähler metrics in conformal geometry

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MD, Paul Tod `arXiv:0901.2261`, Mathematical Proceedings of the Cambridge Philosophical Society (2010).

Three 'can you find' questions

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- Find a non–zero function $\Omega : M \to \mathbb{R}$ such that $\Omega^2 g$ satisfies the Einstein equations $G_{ab} = 0$? Answer: ??? (but lots of neccessary conditions are known: e.g vanishing of the Bach tensor).

Kähler structure

$2n$–dimensional Riemannian manifold (M, g).

- $J : TM \to TM$ is a complex structure: $J^2 = -\mathrm{Id}$ and

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- Important structure : a bridge between Riemannian and symplectic geometry (pure mathematics), necessary for existence of supersymmetries (physics).

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- The general Kähler metric can be locally described by the Kähler potential: there exists a function $\mathcal{K} : M \to \mathbb{R}$ and a holomorphic coordinate system $(z^1, \ldots, z^n)$ such that

$$g = \frac{\partial^2 \mathcal{K}}{\partial z^j \partial \overline{z}^k} dz^j d\overline{z}^k.$$

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- Clearly a 'can you find'–type problem. Obstructions should be given by conformally invariant tensors.
- Important in geometry (Donaldson's uniformisation programme). Spin–off in General Relativity: When does a Lorentizian metric admit a conformal Killing–Yano form?

Summary of results in four dimensions

- One–to–one correspondence between Kähler metrics in the conformal class of g and parallel sections of a certain (canonical) connection $\mathcal{D}$ on a rank ten vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$.

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- If $C_+ = 0$ we get some necessary conditions from the holonomy of the curvature of $\mathcal{D}$. E.g. A metric with $C_+ = 0$ is conformal to Einstein AND conformal to Kähler if and only if it admits an isometry.
- The obstructions are only local. A Kähler metric may not exist globaly on a compact manifold M even in a conformally flat case.

SELF–DUALITY IN FOUR DIMENSIONS

Oriented Riemannian four-manifold (M, g)

- $* : \Lambda^2 \to \Lambda^2, \quad (*f)_{ab} = \frac{1}{2}\varepsilon_{ab}{}^{cd} f_{cd}.$

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- Riemann tensor $R_{abcd} = R_{[ab][cd]}$ gives rise to $\mathcal{R} : \Lambda^2 \to \Lambda^2$.

$$\mathcal{R} = \left(\begin{array}{c|c} C_+ + \frac{R}{12} & \phi \\ \hline \phi & C_- + \frac{R}{12} \end{array} \right).$$

$C_\pm =$ SD/ASD Weyl tensors, $\phi =$ trace-free Ricci curvature, $R =$scalar curvature.

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- Two component spinor notation (love it or hate it): $\mu \in \Gamma(\mathbb{S}), \mu = \mu_A$. Spinor indices $A, B, C, \cdots = 0, 1$. $\mu^A = \varepsilon^{AB}\mu_B, \mu_A = \mu^B\varepsilon_{BA}$. Metric $g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$.

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- Spinors and self–duality. $\Sigma \in \Lambda^2(M), \Sigma_{ab} = \Sigma_{[ab]}$.

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- Spinor curvature decomposition

$$\begin{aligned} R_{abcd} &= \psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \psi_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ &\quad + \phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} + \phi_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ &\quad + \frac{R}{12}(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}). \end{aligned}$$

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- Lemma. The metric g_{ab} is conformal to a Kähler metric if and only if there exists a real, symmetric spinor field $\omega_{A'B'}$ satisfying

$$\nabla_{A(A'}\omega_{B'C')} = 0, \qquad (*)$$

and such that $\omega_{A'B'}\omega^{A'B'} \neq 0$.

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- $(*)$ is the (conformally invariant) twistor equation. Idea: prolong it, look for integrability conditions.

Prolongation of $\nabla_{A(A'}\omega_{B'C')} = 0$

- Drop symmetrisation: $\nabla_{AA'}\omega_{B'C'} - \varepsilon_{A'B'}K_{C'A} - \varepsilon_{A'C'}K_{B'A} = 0$ for some $K \in \Lambda^1(M)$.

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- Differentiate and commute derivatives: $\psi^{E'}{}_{(A'B'C'}\omega_{D')E'} = 0$ and

$$\nabla_{AA'}K_{BB'} + P_{ABA'C'}\omega_{B'}{}^{C'} - \varepsilon_{A'B'}\rho_{AB} = 0$$

(where $P_{ab} = (1/2)R_{ab} - (1/12)Rg_{ab}$) for some $\rho \in \Lambda^2_-(M)$.

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- Geometric interpretation $\Psi = (\omega, K, \rho)$ is a section of a rank–10 vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$ which is parallel with respect to a connection $\mathcal{D}$ determined by the blue equations.

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- Therefore $K^a = 0$ and our assumption was wrong.

Example: Local vs. Global obstructions

- Compact hyperbolic four manifold (M, g). Weyl$= 0$, $R = -1$. All local obstructions vanish.
- Assume globally defined non-degenerate $\omega_{A'B'}$ satisfies the twistor eq.
- $\nabla\omega \neq 0$ so ω defines a Killing vector K_a.
- Killing identity $\Box K_a + R_{ab}K^b = 0$, where $R_{ab} = -g_{ab}/4$.
- Contract with K^a, integrate by parts

$$\int_M |\nabla K|^2 \sqrt{g} d^4x = -\frac{1}{4}\int_M |K|^2 \sqrt{g} d^4x.$$

- Therefore $K^a = 0$ and our assumption was wrong.
- More global obstructions in the ASD case (LeBrun). The only simply connected four–manifolds M that are allowed are $K3$ and $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$.

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  Theorem. Let (M, g) be a 4–manifold such that the self–dual part of the conformal curvature is non–zero. Then there exists a Kähler metric in $[g]$ if and only if C_+ is of type D and

$$\nabla_{A(A'}\psi_{B'C'D'E')} - V_{A(A'}\psi_{B'C'D'E')} = 0, \qquad \nabla_{[a}V_{b]} = 0,$$

where $V_{AA'} = \frac{1}{|\psi|^2}\left(\frac{1}{6}\nabla_{AA'}|\psi|^2 + \frac{4}{3}\psi^{B'C'D'E'}\nabla_{AB'}\psi_{C'D'E'A'}\right)$.

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- Theorem. An anti-self–dual Einstein metric g with $\Lambda \neq 0$ is conformal to a Kähler metric iff g admits a Killing vector.
- Examples of conf. classes with more than one Kähler metrics: Fubini-Study metric on $\mathbb{CP}^2$ with reversed orientation.

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- General approach to overdetermined systems: prolong, construct connection on the prolongation vector bundle, restrict its holonomy. Should be applicable to other 'can you find' type problems.