New Englert type solutions of eleven dimensional supergravity

IPM Workshop on Aspects of Integrable Systems and AdS/CFT
November 1, 2012

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Plan of Talk

- Eleven dimensional supergravity vs $N = 8$ supergravity in $d = 4$
- Field equations
- Squashed and the Englert solutions
- Pope-Warner solution and the Englert solution
- New Englert type solutions
- Symmetries

$d = 11$ supergravity vs $d = 4$ $N = 8$ supergravity

- In 1979, Cremmer and Julia constructed an $N = 8$ supergravity in 4 dimensions by dimensional reduction of the 11 dimensional supergravity on T^7.

- In 1981, de Witt and Nicolai observed that an $SO(8)$ symmetry of the reduced theory can be gauged. However, this also introduced a complicated scalar potentioal into the theory.

- The emergence of $AdS_4 \times S^7$ solution of 11d supergravity changed the sene. In fact, it did not take any longer to conjecture that the de Witt, Nicolai theory should be obtained if one expands the theory around this solution and compactifies it on S^7 and then truncates it to the massless sector.

$d = 11$ supergravity vs $d = 4$ $N = 8$ supergravity

- Further, it was conjectured that critical points of the de Witt, Nicolai potential should be in one-to-one correspondence with the 11d supergravity solutions of the type $AdS_4 \times S^7$.

- In particular, the $SO(8)$ symmetric vacuum corresponds to the Freund-Rubin solution of $AdS_4 \times S^7$, where F_4 has components only along AdS_4.

- Warner identified more critical points of N=8, d=4 supergravity with 11d supergravity solutions: The Englert solution, Pope-Warner solution .., where in all cases the $SO(8)$ symmetry is further broken by components of F_4 along S^7.

Field equations

The dynamical fields in eleven dimensional supergravity consists of

$$g_{MN}\,, \qquad A_{MNP}\,, \qquad \Psi_M$$

The Maxwell field equation for A_{MNP} reads

$$d *_{11} F_4 = -\frac{1}{2} F_4 \wedge F_4\,,$$

For g_{MN} we have the Einstein equations

$$R_{MN} = \frac{1}{12} F_{MPQR} F_N{}^{PQR} - \frac{1}{3 \cdot 48} g_{MN}\; F_{PQRS} F^{PQRS}\,,$$

where $M, N, P, \ldots = 0, 1, \ldots, 10$.

Freund-Rubin solution

Let us start with the Freund-Rubin solution. The 4-form field strength has components only along the four dimensions

$$F_4 = \frac{3}{8} R^3 \epsilon_4 \,, \qquad ds^2 = R^2 (\frac{1}{4} ds^2_{AdS_4} + ds^2_{S^7}) \,, \qquad R_{\mu\nu} = -12/R^2 \, g_{\mu\nu}.$$

The metric on S^7 can be written as an $SU(2)$ bundle over S^4

$$ds^2_{S^7} = \frac{1}{4} (d\mu^2 + \frac{1}{4} \sin^2 \mu \, \Sigma_i^2 + (\sigma_i - \cos^2 \mu/2 \; \Sigma_i)^2) \,,$$

Σ_i's and σ_i's are two sets of left-invariant one-forms

$$\Sigma_1 = \cos\gamma \, d\alpha + \sin\gamma \sin\alpha \, d\beta \,,$$
$$\Sigma_2 = -\sin\gamma \, d\alpha + \cos\gamma \sin\alpha \, d\beta \,,$$
$$\Sigma_3 = d\gamma + \cos\alpha \, d\beta \,,$$

and with a similar expression for σ_i's.

Squashed solution revisited

Squashing corresponds to modifying the round metric on S^7 as

$$ds^2_{S^7} = \frac{1}{4}(d\mu^2 + \frac{1}{4}\sin^2\mu\,\Sigma_i^2 + \lambda^2(\sigma_i - \cos^2\mu/2\,\Sigma_i)^2)\,,$$

with λ the squashing parameter. We take the following ansatz for the 11d metric:

$$ds^2 = \frac{R^2}{4}\left(ds_4^2 \,+\, d\mu^2 + \frac{1}{4}\sin^2\mu\,\Sigma_i^2 + \lambda^2(\sigma_i - \cos^2\mu/2\,\Sigma_i)^2\right)\,,$$

and choose the orthonormal basis of vielbeins as

$$e^0 = \,d\mu\,,\quad e^i = \frac{1}{2}\sin\mu\,\Sigma_i\,,\qquad \hat{e}^i = \lambda(\sigma_i - \cos^2\mu/2\,\Sigma_i)$$

Squashed solution revisited: Ansatz

Let us introduce ω_3, the volume element of the fiber S^3

$$\omega_3 = \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 \,,$$

taking the derivative along with using the Hodge dual we derive

$$d * d\omega_3 = 6\lambda^2 \, \omega_4 - \frac{1}{\lambda} \, d\omega_3 \,, \qquad \omega_4 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \,,$$

ω_4 is the volume element of the base, with $d\omega_4 = 0$. For a linear combination of these two forms we have

$$d * (\alpha \, \omega_4 + \beta \, d\omega_3) = 6\lambda^2 \beta \, \omega_4 + (\alpha - \beta/\lambda) \, d\omega_3$$

i.e., the subspace of ω_4 and $d\omega_3$ is closed under $d*$. This is exactly what we need to construct a consistent ansatz of F_4.

Squashed solution revisited: Ansatz

The above analysis shows that we can take the following ansatz:

$$F_4 = N\epsilon_4 + \alpha\,\omega_4 + \beta\,d\omega_3\,,$$

with N, α, and β constant parameters to be determined by field equations, also note $dF_4 = 0$. Substituting this into the Maxwell equation yields

$$6\lambda^2\,\beta = -\frac{8N}{R^3}\,\alpha\,, \qquad \alpha - \frac{\beta}{\lambda} = -\frac{8N}{R^3}\,\beta\,.$$

A nontrivial solution exists if

$$\lambda\left(\frac{8N}{R^3}\right)^2 - \frac{8N}{R^3} - 6\lambda^3 = 0\,.$$

Squashed solution revisited

With above ansatz, the RHS of the Einstein equations read:

$$R_{\mu\nu}=\left(\frac{4}{R^2}\right)^4\left(-\frac{3!}{12}\,N^2-\frac{4!}{3\cdot 48}(-N^2+\alpha^2+6\lambda^2\beta^2)\right)g_{\mu\nu}\,,$$

$$R_{\alpha\beta}=\left(\frac{4}{R^2}\right)^4\left(\frac{3!}{12}(\alpha^2+3\lambda^2\beta^2)-\frac{4!}{3\cdot 48}(-N^2+\alpha^2+6\lambda^2\beta^2)\right)\delta_{\alpha\beta}\,,$$

$$R_{\hat{\alpha}\hat{\beta}}=\left(\frac{4}{R^2}\right)^4\left(\frac{3!}{12}(4\,\lambda^2\beta^2)-\frac{4!}{3\cdot 48}(-N^2+\alpha^2+6\lambda^2\beta^2)\right)\delta_{\hat{\alpha}\hat{\beta}}\,,$$

with $\mu,\nu=0,\ldots,3,\ \alpha,\beta=4,\ldots 7,$ and $\hat{\alpha},\hat{\beta}=8,9,10$. For the LHS;

$$R_{\alpha\beta}=\left(\frac{4}{R^2}\right)\left(\frac{3(2-\lambda^2)}{2}\right)\,\delta_{\alpha\beta}\,,\qquad R_{\hat{\alpha}\hat{\beta}}=\left(\frac{4}{R^2}\right)\left(\frac{1+2\lambda^4}{2\lambda^2}\right)\,\delta_{\hat{\alpha}\hat{\beta}}\,,$$

these are to be substituted on the LHS of the Einstein equations.

Squashed solution revisited: solution

We can now solve for β, N, λ and α, and get two types of solutions. Those with no internal flux:

$$\alpha = \beta = 0\,, \qquad \lambda^2 = 1 \;\; \text{(round sphere)}\,, \qquad \lambda^2 = 1/5 \;\; \text{(squashed)}.$$

We also get solutions with fluxes:

$$\alpha^2 = 9/5\,, \quad \beta^2 = 9\,, \quad \lambda^2 = 1/5$$

For $\lambda = 1/\sqrt{5}$, $\alpha = -3/\sqrt{5}$, $\beta = 3$, and $N = 3R^3/(4\sqrt{5})$ which represents the squashed S^7 with Einstein metric

$$R_{\alpha\beta} = \left(\frac{4}{R^2}\right)\frac{27}{10}\,\delta_{\alpha\beta}\,, \qquad R_{\hat{\alpha}\hat{\beta}} = \left(\frac{4}{R^2}\right)\frac{27}{10}\,\delta_{\hat{\alpha}\hat{\beta}}\,. \tag{1}$$

This is the squashed solution with torsion obtained in 1980's using the covariantly constant spinors.

$\mathbf{CP}^3$ as an S^2 bundle over S^4

It is possible to write the S^7 metric as a $U(1)$ bundle over $\mathbf{CP}^3$. Further, $\mathbf{CP}^3$ itself can be written as an S^2 bundle over S^4:

$$
\begin{aligned}
ds^2_{S^7} &= d\mu^2 + \frac{1}{4}\sin^2\mu\,\Sigma_i^2 + \lambda^2(\sigma_i - \cos^2\mu/2\ \Sigma_i)^2 \\
&= \lambda^2(d\tau - A)^2 + d\mu^2 + \frac{1}{4}\sin^2\mu\,\Sigma_i^2 + \lambda^2(d\theta - \sin\phi A_1 + \cos\phi A_2)^2 \\
&+ \lambda^2\sin^2\theta\,(d\phi - \cot\theta(\cos\phi A_1 + \sin\phi A_2) + A_3)^2\,,
\end{aligned}
$$

where,

$$A_i = \cos^2\mu/2\ \Sigma_i\,,$$

and,

$$A = \cos\theta\,d\phi + \sin\theta(\cos\phi A_1 + \sin\phi A_2) + \cos\theta A_3\,.$$

Basis

In this new form of the metric, we can further rescale the $U(1)$ fibers:

$$
\begin{aligned}
ds^2_{S^7} &= d\mu^2 + \frac{1}{4}\sin^2\mu\,\Sigma_i^2 + \lambda^2(d\theta - \sin\phi A_1 + \cos\phi A_2)^2 \\
&+ \lambda^2 \sin^2\theta\,(d\phi - \cot\theta(\cos\phi A_1 + \sin\phi A_2) + A_3)^2 + \tilde{\lambda}^2(d\tau - A)^2\,,
\end{aligned}
$$

and choose the following basis

$$
\begin{aligned}
&e^0 = d\mu\,, \qquad e^i = \frac{1}{2}\sin\mu\,\Sigma_i\,, \\
&e^5 = \lambda(d\theta - \sin\phi A_1 + \cos\phi A_2)\,, \;\; e^7 = \tilde{\lambda}(d\tau - A)\,, \\
&e^6 = \lambda\sin\theta(d\phi - \cot\theta(\cos\phi A_1 + \sin\phi A_2) + A_3)\,.
\end{aligned}
$$

In this basis the Ricci tensor is diagonal and reads

$$
\begin{aligned}
&R_{00} = R_{11} = R_{22} = R_{33} = 3 - \lambda^2 - \tilde{\lambda}^2/2\,, \\
&R_{55} = R_{66} = \lambda^2 + 1/\lambda^2 - \tilde{\lambda}^2/2\lambda^4\,, \qquad R_{77} = \tilde{\lambda}^2 + \tilde{\lambda}^2/2\lambda^4\,.
\end{aligned}
$$

Ansatz

A natural 3-form to begin with is $\omega_3 = e^{567}$. We define

$$\begin{aligned} R_1 &= \sin\phi(e^{01}+e^{23}) - \cos\phi(e^{02}+e^{31})\,, \\ R_2 &= \cos\theta\cos\phi(e^{01}+e^{23}) + \cos\theta\sin\phi(e^{02}+e^{31}) - \sin\theta(e^{03}+e^{12})\,, \\ K &= \sin\theta\cos\phi(e^{01}+e^{23}) + \sin\theta\sin\phi(e^{02}+e^{31}) + \cos\theta(e^{03}+e^{12})\,. \end{aligned}$$

These three forms are orthogonal to each other, i.e.,

$$R_1 \wedge R_2 = K \wedge R_1 = K \wedge R_2 = 0\,.$$

Let us also define,

$$\mathrm{Re}\,\Omega = R_1 \wedge e^5 + R_2 \wedge e^6\,, \qquad \mathrm{Im}\,\Omega = R_1 \wedge e^6 - R_2 \wedge e^5\,.$$

Ansatz

We have three independent 4-forms ω_4, $e^7\wedge\mathrm{Im}\,\Omega$, and $e^{56}\wedge K$, which are closed, do not contract into each other, and are closed under $d*$ operation. So a suitable ansatz for F_4 is

$$F_4 = N\epsilon_4 + \alpha\,\omega_4 + \beta\,e^7\wedge\mathrm{Im}\,\Omega + \gamma\,K\wedge e^{56}\,,$$

for α, β, γ three real constants. Taking the Hodge dual we have

$$*_{11}F_4 = N\omega_3\wedge\omega_4 + \epsilon_4\wedge(\alpha\,\omega_3 - \beta\,\mathrm{Re}\,\Omega + \gamma\,K\wedge e^7)\,.$$

Reduced field equations

With this ansatz, we see that the Maxwell equations reduce to

$$\begin{aligned}
&-\alpha\lambda^2 + N\lambda\beta + \gamma = 0\,,\\
&\alpha\tilde{\lambda} + 2\beta/\lambda + (\tilde{\lambda}/\lambda^2 + N)\gamma = 0\,,\\
&N\alpha - 4\lambda\beta + 2\tilde{\lambda}\gamma = 0\,.
\end{aligned}$$

The Einstein equations read

$$\begin{aligned}
&3 - \lambda^2 - \frac{\tilde{\lambda}^2}{2} = \frac{1}{3}(\alpha^2 + \beta^2 + \frac{1}{2}\gamma^2 + \frac{1}{2}N^2)\,,\\
&\lambda^2 + \frac{1}{\lambda^2} - \frac{\tilde{\lambda}^2}{2\lambda^4} = \frac{1}{3}(-\frac{\alpha^2}{2} + \beta^2 + 2\gamma^2 + \frac{1}{2}N^2)\,,\\
&\tilde{\lambda}^2 + \frac{\tilde{\lambda}^2}{2\lambda^4} = \frac{1}{3}(-\frac{\alpha^2}{2} + 4\beta^2 - \gamma^2 + \frac{1}{2}N^2)\,.
\end{aligned}$$

Squashed solution

We can reduce the equations further and find solutions. Let us start by assuming

$$\lambda = \tilde{\lambda} \,,$$

then by the Einstein equations we must have $\beta^2 = \gamma^2$. Taking $\beta = -\gamma$ yields

$$\lambda = \tilde{\lambda} = 1/\sqrt{5} \,, \qquad N = -6/\sqrt{5} \,, \qquad \alpha^2 = \beta^2 = \gamma^2 = 9/5 \,,$$

which is again the squashed solution with torsion and the Ricci tensor

$$R_{\mu\nu} = -45/10 \, g_{\mu\nu} .$$

Englert type solution

For $\beta = \gamma$, we get

$$\lambda = \tilde{\lambda} = 1\,, \qquad N = -2\,, \qquad \alpha^2 = \beta^2 = \gamma^2 = 1\,,$$

this is an Englert type solution with $R_{\mu\nu} = -5/2\, g_{\mu\nu}$. This has the same four-dimensional Ricci tensor as the original solution found by Englert using parallelizing torsions on the 7-sphere, and later by Duff and Pope and Warner using Killing spinors. It was shown that the symmetry of the Englert solution is in fact $SO(7)$. However, as we will discuss the solution with

$$\alpha = -\beta = -\gamma = 1\,,$$

has an $SU(3) \times U(1)$ symmetry.

Pope-Warner solution

To derive the Pope-Warner solution let us begin by defining

$$\mathrm{Re}\, L = -R_1 \wedge e^5 + R_2 \wedge e^6 \,, \qquad \mathrm{Im}\, L = R_1 \wedge e^6 + R_2 \wedge e^5 \,.$$

Further, if we define

$$P = e^{-2i\tau} L \quad \Rightarrow \quad dP = 2/\tilde{\lambda} * P \,.$$

This implies that for the 4-form field strength we can take

$$F_4 = N \epsilon_4 + \eta\, e^7 \wedge (\sin 2\tau\, \mathrm{Re}\, L - \cos 2\tau\, \mathrm{Im}\, L) \,,$$

with η a real constant. The Maxwell and Einstein equations imply

$$N = -2/\tilde{\lambda} \,, \quad \lambda^2 = 1 \,, \quad \tilde{\lambda}^2 = 2 \,, \quad \eta^2 = 2 \,.$$

Pope Warner and the Englert solution

We can construct another consistent ansatz by taking a linear combination of Pope-Warner ansatz and the one introduced before. However, by this we get $T_{56} \neq 0$, unless we set $\beta = 0$. Let us then set

$$F_4 = N\epsilon_4 + \alpha\,\omega_4 + \gamma\, K{\wedge}e^{56} + \eta\, e^7{\wedge}\,(\sin 2\tau\, \mathrm{Re}\, L - \cos 2\tau\, \mathrm{Im}\, L)\,,$$

Maxwell eqs. then require

$$N = -2/\tilde{\lambda}\,, \qquad \lambda^2 = \tilde{\lambda}^2 = 1\,, \qquad \alpha = \gamma\,,$$

while, the Einstein equations imply: $\alpha^2 = \gamma^2 = \eta^2 = 1$.

This is the original Englert solution with $R_{\mu\nu} = -5/2\, g_{\mu\nu}$, and an $SO(7)$ symmetry. The two Englert solutions have the same metric but fluxes have different symmetries.

New Ansatz and solutions

Having defined the three independent forms R_1, R_2, and K, we note that there are still other options to take for F_4. Let us define

$$R = R_1 + iR_2\,, \qquad \hat{e} = e^5 + ie^6\,,$$
$$\tilde{R} = e^{i\tau} R\,, \qquad M = e^{i\tau}\,\hat{e}\wedge K\,,$$

so that we can show that

$$d(e^7\wedge\tilde{R}) \;=\; \frac{i}{\lambda}\,e^7\wedge M + \frac{\tilde{\lambda}}{\lambda^2}\,e^{56}\wedge\tilde{R} \;=\; \frac{1}{\lambda}*M + \frac{\tilde{\lambda}}{\lambda^2}*(e^7\wedge\tilde{R})\,,$$
$$dM \;=\; \frac{i}{\tilde{\lambda}}e^7\wedge M + \frac{1}{\lambda}\,e^{56}\wedge\tilde{R} \;=\; \frac{1}{\tilde{\lambda}}*M + \frac{1}{\lambda}*(e^7\wedge\tilde{R})\,.$$

Hence an ansatz for F_4 could be the following linear combination

$$\xi_1\, d(e^7\wedge\tilde{R}) + \xi_2\, dM\,.$$

New Ansatz and solutions

The bad news is that by the above ansatz we get nonzero components for T_{56}. What about taking a linear combination of all the ansatzs we have obtained so far, namely:

$$F_4 = N\epsilon_4 + \alpha\,\omega_4 + \beta\, e^7\wedge \mathrm{Im}\,\Omega + \gamma\, K\wedge e^{56} + \eta\, e^7\wedge(\sin 2\tau\,\mathrm{Re}\,L - \cos 2\tau\,\mathrm{Im}\,L)$$
$$-\quad \xi\, e^7\wedge(\sin\tau\, e^5 + \cos\tau\, e^6)\wedge K + \xi\, e^{56}\wedge(\cos\tau\, R_1 - \sin\tau\, R_2)\,,$$

Note that if $\alpha \neq 0$ then we must have $\lambda^2 = \tilde{\lambda}^2 = 1$, which is the round sphere. Further, for having $T_{76} = T_{75} = T_{56} = 0$, we need to set

$$\xi\,(\eta - \beta + \gamma) = 0\,, \qquad \xi^2 - 4\eta\beta = 0\,.$$

There is one more condition coming from Maxwell eqs.;

$$\gamma = \alpha + 2\beta\,.$$

New Ansatz and solutions

With the new ansatz, the Einstein equations read

$$\begin{aligned}
\frac{3}{2} &= \frac{1}{3}\,(\alpha^2+\beta^2+\frac{1}{2}\gamma^2+\eta^2+\xi^2)+\frac{2}{3}\,,\\
\frac{3}{2} &= \frac{1}{3}\,(-\frac{1}{2}\alpha^2+\beta^2+2\gamma^2+\eta^2+\frac{5}{2}\,\xi^2)+\frac{2}{3}\,,\\
\frac{3}{2} &= \frac{1}{3}\,(-\frac{1}{2}\alpha^2+4\beta^2-\gamma^2+4\eta^2+\xi^2)+\frac{2}{3}\,,
\end{aligned}$$

where we have set $\lambda^2 = \tilde{\lambda}^2 = 1$, and $N = -2$. Requiring a diagonal T_{MN}, and by using the Maxwell eqs. we see that they collapse to

$$\frac{3}{2} = \frac{1}{3}\alpha^2+\frac{1}{2}\alpha^2+\frac{2}{3}\,, \quad \text{and} \qquad \frac{3}{2} = -\frac{1}{6}\alpha^2+\alpha^2+\frac{2}{3}\,,$$

which fix $\alpha^2 = 1$.

New Ansatz and solutions

Setting $\alpha = 1$, we can fix all other constants in terms of, say, β:

$$\eta = -1 - \beta\,, \quad \gamma = 1 + 2\beta\,, \qquad \xi^2 = -4\beta(\beta + 1)\,,$$

so that we have a real solution whenever

$$-1 \leq \beta \leq 0\,.$$

As for the four-dimensional Ricci tensor we obtain

$$R_{\mu\nu} = -\frac{5}{2}\, g_{\mu\nu}\,,$$

which is identical to that of Englert solution.

New Ansatz and solutions

So we end up with a new set of Englert type solutions of $AdS_4 \times S^7$. The 4-form flux now has a moduli generalizing the known Englert solution. In particular, the Englert type solution with $SU(3) \times U(1)$ symmetry is obtained if

$$\alpha = -\beta = -\gamma = 1\,, \qquad \eta = \xi = 0\,,$$

whereas we obtain the original Englert solution with an $SO(7)$ symmetry by setting

$$\alpha = \gamma = -\eta = 1\,, \qquad \beta = \xi = 0\,.$$

Symmetries

The conventional symmetries are more visible if we write the metric of S^7 as a $U(1)$ bundle over $\mathbf{CP}^3$. Let Z^a, $a = 1, \ldots, 4$, indicate the complex coordinate of $\mathbf{C}^4$ in which S^7 of radius one is embedded:

$$Z^i = \frac{z^i e^{i\tau}}{(1 + \sum_k |z_k|^2)^{1/2}}, \qquad Z^4 = \frac{e^{i\tau}}{(1 + \sum_k |z_k|^2)^{1/2}}, \qquad i = 1, 2, 3$$

so that $\sum_a |Z^a|^2 = 1$. One can now see that e^7 and L are the pullback onto S^7 of the following forms in $\mathbf{C}^4$;

$$\tilde{e}^7 = -\frac{i}{2}(\bar{Z}^a dZ^a + Z^a d\bar{Z}^a), \qquad \tilde{L} = \frac{1}{6}\epsilon_{abcd}\, Z^a\, dZ^b \wedge dZ^c \wedge dZ^d\,.$$

Therefore, e^7 and L, and hence terms proportional to η are invariant under an $SU(4)^+$ under which Z^a transforms in the fundamental representation.

Symmetries

Similarly, we can see that $F = dA$, the Kähler form, is the pullback of

$$F^{+} = i dZ^{a} \wedge d\bar{Z}^{a} \,,$$

which is also invariant under $SU(4)^{+}$. The flux of Englert solution is the pullback of

$$F^{+} \wedge F^{+} + 2\, d\, \mathrm{Re}\tilde{\mathrm{P}} \,,$$

for this particular combination, the flux has an enhanced $SO(7)$ symmetry.

Symmetries

Note that there is a second $SU(4)^-$ under which $(Z^i, \bar{Z}^4)$ transforms in the fundamental representation, and leaves

$$F^- = i(dZ^i \wedge d\bar{Z}^i - dZ^4 \wedge d\bar{Z}^4)\,,$$

invariant. It is not difficult to see that the 4-form flux that we obtained first is proportional to the pullback of

$$(F^- \wedge F^- - F^+ \wedge F^+)/Z^4 \bar{Z}^4 - F^+ \wedge F^+\,,$$

onto S^7. This term, however, is invariant under an $SU(3) \times U(1)$ group (the common subgroup of $SU(4)^+$ and $SU(4)^-$), where Z^i's rotate under $SU(3)$, and $U(1)$ shifts them by a phase.

Symmetries

With a similar argument we can show that the new terms proportional to ξ in our ansatz are the pullback of 4-forms which are anti-self-dual and invariant under $SU(4)^-$. Hence, the largest group which leaves all terms invariant is the $SU(3)$ common subgroup of $SU(4)^+$ and $SU(4)^-$.
So, generically, our new solution has an $SU(3)$ symmetry, though, it gets enhanced to $SO(7)$ at

$$\xi = \beta = 0\,,$$

and to $SU(3) \times U(1)$ at

$$\eta = \xi = 0\,.$$