Recurrent construction of Baxter Q operator for $s\ell(n)$ Heisenberg chain

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The talk is based on the results obtained in collaboration with S. Derkachov, R. Kirschner, P. Valinevich, D. Chicherin

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The goal of the talk

The main goal of this talk is description of the recurrent (by n) construction of Baxter Q-operator for $s\ell(n)$-invariant Heisenberg spin chain.

To do it we present

- Recurrent construction of generators of $s\ell(n)$ in form of differential operators.
- Recurrent construction of the Lax operator for $s\ell(n)$-invariant Heisenberg spin chain.
- Construction the building blocks for particular R-operators and Baxter Q-operators.

The talk is based on the publications: *(1)* Chicherin, Derkachov, Karakhanyan, Kirschner, *Baxter operators for arbitrary spin I and II*, *Nucl.Phys.* **B854** [FS] (2012) 393432, 433-465; *(2)* Karakhanyan, Kirschner, *Jordan-Schwinger Representations and Factorized Yang-Baxter Operators*, *SIGMA* 6:029,2010; *(3)* Derkachov, Karakhanyan, Kirschner, Valinevich, *Iterative construction of $U_q(s\ell(n+1))$ representations and Lax matrix factorization*, *Lett.Math.Phys.* **85**(2007) 221-234. *(4)* Derkachov, Karakhanyan, Kirschner, *Yang-Baxter R operators and parameter permutations*, *Nucl.Phys.* **B785** (2007) 263-285. *(5)* Derkachov, Karakhanyan, Kirschner, *Baxter Q-operators of the XXZ chain and R-matrix factorization*, *Nucl.Phys.* **B738** (2006) 368-390.

Motivation

- It is generally believed that the key to strong coupling problem lies in integrability.
- The Yang-Baxter relation, Lax operator, Baxter Q-operator are the crucial objects of the theory of integrable models.
- Besides of the integrable structures discovered recently in Quantum Field Theory and the possible applications in Condensed Matter Physics, the study of Heisenberg Model has its own interest.

Yang-Baxter relation

We work in $s\ell(n)$-module $V = \mathbb{C}[x_{ik}]$ space of polynomials of $n(n-1)/2$, $i < k$ complex variables.

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v),$$

Defining relation for Universal $\mathbb{R}$-operator:

$$\check{\mathbb{R}}_{12}(u-v)L_1(u)L_2(v) = L_1(v)L_2(u)\check{\mathbb{R}}_{12}(u-v), \; \check{\mathbb{R}}_{12}(u-v) = \mathbb{P}_{12}\mathbb{R}_{12}(u-v). \quad (1)$$

R-operator acts on tensor product of two representation spaces $V_1 \otimes V_2$, $\mathbb{P}_{12}$ is permutation operator: $\mathbb{P}_{12}(a \times b) = (b \times a)$, $a \in V_1$, $b \in V_2$.
The Lax operator in fact depends on n parameters: spectral parameter u and $n-1$ quantum numbers ℓ_k, characterizing representation of $s\ell(n)$:

$$L = L(u_1, u_2, \ldots u_n), \quad u_k = u + \ell_k, \quad \sum_{k=1}^{n} \ell_k = 0.$$

The recurrent construction of $s\ell(n)$ generators

A particular differential representation for $g\ell(n)$ generators E_{ij}: $[E_{ij}; E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}$ is given by Jordan-Schwinger constriction:

$$E_{ij} = x_i \partial_j , \qquad i = 1, \dots n.$$

It provides transition $g\ell(n-1) \to g\ell(n)$:

$$E_{\alpha\beta} = x_\alpha \partial_\beta + \varepsilon_{\alpha\beta} , \quad E_{n\beta} = \partial_\beta , \tag{2}$$

$$E_{\alpha n} = x_\alpha \ell_n - \sum_{\beta=1}^{n-1} x_\beta (\varepsilon_{\alpha\beta} + x_\alpha \partial_\beta), \quad E_{nn} = \ell_n - \sum_{\beta=1}^{n-1} x_\beta \partial_\beta ,$$

here homogeneity constraint $\sum_{k=1}^{n} x_\beta \partial_\beta = \ell_n$ imposed and solved by $x_\alpha \to x_\alpha / x_n$. Taking in mind the recurrent procedure it is reasonable to change notations: $x_\alpha \to x_{\alpha n}$.

$g\ell(n)$ generators

Taking as generator of $g\ell(1)$ constant ℓ_1 one obtains for $g\ell(2)$:

$$\varepsilon_{11} = x_{12}\partial_{12} + \ell_1, \quad \varepsilon_{21} = \partial_{12},$$
$$\varepsilon_{12} = -x_{12}(x_{12}\partial_{12} + \ell_1 - \ell_2), \quad \varepsilon_{22} = \ell_2 - x_{12}\partial_{12}.$$

Then for $g\ell(3)$ one obtains:

$$\mathcal{E}_{\alpha\beta} = \varepsilon_{\alpha\beta} + x_{\alpha 3}\partial_{\beta 3}, \quad \mathcal{E}_{3\beta} = \partial_{\beta 3}, \quad \alpha, \beta = 1, 2,$$
$$\mathcal{E}_{\alpha 3} = x_{\alpha 3}\ell_3 - \sum_{\beta=1}^{2} x_{\beta 3}(\varepsilon_{\alpha\beta} + x_{\alpha 3}\partial_{\beta 3}), \quad \mathcal{E}_{33} = \ell_3 - x_{13}\partial_{13} - x_{23}\partial_{23},$$

etc.

The above procedure can be pulled to the Lax operator level. The general $g\ell(n)$ Lax operator can be constructed by means of fusion procedure using particular Jordan-Schwinger Lax operator $L^{JS}_{ij}(u) = {}_xL(u) = u\delta_{ij} + x_j\partial_i$ and $g\ell(n-1)$ Lax operator ${}_\varepsilon L_{jk}(u) = u\delta_{jk} + \varepsilon_{\delta\gamma}(e_{\gamma\delta})_{jk} + \mathcal{A}_\gamma(e_{n\gamma})_{jk}$, pulled to n-dimensional auxiliary space. The unknown last row of latter is determined by projection on symmetric subspace of $V_x \otimes V_\varepsilon$, which implies that $L(u)$ is linear by u:

$$L_{ik}(u+1) = {}_xL_{ij}(u+1)_\varepsilon L_{jk}(u), \qquad L_{ik}(1) = 0 \;\Rightarrow\; \mathcal{A}_\alpha = -\sum_{\beta=1}^{n-1} \frac{x_\beta}{x_n}\varepsilon_{\alpha\beta}. \quad (3)$$

So finally one obtains:

$$L_{ik}(u) = u\delta_{ik} + \varepsilon_{\delta\gamma}(e_{\gamma\delta})_{ik} + x_k\partial_i - \sum_{\beta=1}^{n-1} \frac{x_\beta}{x_n}\varepsilon_{\alpha\beta},$$

in agreement with (2).

Factorized form of the Lax operator

The x-dependence in both factors of (3) can be extracted in form of triangular matrices:

$$ {}_xL_{ij}(u+1) = (\delta_{ik} - \frac{x_\alpha}{x_n}(e_{n\alpha})_{ik})[u\delta_{kl} + (e_{nn})_{kl}(1 + x_a\partial_a) + (e_{\beta n})_{kl}x_n\partial_\beta](\delta_{lj} + \frac{x_\gamma}{x_n}(e_{n\gamma})_{lj}), $$

$$ {}_\varepsilon L_{ij}(u) = (\delta_{ik} - \frac{x_\alpha}{x_n}(e_{n\alpha})_{ik})[u\delta_{kl} + (e_{\alpha\beta})_{kl}\varepsilon_{\beta\alpha}](\delta_{lj} + \frac{x_\gamma}{x_n}(e_{n\gamma})_{lj}). \qquad (4) $$

As result the general Lax can be rewritten:

$$ L_{ij}(u+1) = u(\delta_{ik} - x_\alpha(e_{n\alpha})_{ik})[u\delta_{kl} + (e_{\gamma\beta})_{kl}\varepsilon_{\beta\gamma} + \ell_n(e_{nn})_{kl} + (e_{\beta n})_{kl}\partial_\beta](\delta_{lj} + x_\delta(e_{n\delta})_{lj}) $$

which allows to prove by induction the factorization property:

$$ L_{ij}(u) = X^{-1}_{ik}(u\delta_{kl} + D_{kl})X_{lj}, \qquad (5) $$

where $X_{ik} = x_{ki},\ i > k,\ X_{ii} = 1,\ X_{ik} = 0,\ i < k,$
$D_{kl} = \sum_{i=k}^{n} x_{ki}\partial_{il},\ k < l,\ D_{kk} = \ell_k,\ D_{kl} = 0,\ k > l.$

Factorization of the $\mathbb{R}$-operator

The Universal $\mathbb{R}$-operator can be represented in factorized form $\mathbb{R} = \mathcal{R}_1\mathcal{R}_2\ldots\mathcal{R}_n$, or permutation of two sets $(u_1, u_2, \ldots u_n)$ and $(v_1, v_2, \ldots v_n)$ can be can be achieved step by step (S.Derkachov, arXiv:Math/0503396):

$$\mathcal{R}_k L_1(u_1,\ldots u_k,\ldots u_n)L_1(v_1,\ldots v_k,\ldots u_v) = L_1(u_1,\ldots v_k,\ldots u_n)L_1(v_1,\ldots u_k,\ldots v_n)\mathcal{R}_k$$

It allows to atomize the procedure and to reduce $\mathbb{R}$- operator to combination of elementary permutation operators (S. Derkachov, D. Karakhanyan, R. Kirschner, Nucl.Phys.B785:263-285, 2007):

$$\begin{gathered}\mathbb{W}_k L(u_1,\ldots, u_k, u_{k+1},\ldots, u_n) = L(u_1,\ldots, u_{k+1}, u_k,\ldots u_n)\mathbb{W}_k, \quad \mathbb{W}_k = D_{k+1k}^{\ell_{k+1}-\ell_k}, \\ \mathbb{S}(u_1 - v_n)L_1(u_1,\ldots, u_n)L_2(v_1,\ldots, v_n) = L_1(v_n,\ldots, u_n)L_2(v_1,\ldots, u_1)\mathbb{S}(u_1 - v_n).\end{gathered} \tag{6}$$

Using these intertwining operators one can reproduce an arbitrary permutation of parameters $\{u_i, v_i\}_{i=1}^n$.

Intertwining operators

The simplest $s\ell(2)$-module $V_\ell = \mathbb{C}$ is given by the space of polynomials of single complex variable spanned by $\{x^n\}$ is irreducible at general complex ℓ. However for positive integer 2ℓ ($\ell_2 - \ell_1$ in terms of $g\ell(2)$) V_ℓ contains the invariant subspace $v_{2\ell} = ker\mathbb{W}_\ell$, $(\mathbb{W}_\ell = \partial_x^{2\ell})$: $V_\ell = v_{2\ell+1} \bigcup V_{1-\ell}$. Or in terms $g\ell(2)$:

$$V_{\ell_1\ell_2} \xrightarrow{\mathbb{W}_{\ell_2-\ell_1}} V_{\ell_2\ell_1}/v_{\ell_2-\ell_1}, \quad at \quad \ell_2 - \ell_1 \in \mathbb{N}.$$

The similar degeneracy takes place in general case of $g\ell(n)$ when $\ell_i - \ell_k \in \mathbb{N}$.

The intertwining operators satisfy:

$$\mathbb{W}_k \mathbb{W}_k = \mathbb{I},$$

$$\mathbb{W}_k \mathbb{W}_j = \mathbb{W}_j \mathbb{W}_k, \quad |k-j| > 1,$$

$$\mathbb{W}_k \mathbb{W}_{k+1} \mathbb{W}_k = \mathbb{W}_{k+1} \mathbb{W}_k \mathbb{W}_{k+1}.$$

Exchange operator $\mathbb{S}$

Operator $\mathbb{S}$, exchanging parameters u_1 and v_n is given by function $z^{u_1 - v_n}$ $z = (X^{-1}Y)_{n1}$, where according to (5): $L_1(u) = X^{-1}(u + {}_xD)X$ and $L_2(v) = Y^{-1}(v + {}_yD)Y$. Then one easily deduces from (5) that the column X^{-1}_{j1} and the row X_{ni} are eigenvectors of the Lax operator:

$$ {}_yL_{ij}(v) \cdot Y^{-1}_{j1} = v_1 Y^{-1}_{i1}, \qquad X_{ni} \cdot {}_xL_{ij}(u) = u_n X_{nj}. $$

Then one calculates:

$$ \mathbb{S}(L_1(u))_{ij}\mathbb{S}^{-1} = (X^{-1}(u - (u_1 - v_n)e_{11} + {}_xD)X)_{ij} + \frac{u_1 - v_n}{(X^{-1}Y)_{n1}} X^{-1}_{ni} Y_{j1}, $$

$$ \mathbb{S}(L_2(v))_{ij}\mathbb{S}^{-1} = (Y^{-1}(v + (u_1 - v_n)e_{nn} + {}_yD)Y)_{ij} - \frac{u_1 - v_n}{(X^{-1}Y)_{n1}} X^{-1}_{ni} Y_{j1}, $$

which proves (6) due to:

$$ (X^{-1}(u - (u_1 - v_n)e_{11} + {}_xD)X)_{ij} = {}_xL_{ij}(v_n, u_2 \ldots, u_n), $$

$$ (Y^{-1}(v + (u_1 - v_n)e_{nn} + {}_yD)Y)_{ij} = {}_yL_{ij}(v_1, u_2 \ldots, u_1). $$

Particular $\mathcal{R}$-operators

Now the particular $\mathcal{R}$-operator, permuting u_k and v_k can be represented using elementary operators (6) as follows:

$$\mathcal{R}_k(u_k - v_k) = (\mathbb{W}^x_{k-1}\ldots\mathbb{W}^x_1)(\mathbb{W}^y_k\ldots\mathbb{W}^y_{n-1})\mathbb{S}(\mathbb{W}^n_{n-1}\ldots\mathbb{W}^x_k)(\mathbb{W}^x_1\ldots\mathbb{W}^x_{k-1}).$$

It satisfies:

$$\mathcal{R}^{(k)}_{12}(0) = \mathbb{I},$$

$$\mathcal{R}^{(k)}_{12}(\lambda)\mathcal{R}^{(k)}_{12}(\mu) = \mathcal{R}^{(k)}_{12}(\lambda+\mu),$$

$$\mathcal{R}^{(k)}_{12}(\lambda)\mathcal{R}^{(k)}_{23}(\lambda+\mu)\mathcal{R}^{(k)}_{12}(\mu) = \mathcal{R}^{(k)}_{23}(\mu)\mathcal{R}^{(k)}_{12}(\lambda+\mu)\mathcal{R}^{(k)}_{23}(\lambda),$$

$$\mathcal{R}^{(k)}_{12}(\lambda)\mathcal{R}^{(j)}_{23}(\mu) = \mathcal{R}^{(j)}_{23}(\mu)\mathcal{R}^{(k)}_{12}(\mu),$$

Particular Q-operators

We define:

$$Q_k = tr_0(\mathcal{R}^{(k)}_{01} \dots \mathcal{R}_{0L}). \qquad (7)$$

Note that $\mathcal{R}^{(k)}_{0a}$ changes "spins" of the representation space:

$$\mathcal{R}^{(k)}_{12} : V_{\ell_1 \dots \ell_n} \otimes V_{\rho_1 \dots \rho_n} \to V_{\ell_1 \dots \ell_k + \xi_k \dots \ell_n} \otimes V_{\rho_1 \dots \rho_k - \xi_k \dots \rho_n}, \quad \xi_k = (u_k - v_k)/2.$$

In simplest case of $s\ell(2)$

$$\mathcal{R}^{(2)}_{12}(u_1, u_2 | v_2) = \frac{\Gamma(u_1 - u_2)}{\Gamma(u_1 - v_2)} \frac{\Gamma(x_{12}\partial_1 + u_1 - v_2)}{\Gamma(x_{12}\partial_1 + u_1 - u_2)},$$

$$\mathcal{R}^{(2)} \leftrightarrow \mathcal{R}^{(1)},\ 1 \leftrightarrow 2,\ u_1 \leftrightarrow v_2,\ u_2 \leftrightarrow v_1.$$

$$Q^{(2)}(u) = tr_{V_0}(\mathbb{P}_{10}\mathcal{R}^{(2)}_{10}(u_1, u_2|0) \dots \mathbb{P}_{L0}\mathcal{R}^{(2)}_{L0}(u_1, u_2|0)) = \qquad (8)$$

$$= \left(\frac{\Gamma(1 - \ell - u)}{\Gamma(-2\ell)} \right)^L tr_{V_0}(\mathbb{P}_{10} \frac{\Gamma(x_{01}\partial_0 - 2\ell)}{\Gamma(x_{01}\partial_0 + 1 - \ell - u)} \dots \mathbb{P}_{L0} \frac{\Gamma(x_{0L}\partial_0 - 2\ell)}{\Gamma(x_{0L}\partial_0 + 1 - \ell - u)})$$

Regularization

The trace over infinite-dimensional space does not converge and requires a regulator, which breaks $s\ell(n)$ to its diagonal subgroup:

$$\mathbb{R}_{12}(u) \to \mathbb{R}_{12}(u,q) = \mathbb{R}_{12}(u,q_1,\ldots q_{n-1}) = \mathbb{I}_1 \otimes (\prod_{k=1}^{n-1} q_k^{H_k})_2 \cdot \mathbb{R}_{12}(u),$$

where H_k are elements of Carthan subalgebra ($|q| < 1$), acting on second (auxiliary) space. It satisfies YBE:

$$\mathbb{R}_{12}(u-v)\mathbb{R}_{13}(u,q)\mathbb{R}_{23}(v,q) = \mathbb{R}_{23}(v,q)\mathbb{R}_{13}(u,q)\mathbb{R}_{12}(u),$$

and gives rise a family of commuting transfer matrices:

$$T_\rho(u,q) = tr_\rho(\mathbb{R}_{10}(u,q)\ldots\mathbb{R}_{L0}(u,q), \quad [T_{\rho_1}(u,q); T_{\rho_2}(v,q)] = 0.$$

The regularization is lifted by $q_k \to 1$.

General transfer matrices

The general transfer matrix $T(u) = tr_0(\mathbb{R}_{10}(u) \dots \mathbb{R}_{L0}(u))$ can be represented in factorized form:

$$T(u) = Q_1(u + \ell_1)\mathcal{P}^{-1} \dots \mathcal{P}^{-1} Q_n(u + \ell_n),$$

where $\mathcal{P}$ is cyclic shift operator.

This relation, as well as commutativity: $[Q_k(u); Q_j(v)] = 0$ is proved using commutative diagrams like defining relation (1) is used to prove YBE:

$$\mathbb{R}_{12}(u - v)\mathbb{R}_{13}(u - w)\mathbb{R}_{23}(v - w) = \mathbb{R}_{23}(v - w)\mathbb{R}_{13}(u - w)\mathbb{R}_{12}(u - v),$$

by transformation product $L_1(u)L_2(v)L_3(w)$ to $L_3(w)L_2(v)L_1(u)$ in two possible ways.

Transfer matrices for reducible modules

As is already mentioned above at $\ell_k - \ell_j \in \mathbb{N}$ the $s\ell(n)$-module V have at least one invariant subspace $v \subset V$ and the R-operator, acting on tensor product $V' \otimes V$ has a triangular form:

$$R(u) = \begin{pmatrix} r(u) & * \\ 0 & \bar{R}(u) \end{pmatrix},$$

here $\bar{R}(u)$ acts on $V' \otimes \bar{V}$, $\bar{V} = V/v$.
As result one obtains the relation:

$$T(u) = \bar{T}(u) + t(u),$$

where $t(u) = tr(r_1(u) \dots r_L(u))$ is transfer matrix with trace over finite-dimensional auxiliary space.

Singularities at integer points

The expressions for universal $\mathbb{R}$-matrix, Q-operators and general transfer-matrices well-defined at general complex values of representation parameters become singular at integer points corresponding to appearance of finite-dimensional invariant subspaces.
It can be seen from (8) that operator Q_2 is ill-defined at positive half-integer ℓ.
The particular $\mathcal{R}$-operators map the infinite-dimensional representation space into finite-dimensional invariant subspace (degenerate into projection operators) or become singular at integer points. However all physically comprehensive objects: R-matrix, Lax operator etc. leave representation spaces unchanged, have regular limit at mentioned points and turn to their well-known expressions.

A similar approach to construction of Q-operator, based on DST (degenerate self-trapping) Sklyanin's model (E.K. Sklyanin (Backlund transformations and Baxters Q-operator)),nlin/0009009, was developed by Bazhanov and Staudacher A Shortcut to the Q-Operator, J. Stat. Mech.1011 (2010) P11002.

The simplest DST-model involves two 2-dim. Lax operators:

$$L^{+}(u)=\left(\begin{array}{cc} u+\partial x & -\partial \\ -x & 1 \end{array}\right), \qquad L^{-}(u)=\left(\begin{array}{cc} 1 & -\partial \\ x & u-x\partial \end{array}\right),$$

correspond to special limiting form of $s\ell(2)$ Lax operator $L(u_1, u_2)$ at $u_{1,2} \longrightarrow \infty$.

In $g\ell(n)$ case one has n solutions of $R(u-v)L(u)L(v)=L(v)L(u)R(u-v)$ with $R(u)=uI+P$, in form of $n\times n$ matrices $L(u)$ with spectral parameter entering in form ue_{ii}.

Summary and discussion

The universal $\mathbb{R}$-operator for $s\ell(n)$-symmetric Heisenberg chain can be constructed by means of recurrent procedure by n.

- The parameters, characterizing representation V of $s\ell(n)$ are combined with spectral parameter u, which is similar to n-th parameter, completing $s\ell(n)$ to $g\ell(n)$.
- The action of $\check{\mathbb{R}}$-operator: $\check{\mathbb{R}} L_1(u_1, \dots u_n) L_2(v_1, \dots v_n) = L_1(v_1, \dots v_n) L_2(u_1, \dots u_n) \check{\mathbb{R}}$ can be made in n steps: $\check{\mathbb{R}} = \mathcal{R}_1 \dots \mathcal{R}_n$.
- Moreover, each operator $\mathcal{R}_i$ can be represented as a product elementary permutation operators $\mathbb{W}_k(\lambda)$ and $\mathbb{S}(\lambda)$.
- The particular $mathcalR_k$-operators are building blocks for particular Q_k-operators at general values of representation parameters.
- The action of particular Q_k-operators is presented explicit form.