Plasma expansion into vacuum and ion acceleration problem in terms of self-similar solution

M. Murakami
Institute of Laser Engineering, Osaka University, Japan

Contents of talk

- Introduction to self-similar analysis
- Self-similar solution with full account of charge separation
- Ion energy spectrum & comparison with experiment
- Maximum ion energy
- Application to Coulomb explosion
Introduction to self-similar analysis
Similarity between different scale systems
(1) Karman’s Eddy

\[
\frac{500\text{km}}{5\text{cm}} = 10^7
\]
Similarity between different scale systems

(2) Blast Wave

\[
\frac{100\text{m}}{1\text{cm}} = 10^4 \\
\frac{3\text{pc}}{100\text{m}} = 10^{15}
\]
Representative tools for finding self-similarity

The following two methods are systematic and powerful methods to find self-similarity of the system under consideration, which may work complementarily.

(1) Π-theorem

Even without knowing the system equations, one can introduce combined new variables by dimensional analysis, with which the number of the independent parameters are reduced, and the system can be significantly simplified.

(2) Lie group analysis

Given the differential equations system, one can find combined new variables by appropriate variable transformation, with which the number of the independent parameters are reduced, and the system can be significantly simplified. One can even find linearity indwelling in the nonlinear system.
If a pig is assumed to be spherical, Then the only parameter is the radius.

$M \propto R^3$

$V \propto R^3$

$S \propto R^2$

Try to simplify the system as much as possible, but be careful not to lose the essence.
In many cases, similarity solutions can be obtained by dimensional analysis. This is based on the fact that the equations of physics are homogeneous with respect to dimensions, and writing them in dimensionless form may lead to a reduction of the number of independent variables.

Suppose a physical quantity \( a \) as a function of other \( n \) quantities, \( a_1, a_2, \ldots, a_n \)

\[
a = F(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)
\]

have independent dimensions can be expressed dimensionally in terms of the variables of the 1st group
Dimensions denoted by [ ]:

\[ [a] = [a_1]^{q_1} [a_2]^{q_2} \cdots [a_k]^{q_k} \]

\[ [a_{k+j}] = [a_1]^{q_{1j}} [a_2]^{q_{2j}} \cdots [a_k]^{q_{kj}}, \quad (j = 1, 2, \ldots, n - k) \]

Dimensionless variables:

\[ \Pi = \frac{a}{a_1^{q_1} a_2^{q_2} \cdots a_k^{q_k}} \]

\[ \Pi_j = \frac{a_{k+j}}{a_1^{q_{1j}} a_2^{q_{2j}} \cdots a_k^{q_{kj}}}, \quad (j = 1, 2, \ldots, n - k) \]
Substitution $\Pi$ for $a$ and $\Pi_j$ for $a_{k+j}$:

$$\Pi = \frac{F\left(a_1, \ldots, a_k, \Pi_1, a_{q_{i1}}^1 \cdots a_{q_{k1}}^1, \ldots, \Pi_{n+k}, a_{q_{1_{n+k}}}^1 \cdots a_{q_{k_{n+k}}}^1\right)}{a_1^{q_1} \cdots a_k^{q_k}}$$

Here the right-hand-side must not depend on with independent dimensions, because the left-hand-side does not. Otherwise, invariance with respect to system of units would be violated. Therefore:

$$\Pi = \Phi(\Pi_1, \Pi_2, \ldots, \Pi_{n-k})$$  \hspace{1cm} (\Pi\text{-theorem})

$\Rightarrow$ for $k = n$, \hspace{0.5cm} $\Pi = \Phi_0$

$\Rightarrow$ for $k = n - 1$, \hspace{0.5cm} $\Pi = \Phi(\Pi_1)$ similarity solution
A typical example of \( \Pi \)-theorem

The number of parameters that characterize the system: \( n = 4 \)

Radius (cm) \( : [r] = L \)
Time (sec) \( : [t] = T \)
Density (g/cm\(^3\)) \( : [\rho_0] = ML^{-3} \)
Energy (erg) \( : [E] = ML^2T^{-2} \)

The number of dimensions of the system: \( T, M, L \ (k = 3) \)

The number of nondimensional parameters: \( n - k = 1 \)

\( \Pi \)-Theorem

\[
\Pi_1 = \left( \frac{\rho_0}{E} \right)^{1/5} \frac{r}{t^{2/5}}
\]
Hydrodynamic fields: $\rho(r, t), u(r, t), p(r, t)$

Similarity ansatz:

$$\begin{align*}
\rho &= \rho_0 G(\Pi_1) \\
\frac{r}{t} U(\Pi_1) \\
p &= \rho_0 \left(\frac{r}{t}\right)^2 P(\Pi_1)
\end{align*}$$

where $G(\Pi_1), U(\Pi_1), P(\Pi_1)$ are still unknown functions.

Inserting ansatz into hydro-equations (PDE’s)
leads to a set of ordinary equations (ODE’s).
Taylor-Sedov (-Von Neumann) Blast Wave

\[ r_f = \xi_0 \left( \frac{E t^2}{\rho_0} \right)^{1/5} \]
Lie Group Analysis - symmetry of differential equations -

Discovery of the symmetry

Introduction of a new combined variable

Partial differential system → Ordinary differential system
Higher order system → Lower order system
Nonlinear system → Linear system

Finding of self-similar solution

Marius Sophus Lie
Norwegian mathematician
End of 19th century
The system can be significantly simplified, if it is converted into a symmetric system under a specific variable transformation.

\[
F \left( x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \ldots \right) = 0
\]

\[
\begin{align*}
\hat{x} &= \hat{x}(x, y, z; \lambda) \\
\hat{y} &= \hat{y}(x, y, z; \lambda) \\
\hat{z} &= \hat{z}(x, y, z; \lambda)
\end{align*}
\]

\[
F \left( \hat{x}, \hat{y}, \hat{z}, \frac{\partial \hat{z}}{\partial \hat{x}}, \frac{\partial \hat{z}}{\partial \hat{y}}, \frac{\partial^2 \hat{z}}{\partial \hat{x}^2}, \frac{\partial^2 \hat{z}}{\partial \hat{x} \partial \hat{y}}, \frac{\partial^2 \hat{z}}{\partial \hat{y}^2}, \ldots \right) = 0
\]
Application to the linear heat conduction problem (stretching groups)

\[ T_t = T_{xx}, \quad T(x, 0) = \delta(x), \quad -\infty < x < \infty, \quad t \geq 0 \]

\[ \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \]
\[ \hat{x} = \lambda x \quad \hat{t} = \lambda^\alpha t \quad \hat{T} = \lambda^\beta T \]

\[ \frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial^2 \hat{T}}{\partial \hat{x}^2} \quad \text{invariant} \]

\[ \left\{ \begin{array}{c}
\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial (\lambda^\beta T)}{\partial (\lambda^\alpha t)} = \lambda^{-\alpha} \frac{\partial T}{\partial t} \\
\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} = \frac{\partial (\lambda^\beta T)}{[\partial (\lambda x)]^2} = \lambda^{\beta-2} \frac{\partial^2 T}{\partial x^2}
\end{array} \right. \]

1st eigenvalue \(\alpha = 2\)

\[ \begin{cases} 
\hat{x} = \lambda x \\
\hat{t} = \lambda^2 t 
\end{cases} \quad \Rightarrow \quad \frac{\hat{x}}{\sqrt{\hat{t}}} = \frac{x}{\sqrt{t}} \equiv \xi \]
Application to the linear heat conduction problem (cont.)

\[ \hat{x} = \lambda x \]
\[ \hat{t} = \lambda^\alpha t \]
\[ \hat{T} = \lambda^\beta T \]

\[ T(\lambda x, \lambda^\alpha t) = \lambda^\beta T(x, t) \]
\[ T(x, t) = t^{\beta/\alpha} \Phi \left( \frac{x}{t^{1/\beta}} \right) \]

2nd eigenvalue
\[ \int_{-\infty}^{\infty} T \text{d}x = \text{const} \Rightarrow \beta = -1 \Rightarrow T = t^{-1/2} \Phi(\xi) \]

Determination of the structure

2nd-order ODE
\[ 2 \Phi'' + \xi \Phi + \Phi = 0 \]

Solution
\[ T(x, t) = \frac{\exp(-x^2/4t)}{\sqrt{4\pi t}} \]

\[ T \]
\[ x \]

\[ t = t_0 \]
\[ t = 2t_0 \]
\[ t = 4t_0 \]
Simple planar (SP) self-similar solution of isothermal rarefaction wave

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0
\]
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{c_s^2}{\rho} \frac{\partial \rho}{\partial x}
\]

\[\xi = \frac{x}{c_s t}\]

\[\frac{v}{c_s} = \xi - 1\]

\[\frac{\rho}{\rho_0} = \exp(\xi - 1)\]

(Grevich 1966)
Comparison between the present model and the semi-infinite model
1. Self-similar solution of finite-mass & quasi-neutral plasma

1. Fluid equations for ions:

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \mathbf{v} = 0
\]
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{Ze}{m_i} \nabla \phi
\]

2. Boltzmann statistics for electrons:

\[
n_e(r, t) = Zn_i(r, t) = n_{ec}(t) \exp\left(\frac{e \phi(r)}{T_e}\right)
\]

3. Reduced partially differential equation system:

\[
\begin{cases}
\frac{\partial n}{\partial t} + \frac{1}{r^{v-1}} \frac{\partial}{\partial r} \left( r^{v-1} n_v \right) = 0 \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial r} = -\frac{c_s^2}{n} \frac{\partial n}{\partial r}
\end{cases}
\]

where \( c_s \equiv \sqrt{\frac{Z T_e}{m_i}} \) is the sound speed.

\[
v = \begin{cases} 
1 : \text{planar} \\
2 : \text{cylinder} \\
3 : \text{spherical}
\end{cases}
\]
4. Similarity ansatz:

\[ \xi = \frac{r}{R(t)}, \quad v = \dot{R}\xi, \quad n = n_0 \left( \frac{R_0}{R} \right)^\nu N(\xi) \]

5. Reduced ordinary differential equation:

\[ \frac{R\ddot{R}}{c_s^2} = -\frac{N'}{\xi N} = 2 \quad \text{Variable separated!} \]

6. Self-similar solution:

\[ N = \exp(-\xi^2), \quad \frac{e\phi}{T} = -\xi^2 \]

\[ \dot{R} = 2c_s \sqrt{\ln\left(\frac{R}{R_0}\right)} \]
Self-similar solution of finite-mass quasi neutral plasma expansion

(planar, cylinder, sphere)

\[ N = \exp(-\xi^2) \]

\[ V = \xi \]

![Graph showing density and velocity as functions of similarity coordinate \( \xi \).]
Temporal evolution of the system characteristic scale

\[ R(t)/R_0 \]

\[ v = 1, 2, 3: \text{ Isothermal case} \]

\[ R \propto t \]
Ion energy spectra obtained from quasi-neutral model

(1) Isothermal expansion

(a) Finite mass
\[ \frac{dN}{d\varepsilon} \propto \varepsilon^{(\nu-2)/2} \exp(-\varepsilon) = \begin{cases} \exp(-\varepsilon)/\sqrt{\varepsilon} & (\nu = 1) \text{ Plane} \\ \exp(-\varepsilon) & (\nu = 2) \text{ Cylinder} \\ \exp(-\varepsilon) \sqrt{\varepsilon} & (\nu = 3) \text{ Sphere} \end{cases} \]

(b) Half-infinitely-stretched plane
\[ \frac{dN}{d\varepsilon} \propto \exp(-\sqrt{\varepsilon})/\sqrt{\varepsilon} \]

(2) Adiabatic expansion (planar/initially isentropic distribution)

(a) Finite mass
\[ \frac{dN}{d\varepsilon} \propto (1 - \varepsilon)^{1/(\gamma-1)} \sqrt{\varepsilon} \]

(b) Half-infinitely-stretched plane
\[ \frac{dN}{d\varepsilon} \propto \left(1 - \frac{\gamma - 1}{2} (\sqrt{\varepsilon} - 1)\right)^{2/(\gamma-1)}/\sqrt{\varepsilon} \]

(3) Others
(a) Maxwellian
\[ \frac{dN}{d\varepsilon} \propto \exp(-\varepsilon) \sqrt{\varepsilon} \]
The analytical model excellently reproduces the experimental results on ion kinetic energy spectrum (planar geometry)

The analytical model excellently reproduces the experimental results on ion kinetic energy spectrum (spherical geometry)

At $t = 0$ the electron component of a finite plasma mass is heated to a uniform temperature $T_e(r, 0) = T_{e0}$. Hot electrons expand and create an ambipolar electric field $E(r, t)$, which drags the cold ions.

**Assumptions**

- There are no collisions between electrons and ions.
- At all times, the electron temperature is very quickly leveled off across the plasma volume: $T_e(r, t) = T_e(t)$. 

Governing equations

In the framework of two fluid approximation, expansion of the considered plasma is governed by the following system of equations,

\[
\begin{align*}
\frac{\partial n_i}{\partial t} + \frac{1}{r^{v-1}} \frac{\partial}{\partial r} \left( r^{v-1} n_i v_i \right) &= 0 \quad (1) \\
\frac{\partial n_e}{\partial t} + \frac{1}{r^{v-1}} \frac{\partial}{\partial r} \left( r^{v-1} n_e v_e \right) &= 0 \quad (2) \\
\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial r} &= -\frac{Ze}{m_i} \frac{\partial \Phi}{\partial r} \quad (3) \\
\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial r} &= \frac{e}{m_e} \frac{\partial \Phi}{\partial r} - \frac{T_e}{m_e n_e} \frac{\partial n_e}{\partial r} \quad (4) \\
\frac{1}{r^{v-1}} \frac{\partial}{\partial r} \left( r^{v-1} \frac{\partial \Phi}{\partial r} \right) &= 4\pi e (n_e - Z n_i) \quad (5)
\end{align*}
\]

where \( v \) stands for the geometrical index: \[
\begin{align*}
v &= 1, \quad \text{planar} \\
v &= 2, \quad \text{cylindrical} \\
v &= 3, \quad \text{spherical}
\end{align*}
\]
Similarity ansatz

The key assumption is that the velocity, \( v(r, t) \), is linear in radius. This is always correct for the asymptotic stage of free expansion of a finite mass.

\[
\xi = \frac{r}{R(t)}, \quad \dot{R} \equiv \frac{dR}{dt}
\]  

(6)

\[
v_e(r, t) = v_i(r, t) = \dot{R} \xi
\]

(7)

\[
n_e(r, t) = n_{e0} \left( \frac{R_0}{R(t)} \right)^v N_e(\xi), \quad N_e(0) = 1
\]

(8)

\[
Z n_i(r, t) = n_{i0} \left( \frac{R_0}{R(t)} \right)^v N_i(\xi), \quad N_i(0) \neq 1
\]

(9)

- Cold ions preserve a sharp edge at \( \xi = \xi_f \) (still unknown).

- Functions, \( v_i \) and \( N_i \), are then defined only for \( 0 \leq \xi \leq \xi_f \).
Coherency of the $R(t)$ and $\lambda_D(t)$ for self-similarity

- The present system contains two characteristic scale lengths, $R$ and $\lambda_D$

- In general, a self-similar solution cannot be found, if the considered system has two (or more) temporal characteristic scales.

- This can be interpreted in turn that one can find a self-similar solution if the system scale lengths coherently evolve in time, i.e.,

$$\Lambda = \frac{R(t)}{\lambda_D(0,t)} = \frac{R(0)}{\lambda_D(0,0)} = R \left( \frac{4\pi n_e e^2}{T_e} \right)^{1/2} = \text{const}$$

recalling mass conservation, $n_e R^\nu = \text{const}$

$$T_e(t) \propto R(t)^{2-\nu} \propto n_e^{1-2/\nu}$$
Self-similar solution

- Continuity equations (1) and (2) are automatically satisfied for any \( R(t), N_e(\xi) \) and \( N_i(\xi) \).
- The electron equation of motion (4) yields

\[
N_e = \exp(\phi - \mu_e \xi^2) \quad \text{for} \quad 0 < \xi < \infty \quad \text{where} \quad \begin{cases} 
\phi = e\Phi / T_e \\
\mu_e = Zm_e / m_i 
\end{cases}
\]

This is the well-known Boltzmann relation generalized to the case of \( \mu_e > 0 \).

- The ion equation of motion (3) yields

Spatial profile \( \phi(\xi) = -\xi^2 \quad \text{for} \quad \xi < \xi_f \)

Temporal evolution \( \dot{R}(t) = \begin{cases} 
2c_{s0}^2 \frac{t}{R_0}, & v = 1 \\
2c_{s0} \sqrt{\ln\left(\frac{R(t)}{R_0}\right)}, & v = 2 \\
2c_{s0} \sqrt{1 - R_0 / R(t)}, & v = 3 
\end{cases} \)

\[
\frac{c_{s0}}{m_i} = \sqrt{\frac{Z T_{e0}}{m_i}}
\]
The Poisson equation (5) is reduced to

\[ N_i = N_e + \frac{2v}{\Lambda^2} = \exp\left(-(1 + \mu_e)\xi^2\right) + \frac{2v}{\Lambda^2} \quad \text{for} \quad 0 \leq \xi \leq \xi_f \]

and a boundary-value problem

\[
\begin{cases}
\frac{1}{\xi^{v-1}} \frac{d}{d\xi} \left( \xi^{v-1} \frac{d\phi}{d\xi} \right) = \Lambda^2 \exp(\phi - \mu_e \xi^2) , & \text{for} \quad \xi_f < \xi < \infty \\
\phi(\xi_f) = -\xi_f^2 , \quad \frac{d\phi(\xi_f)}{d\xi} = -2\xi_f , \quad \lim_{\xi \to \infty} \xi^{v-1} \frac{d\phi}{d\xi} = 0
\end{cases}
\]

which allows us to calculate the position of the ion front \( \xi = \xi_f \).
Main results of the solution (1) Ion energy spectrum

\[
\frac{dN_i}{d\varepsilon_i} = \frac{\Lambda}{\varepsilon_0} \left( \frac{\varepsilon_i}{\varepsilon_0} \right) \left\{ \frac{2v}{\Lambda^2} + \exp \left[ -\left( 1 + \mu_e \right) \frac{\varepsilon_i}{\varepsilon_0} \right] \right\} \quad 0 \leq \varepsilon_i \leq \varepsilon_{i,\text{max}}
\]

\[
\frac{dN_i}{d\varepsilon_i} \propto \begin{cases} 
\tilde{\varepsilon}^{1/2}, & \Lambda \ll 1 \iff \text{Coulomb explosion} \\
\tilde{\varepsilon}^{1/2} \exp(-\tilde{\varepsilon}), & \Lambda \gg 1
\end{cases}
\]

Normalized ion kinetic energy \( \varepsilon_i / \varepsilon_0 \)

\( \Lambda_{ss} = 0.25, 0.5, 1, 2, 4, 8, 16, 32, 64, 128, 256 \)

[\( \mu_e^{-1} = 1000 \)]
Main results of the solution (2) Maximum ion energy

\[ \varepsilon_{i,\text{max}} = \varepsilon_0 \xi_f^2 \]

\[ \varepsilon_0 = \frac{1}{2} m_i v^2 = \begin{cases} 2ZT_e0 \ln \frac{R(t)}{R_0}, & v = 2 (\gamma = 1, \text{ isothermal}) \\ 2ZT_e0, & v = 3 (\gamma = 4/3, \text{ adiabatic}) \end{cases} \]

\( v_\infty \) is the bulk fluid velocity of the expanding plasma (can be calculated under the condition of quasi-neutrality)

\[ \xi_f^2 = \begin{cases} W \left( \pi^{1/3} \Lambda^{4/3} / 2\mu_e \right) / 2, & \Lambda << 1 \\ W(\Lambda^2 / 2), & \Lambda >> 1 \end{cases} \]

\( W(x) \) is the Lambert W-function defined by

\[ x = W \exp(W) \Rightarrow W = \begin{cases} x, & x \ll 1 \\ \ln(x / \ln x), & x \gg 1 \end{cases} \]
The maximum energy of accelerated ions is proportional to $\xi_f^2$, where $\xi_f = \xi_f(\Lambda, \mu_e, \nu)$ is the position of the ion front; $\xi_f^2$ may be called the acceleration factor.

\[
\begin{align*}
\mu_e^{-1} &= 10^5 \\
\mu_e^{-1} &= 2000 \\
\frac{\Lambda^2}{2}, & \quad \Lambda \gg 1, \quad \nu = 1, 2, 3 \\
\frac{1}{2} W \left( \frac{\pi^{1/3}}{2 \mu_e} \Lambda^{4/3} \right), & \quad \Lambda \ll 50, \quad \nu = 3
\end{align*}
\]
Generalization of the main result

We believe our result for $\varepsilon_{i,\text{max}}$ to be more general than the self-similar solution itself.

Argument:
For $\mu_e << 1$ and $\Lambda >> 1$ one can integrate the Poisson equation over the electron sheath to obtain the electric field at the ion front and the maximum ion energy:

Poisson equation: \[ \frac{\partial^2 \phi}{\partial x^2} = 4\pi e n_e \] \[ \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 4\pi e n_{e0} \exp \left( \frac{e\phi}{T_e} \right) \]

Boltzmann distribution: \[ n_e = n_{e0} \exp \left( \frac{e\phi}{T_e} \right) \]

Energy balance: \[ \frac{1}{2} E_f^2 = 4\pi n_{e\text{f}} T_e \]

\[ E_f = \sqrt{8\pi n_{e\text{f}} T_e} \propto n_{e\text{f}} \propto r_{e\text{f}}^{-\gamma/2} \Rightarrow \varepsilon_{i,\text{max}} = Z e \int_{r_{f0}}^{\infty} E_f \, dr_f \propto \int_{r_{f0}}^{\infty} r_f^{-\gamma/2} \, dr_f \]

The integral converges whenever $\gamma \nu > 2$, which means that all the acceleration occurs during the initial phase of expansion.
Generalized form of the maximum ion energy

For any $\gamma \geq 1$, the maximum ion energy in cylindrically and spherically expanding plasmas is given by

$$\varepsilon_{i,\text{max}} \approx \varepsilon_0 W(\Lambda_0^2 / 2)$$

where

$$\varepsilon_0 = \frac{1}{2} m_i v_\infty^2 = \begin{cases} 
2ZT_{e0} \ln \frac{R(t)}{R_0}, & \gamma = 1 \text{ (isothermal)} \\
2ZT_{e0} / \nu(\gamma - 1), & \gamma > 1 \text{ (adiabatic)}
\end{cases}$$

where $\Lambda_0 = \Lambda(t_0)$ is measured at the onset of the asymptotic phase of expansion.
The simple planar solution is the plane-parallel isothermal expansion of a plasma, initially occupying a half-space.

- Self-similar solution exists only in the framework of the quasi-neutral hydrodynamics.
- P.Mora (PRL, 2003) obtained an accurate numerical solution for $\mu_e = 0$.
- In the limit of $t \to \infty$ the ions at the front are accelerated to the infinite energy.

\[
\varepsilon_{i,\text{max}} = 2ZT_e^0 \left( \ln \frac{c_s t}{\lambda_D} \right)^{2} = 2ZT_e^0 \ln \frac{c_s t}{\lambda_D} \cdot \ln \frac{c_s t}{\lambda_D} \quad \downarrow \\
\frac{1}{2} m_i v_{\text{bulk}}^2
\]
**Conjecture**

As a final conclusion, we may conjecture that in all situations, where ions are accelerated due to the free plasma expansion, the maximum ion energy is given by

\[ \varepsilon_{i,\text{max}} = \varepsilon_0 G_{pl} \]

where \( \varepsilon_0 = \frac{1}{2} m_i v_{\text{bulk}}^2 \approx \begin{cases} 2ZT_{e0} \ln \frac{R(t)}{R_0}, & \text{isothermal} \\ 2ZT_{e0}, & \text{adiabatic} \end{cases} \)

and the factor, \( G_{pl} \) is a weak (logarithmic) function of the plasma parameters; it may be called a plasma logarithm (by analogy with the Coulomb logarithm). For \( D \gg \lambda_D \) we have

\[ G_{pl} \approx \ln \frac{D}{\lambda_D} = 5 - 30 \quad \text{where} \quad D = \begin{cases} \text{R0 = the plasma size, or} \\ \text{the laser focal-spot size, or} \\ \text{the distance to the ion detector.} \end{cases} \]

- **Our self-similar solution**
  \[ G_{pl} = W\left[ \frac{1}{2} \left( \frac{R_0}{\lambda_D} \right)^2 \right] \]

- **The SP solution**
  \[ G_{pl} = \ln \frac{c_s t}{\lambda_D} \]
Application to Coulomb explosion
The self-similar solution can describe plasma expansions of any size

\( \Lambda \sim 1 \)

\( \Lambda \sim 10^6 \)
Energy transfer efficiency from electrons to ions

Simulation
($\mu_e^{-1} = 100$)

- $n_e^0 = 10^{21}$ cm$^{-3}$
  - $R_{f0} = 10$ nm

- $n_e^0 = 10^{23}$ cm$^{-3}$
  - $R_{f0} = 2.2$ nm

Theory

- $\mu_e^{-1} = 100$
- $\mu_e^{-1} = 1000$
Summary

- A new self-similar solution is found to describe the plasma expansion into vacuum with full account of the charge separation.

- The experimental results have turned out to be excellently reproduced by the present solution.

- The maximum ion energy has been also obtained in a simple analytical form, for example for spherical geometry, $\varepsilon_{i,\text{max}} = 2ZT_e0 \ln(\Lambda^2 / 2)$
Thermodynamic property in terms of the adiabatic index $\gamma$

- $T \sim n^{1-\nu/2}$ defines the polytropic characteristic of the system in the form,

$$T \propto n_e^{\Gamma - 1} \quad \text{with} \quad \Gamma = \begin{cases} 
0, & \nu = 1 \\
1, & \nu = 2 \\
4/3, & \nu = 3 
\end{cases}$$

Since $\gamma > 1$,

(A) $\nu = 1$ (planar case)

The system is needed to be kept heated such that $T(t) \sim R(t)$ for any $\gamma$.

(B) $\nu = 2$ (cylindrical case)

The system is needed to be kept heated such that $T(t) \sim \text{const (isothermal)}$ for any $\gamma$.

(C) $\nu = 3$ (spherical case)

$$\begin{cases} 
\gamma > 4/3 & \text{cooling (mono – atomic nonrelativistic plasmas)} \\
\gamma = 4/3 & \text{adiabatic (relativistic electrons)} \\
\gamma < 4/3 & \text{heating (multi – ionized nonrelativistic plasmas)} 
\end{cases}$$