Poisson-Lie T-Duality and supermanifolds

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Contents

2-dimensional σ-model

Field

$$\varphi : (\Sigma, \eta, \omega) \to (M, G, B)$$

with action

$$S_\varphi := \int \frac{1}{2}\left[\eta^{ab}(\varphi^* G)_{ab} + \omega^{ab}(\varphi^* B)_{ab}\right]\omega = \int \mathcal{L}\,\omega$$

$$\eta = d\tau \otimes d\tau - d\sigma \otimes d\sigma \quad \& \quad \omega = d\tau \wedge d\sigma$$

Euler & Lagrange equations:

$$S_\varphi \leq S_{\varphi + t\psi} \quad \Rightarrow \quad \frac{d}{dt} S_{\varphi + t\psi}(0) = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \varphi^\mu} - \partial_a \frac{\partial \mathcal{L}}{\partial\left(\partial_a \varphi^\mu\right)} = 0$$

If it is not possible to solve it. . . ?

PLT-Duality

1. Rewrite the Lagrangian in lightcone coordinates $x_{\pm} = \sigma \pm \tau$

$$\rightarrow \qquad \bar{\mathcal{L}} = \mathcal{F}_{\mu\nu} \partial_- \varphi^\mu \partial_+ \varphi^\nu \qquad \mathcal{F} = G + B$$

2. If there is an action of some Lie group $\mathcal{G}$ on M s.t. $M \approx \mathcal{G} \times M / \mathcal{G}$
3. Try to find a Drinfel'd double $\mathcal{D} = (\mathcal{G} \mid \widetilde{\mathcal{G}})$
   - it's Lie algebra $\mathfrak{d} = \mathfrak{g} + \widetilde{\mathfrak{g}}$
   - $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ maximally isotropic w.r.t. *ad*-invariant form $\langle \, | \, \rangle_{\mathfrak{d}}$
4. Check the condition

$$(\mathfrak{L}_{V_i} \mathcal{F})_{\mu\nu} = \mathcal{F}_{\mu\rho} V_j^\rho \widetilde{f}_i^{jk} V_k^\lambda \mathcal{F}_{\lambda\nu}$$

   - V_i left invariant fields on $\mathcal{G}$
   - $\widetilde{f}_i^{jk}$ structure constants of $\widetilde{\mathfrak{g}}$
5. Then we can construct the σ-model $\widetilde{\varphi}$ on $\widetilde{\mathcal{G}} \times M / \mathcal{G}$ and map the solution $\varphi \leftrightarrow$ the solution $\widetilde{\varphi}$

Are we able to “super” it?

What’s hidden behind the words?

- superfield
- supermanifold
- superspace
- superalgebra

And how to work with it?

References

- (F.) A. Rogers: A global theory of supermanifolds, J. Math. Phys. 21(6) (June 1980) 1352-1365
- (F.) A. Rogers: Super Lie groups: global topology and local structure, J. Math. Phys. 22(5) (May 1981) 939-945.
- (F.) A. Rogers: Supermanifolds: theory and apllications, World Scientific Publishing Co. Pte. Ltd., Singapore © 2007
- J. Cook & R. Fulp: Infinite dimensional super Lie groups, Differ.Geom.Appl.26:463-482,2008
- А. Ю. Хренников: Суперанализ, Физматлит, Москва 2005
  A. Yu. Khrennikov: Superanalysis, Kluwer Academic Publishers, 1999

Supernumbers

Grassmann algebra

$$\Lambda = \{\lambda = \sum_{k\in\mathbb{N}_0} \sum_{i_1<\ldots<i_k} \lambda_{i_1\ldots i_k} q_{i_1}\ldots q_{i_k} \mid \lambda_{i_1\ldots i_k} \in \mathbb{R}\}$$

with generators

$$1 \cup \{q_n\}_{n\in\mathbb{N}}, \quad \forall k, n \in \mathbb{N}\,(q_n q_k = -q_k q_n), \quad 1\, q_k = q_k\, 1$$

and a condition

$$\|\lambda\| := \sum_{k\in\mathbb{N}_0} \sum_{i_1<\ldots<i_k} |\lambda_{i_1\ldots i_k}| < +\infty$$

is $\mathbb{Z}_2$-graded Banach algebra

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \sim l^1(\mathbb{R}),$$

which is supercommutative $\quad (\alpha = \alpha_0 + \alpha_1)$

$$\alpha_i \beta_j = (-1)^{ij} \beta_j \alpha_i$$

Superspace

- Superspace

$$\mathbb{R}_{\Lambda}^{p|\pi} := \Lambda_0^p \times \Lambda_1^\pi$$

with a norm $\|x\| := \sum_{\mathbf{a}} \|x_{\mathbf{a}}\|$ is a Banach space.

- G-differentiability of a mapping $f : \mathbb{R}_{\Lambda}^{p|\pi} \to \Lambda$ at point x:

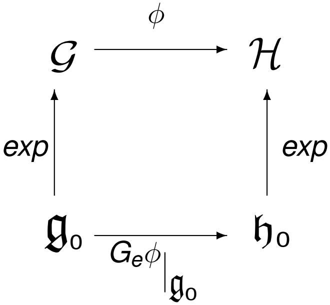
$$\forall \mathbf{a} \in \{1, ..., p+\pi\} \quad \exists G_{\mathbf{a}} f(x) \in \Lambda \quad \exists \omega : \mathbb{R}_{\Lambda}^{p|\pi} \to \Lambda$$

$$\begin{aligned} f(x+h) &= f(x) + h^{\mathbf{a}} G_{\mathbf{a}} f(x) + \|h\| \, \omega(h), \qquad \omega(0) = 0 \\ \text{Fréchet: } f(x+h) &= f(x) + f'(x) \cdot h + \|h\| \, \omega(h) \end{aligned}$$

$$f'(x) \cdot h = h^1 G_1 f(x) + ... + h^{p+\pi} G_{p+\pi} f(x)$$

Supermanifolds & supergroups

- Supermanifold $M \sim \mathbb{R}_{\Lambda}^{p|\pi}$
- $(G^\infty \Rightarrow C^\infty) \implies$ (Supermanifolds $\subseteq$ Banach manifolds)
  (Supergroups $\subseteq$ Banach Lie groups)



$N = (1,1)$ SUSY σ-model

$$\varphi : \mathbb{R}^{2|2}_{\Lambda} \to \mathcal{G}^{n|O}$$

$$\int d\theta_r\, \theta_s = \delta_{rs}, \quad \int d^2\theta = \int d\theta_- \int d\theta_+, \quad D_\pm = \pm i \frac{\partial}{\partial\theta_\mp} \pm \theta_\mp \partial_\pm$$

$$S_\varphi = \frac{1}{2i} \int d^2x d^2\theta \left[\mathbf{G}_{\mu\nu}(\varphi) \overline{D\varphi^\mu} D\phi^\nu + \mathbf{B}_{\mu\nu}(\varphi) \overline{D\varphi^\mu} (\gamma^5 D)\varphi^\nu \right]$$

- PLT-Duality

Target space is not purely even supermanifold

- $\varphi : \mathbb{R}_{\Lambda}^{2|0} \rightarrow \mathcal{G}^{p|\pi}$ & PLT-Duality

$$S_{\varphi} = \int \big[(\eta, \varphi^* G) - (\omega, \varphi^* B)\big]\omega$$

A. Eghbali & A. Rezaei-Aghdam:
Poisson-Lie T-dual sigma models on supermanifolds,
JHEP 0909:094, 2009

- Action

$$\mathcal{G} \times M \rightarrow M$$

$$M \approx \mathcal{G} \times M / \mathcal{G}$$

Thank you for your attention