Kähler metrics in conformal geometry

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Let \((M, g = g_{ab}(x)dx^a dx^b)\) be a Lorentzian four-manifold. Can you

- Find a local coordinate system \((t, x, y, z)\) such that

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g = dt^2 - dx^2 - dy^2 - dz^2
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Answer: Need \(R_{abc}^d = 0\) (Riemann curvature).

- Find a local coordinate system \((t, x, y, z)\) and a non–zero function \(\Omega = \Omega(t, x, y, z)\) such that

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g = \Omega^2(dt^2 - dx^2 - dy^2 - dz^2)
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Answer: Need \(C_{abcd} = 0\) (Weyl curvature).
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Three ‘can you find’ questions

Let \( (M, g = g_{ab}(x)dx^a dx^b) \) be a Lorentzian four-manifold. Can you

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3. Find a non–zero function \(\Omega : M \to \mathbb{R}\) such that \(\Omega^2 g\) satisfies the Einstein equations \(G_{ab} = 0\)?

\[\text{but lots of necessary conditions are known: e.g vanishing of the Bach tensor}\]
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2n–dimensional Riemannian manifold $(M, g)$.

- $J : TM \to TM$ is a complex structure: $J^2 = -\text{Id}$ and

$$[T^{(1,0)}, T^{(1,0)}] \subset T^{(1,0)},$$

where $T^{(1,0)} = \{ X \in TM \otimes \mathbb{C}, J(X) = iX \}$. 

Fundamental two–form $\Sigma(X,Y) = g(X,JY)$ is closed.

Alternative characterisation: Holonomy group of $g$ is $U(n)$.

Important structure: a bridge between Riemannian and symplectic geometry (pure mathematics), necessary for existence of supersymmetries (physics).
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**Kähler structure**

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- Important structure: a bridge between Riemannian and symplectic geometry (pure mathematics), necessary for existence of supersymmetries (physics).
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- The general Kähler metric can be locally described by the Kähler potential: there exists a function $\mathcal{K} : M \rightarrow \mathbb{R}$ and a holomorphic coordinate system $(z^1, \ldots, z^n)$ such that

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- The difference between the number of arbitrary functions is $2n^2 - n - 2$, which is positive if $n > 1$ (every metric in 2D is Kähler).
Conformal to Kähler problem

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- Clearly a ‘can you find’–type problem. Obstructions should be given by conformally invariant tensors.

- Important in geometry (Donaldson’s uniformisation programme). Spin–off in General Relativity: When does a Lorentzian metric admit a conformal Killing–Yano form?
One–to–one correspondence between Kähler metrics in the conformal class of $g$ and parallel sections of a certain (canonical) connection $\mathcal{D}$ on a rank ten vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$. If the self–dual (SD) Weyl tensor $\mathcal{C}$ of $g$ is non–zero we find the necessary and sufficient conditions: The SD Weyl spinor must be of algebraic type $D$ and a tensor obstruction of order $4$ in $g$ must vanish. If $\mathcal{C} = 0$ we get some necessary conditions from the holonomy of the curvature of $\mathcal{D}$. E.g. A metric with $\mathcal{C} = 0$ is conformal to Einstein AND conformal to Kähler if and only if it admits an isometry. The obstructions are only local. A Kähler metric may not exist globally on a compact manifold $M$ even in a conformally flat case.
Summary of results in four dimensions

- One-to-one correspondence between Kähler metrics in the conformal class of $g$ and parallel sections of a certain (canonical) connection $\mathcal{D}$ on a rank ten vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$.

- If the self-dual (SD) Weyl tensor $C_+$ of $g$ is non-zero we find the necessary and sufficient conditions: The SD Weyl spinor must be of algebraic type $D$ and a tensor obstruction of order 4 in $g$ must vanish.

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Oriented Riemannian four-manifold $(M, g)$

$\ast: \Lambda^2 \rightarrow \Lambda^2, \quad (\ast f)_{ab} = \frac{1}{2} \varepsilon_{ab}^{\quad cd} f_{cd}$. 

$R_{abcd} = R[ab]^{\quad cd}$ gives rise to $R: \Lambda^2 \rightarrow \Lambda^2$.

$C_{\pm} =$ SD/ASD Weyl tensors, $\phi =$ trace-free Ricci curvature, $R =$ scalar curvature.
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- \(* : \Lambda^2 \to \Lambda^2\), \((*f)_{ab} = \frac{1}{2} \epsilon_{ab}^{\ cd} f_{cd}\).
- \(*^2 = \text{Id}\), \(\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-\).
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- Riemann tensor \(R_{abcd} = R_{[ab][cd]}\) gives rise to \(\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\).

\[
\mathcal{R} = \begin{pmatrix}
C_+ + \frac{R}{12} & \phi \\
\phi & C_- + \frac{R}{12}
\end{pmatrix}.
\]

\(C_\pm = \text{SD/ASD Weyl tensors},\) \(\phi = \text{trace-free Ricci curvature},\)
\(R = \text{scalar curvature}.\)
Spinors in four dimensions

\( C \otimes TM \cong S \otimes S' \), where \((S, \varepsilon), (S', \varepsilon')\) are rank–two complex symplectic vector bundles (spin bundles) over \( M \).
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\[ g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon(v_1, v_2)\varepsilon'(w_1, w_2), \text{ where } v_1, v_2 \in \Gamma(S), \]
\[ w_1, w_2 \in \Gamma(S'). \]
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Two component spinor notation (love it or hate it):
\[ \mu \in \Gamma(S), \mu = \mu_A. \] Spinor indices \( A, B, C, \cdots = 0, 1. \)
\[ \mu^A = \varepsilon^{AB} \mu_B, \mu_A = \mu^B \varepsilon_{BA}. \] Metric \( g_{ab} = \varepsilon_{AB} \varepsilon_{A'B'}. \)
Spinors in Four Dimensions

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  - $\mu^A = \varepsilon^{AB} \mu_B, \mu_A = \mu^B \varepsilon_{BA}$. Metric $g_{ab} = \varepsilon_{AB} \varepsilon_{A'B'}$.
- Spinors and self–duality. $\Sigma \in \Lambda^2(M), \Sigma_{ab} = \Sigma_{[ab]}$.
  \[
  \Sigma_{AA'BB'} = \omega_{AB} \varepsilon_{A'B'} + \omega_{A'B'} \varepsilon_{AB},
  \]
  where $\omega_{AB} = \omega_{(AB)}$ and $\omega_{A'B'} = \omega_{(A'B')}$.
Spinors in Four Dimensions

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- Spinor curvature decomposition
  \[
  R_{abcd} = \psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \psi_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD}
  \]
  \[
  + \phi_{ABCD} \varepsilon_{A'B'} \varepsilon_{CD} + \phi_{A'B'C'D} \varepsilon_{AB} \varepsilon_{C'D'}
  \]
  \[
  + \frac{R}{12} \left( \varepsilon_{AC} \varepsilon_{BD} \varepsilon_{A'C'} \varepsilon_{B'D'} - \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'D'} \varepsilon_{B'C'} \right).
  \]
Twistor equation

- $J^2 = -Id^2 \rightarrow \Sigma$ is SD or ASD. Make a choice:

$$\Sigma_{ab} = \omega_{A'B'}\varepsilon_{AB}, \quad \text{(self–dual)}.$$
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- Conformal rescalling

$$g_{ab} \rightarrow \Omega^2 g_{ab}, \quad \Sigma_{ab} \rightarrow \Omega^3 \Sigma_{ab}, \quad \text{so} \quad \omega_{A'B'} \rightarrow \Omega^2 \omega_{A'B'}.$$
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\]

- Lemma. The metric \( g_{ab} \) is conformal to a Kähler metric if and only if there exists a real, symmetric spinor field \( \omega_{A'B'} \) satisfying

\[
\nabla_A(A'\omega_{B'C'}) = 0, \quad \text{(*)}
\]

and such that \( \omega_{A'B'}\omega^{A'B'} \neq 0 \).
Twistor equation

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- Conformal rescalings

  $$g_{ab} \rightarrow \Omega^2 g_{ab}, \quad \Sigma_{ab} \rightarrow \Omega^3 \Sigma_{ab}, \quad \text{so} \quad \omega_{A'B'} \rightarrow \Omega^2 \omega_{A'B'}.$$  

- Lemma. The metric $g_{ab}$ is conformal to a Kähler metric if and only if there exists a real, symmetric spinor field $\omega_{A'B'}$ satisfying

  $$\nabla_A (A' \omega_{B'C'}) = 0, \quad (*)$$

  and such that $\omega_{A'B'} \omega^{A'B'} \neq 0$.

- $(*)$ is the (conformally invariant) twistor equation. Idea: prolong it, look for integrability conditions.
Drop symmetrisation: \[ \nabla_{AA'} \omega_{B'C'} - \varepsilon_{A'B'} K_{C'A} - \varepsilon_{A'C'} K_{B'A} = 0 \] for some \( K \in \Lambda^1(M) \).
Prolongation of $\nabla_A (A' \omega_{B'C'}) = 0$

- Drop symmetrisation: $\nabla_{AA'} \omega_{B'C'} - \varepsilon_{A'B'} K_{C'A} - \varepsilon_{A'C'} K_{B'A} = 0$ for some $K \in \Lambda^1(M)$.
- Differentiate and commute derivatives: $\psi^{E'_E} (A'B'C' \omega_{D'}) = 0$ and

\[
\nabla_{AA'} K_{BB'} + P_{ABA'C'} \omega_{B'}^C' - \varepsilon_{A'B'} \rho_{AB} = 0
\]

(where $P_{ab} = (1/2) R_{ab} - (1/12) R g_{ab}$) for some $\rho \in \Lambda^2_-(M)$. 

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- Differentiate and commute derivatives:

$$\nabla_{AA'}\rho_{BC} - \omega_{A'E'}\nabla_{E'}D\psi_{ABCD} + K_{A'D}\psi_{ABCD} - 2P_{A'E'}A(BK_C)^{E'} = 0.$$
Prolongation of $\nabla_A (A' \omega_{B'C'}) = 0$

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- Differentiate and commute derivatives: $\psi^{E'}_{(A'B'C'\omega_{D'})E'} = 0$ and

\[ \nabla_{AA'} K_{BB'} + P_{ABA'C'} \omega_{B'C''} - \varepsilon_{A'B'} \rho_{AB} = 0 \]

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- Now the system is closed: All derivatives of ‘unknowns’ have been determined.
Prolongation of $\nabla_A(A'\omega_{B'C'}) = 0$

- Drop symmetrisation: $\nabla_{AA'}\omega_{B'C'} - \varepsilon_{A'B'}K_{C'A} - \varepsilon_{A'C'}K_{B'A} = 0$ for some $K \in \Lambda^1(M)$.

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- Differentiate and commute derivatives:

$$\nabla_{AA'}\rho_{BC} - \omega_{A'E'}E' D \psi_{ABCD} + K_A D \psi_{ABCD} - 2P_A E' A (B K_C)^{E'} = 0.$$

- Now the system is closed: All derivatives of ‘unknowns’ have been determined.

- Geometric interpretation $\Psi = (\omega, K, \rho)$ is a section of a rank–10 vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$ which is parallel with respect to a connection $\mathcal{D}$ determined by the blue equations.
Example: Local vs. Global obstructions

- Compact hyperbolic four manifold \((M, g)\). Weyl = 0, \(R = -1\). All local obstructions vanish.
**Example: Local vs. Global obstructions**

- Compact hyperbolic four manifold \((M, g)\). Weyl = 0, \(R = -1\). All local obstructions vanish.
- Assume globally defined non-degenerate \(\omega_{A'B'}\) satisfies the twistor eq.
Compact hyperbolic four manifold \((M, g)\). Weyl\(= 0\), \(R = -1\). All local obstructions vanish.

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- More global obstructions in the ASD case (LeBrun). The only simply connected four–manifolds \(M\) that are allowed are \(K3\) and \(\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}, \ k \geq 10\).
**Generic case \( C_+ \neq 0 \)**

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- Differentiate $(*)$, impose the twistor equation.

**Theorem.** Let $(M, g)$ be a 4–manifold such that the self–dual part of the conformal curvature is non–zero. Then there exists a Kähler metric in $[g]$ if and only if $C_+$ is of type $D$ and

$$\nabla_A (A' \psi_{B'C'D'E'}) - V_A (A' \psi_{B'C'D'E'}) = 0, \quad \nabla_{[a} V_{b]} = 0,$$

where $V_{AA'} = \frac{1}{|\psi|^2} \left( \frac{1}{6} \nabla_{AA'} |\psi|^2 + \frac{4}{3} \psi^{B'C'D'E'} \nabla_{AB'} \psi_{C'D'E'A'} \right)$. 

Dunajski (DAMTP, Cambridge)
**Theorem.** Parallel sections $\Psi$ of $\mathcal{D}$ on a rank 10 vector bundle $E \to M$ correspond to Kähler metrics in a conformal class.

Integrability conditions:

$F \Psi = 0$

where

$F = [D, D]$

(there are some indices, but let’s not write them down).

If $F = 0$ then $g$ is conformally flat. Otherwise differentiate:

$(DF) \Psi = 0$, $(DDF) \Psi = 0$, ...

After $K$ steps $F^K \Psi = 0$, where $F^K$ is a matrix of linear blue eqn.

Stop when rank $(F^K) = \text{rank} (F^{K+1})$.

The space of parallel sections has dimension $(10 - \text{rank} (F^K))$.

**Theorem.** An anti-self-dual Einstein metric $g$ with $\Lambda \neq 0$ is conformal to a Kähler metric iff $g$ admits a Killing vector.

Examples of conf. classes with more than one Kähler metrics: Fubini-Study metric on $\mathbb{C}P^2$ with reversed orientation.

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If $\mathcal{F} = 0$ then $g$ is conformally flat. Otherwise differentiate:

$$(\mathcal{D}\mathcal{F})\Psi = 0, \quad (\mathcal{D}\mathcal{D}\mathcal{F})\Psi = 0, \ldots$$

After $K$ steps $\mathcal{F}^K\Psi = 0$, where $\mathcal{F}^K$ is a matrix of linear equations. Stop when rank $(\mathcal{F}^K) = \text{rank} (\mathcal{F}^{K+1})$.

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Summary and Outlook


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