Lesson One

♠ Symmetries in Quantum Field Theory ♠

A Theoretical Excursus

Néda Sadooghi (SUT and IPM) Tehran-Iran

September 2005

Content

- Symmetries and Conservation Laws; Noether Theorem
- Spontaneous Symmetry breaking; Goldstone Theorem
- Spontaneously Broken Approximate Symmetries; Vacuum Alignment
- Hypothesis of Spontaneous Chiral Symmetry Breaking in QCD

Lesson Two

♠ QCD at High Temperature and Finite Density ♠

A Phenomenological Excursus

Néda Sadooghi (SUT and IPM) Tehran-Iran

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Content

- QCD at Zero Temperature and Density
- QCD under Extreme Conditions; Motivation
- QCD Phase Diagram
  - ◇ Phases of QCD at high temperature and zero chemical potential
  - ◇ Lattice QCD in 3 minutes
  - ◇ Color deconfinement and chiral symmetry restoration from lattice
  - ◇ Phases of QCD at high temperature and finite chemical potential
- What about Experiments? (SPS, RHIC, LHC, VLHC-1, VLHC-2)

Lesson One: Symmetries in QFT

(I) Symmetries and Conservation Laws; Noether Theorem

In classical Mechanics

$$L = L(q_i, \dot{q}_i; t), \qquad \text{EoM:} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

If q_j is a cyclic coordinate

$$\frac{\partial L}{\partial q_j} = 0 \Longrightarrow \dot{p}_j = 0, \qquad \text{with} \qquad p_j \equiv \frac{\partial L}{\partial \dot{q}_j} \qquad \Longrightarrow p_j = \text{const.}$$

Example:

Translational invariance $\frac{\partial L}{\partial x_i} = 0 \Longrightarrow p_i$ is constant of motion

Rotational invariance $\frac{\partial L}{\partial \theta_i} = 0 \Longrightarrow L_{\theta_i}$ is constant of motion

Noether Theorem for global and continuous symmetries

- Space-time transformation $\longrightarrow$ Lorentz transformation
- Internal symmetries ♠

Example 1: Internal Global Phase Transformation

The Lagrangian density

$$\mathcal{L} = \partial_\mu \varphi^\star \partial^\mu \varphi - m^2 \varphi^\star \varphi$$

invariant under global $U(1)$ transformation

$$\varphi \longrightarrow e^{+i\alpha}\varphi, \qquad \varphi^\star \longrightarrow e^{-i\alpha}\varphi^\star$$

The corresponding conserved current

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)} \Delta\varphi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi^\star\right)} \Delta\varphi^\star$$

with

$$\Delta\varphi = +i\varphi, \qquad \text{and} \qquad \Delta\varphi^\star = -i\varphi^\star$$

Conserved Current

$$j^\mu = i\left(\varphi\partial^\mu\varphi^\star - \varphi^\star\partial^\mu\varphi\right), \qquad \text{with} \qquad \partial_\mu j^\mu = 0$$

leads to a constant charge corresponding to the global U(1) transformation

$$0 = \int_V d^3x\, \partial_\mu j^\mu = \partial_0 \int d^3x\, j^0 + \text{surface term.}$$

Defining

$$Q \equiv \int d^3x\, j^0, \qquad \text{we will have} \qquad \dot{Q} = 0 \qquad \Longrightarrow Q = \text{const.}$$

Example 2: Internal Local Symmetries

Under local $U(1)$ gauge transformation

$$\psi \to e^{+i\alpha(x)}\psi, \qquad \bar{\psi} \to e^{-i\alpha(x)}\bar{\psi}$$

the QED Lagrangian density

$$\mathcal{L} = \bar{\psi}(\, i\partial\!\!\!/\,)\psi - m\bar{\psi}\psi$$

is invariant only when we introduce a gauge field by minimal coupling

$$\partial_\mu \to D_\mu = \partial_\mu + ig\,A_\mu, \qquad \text{with} \qquad A_\mu \to A_\mu - \frac{1}{g}\partial_\mu\alpha(x)$$

The modified Lagrangian is

$$\mathcal{L} = \bar{\psi}(\, iD\!\!\!\!/\,)\psi - m\bar{\psi}\psi$$

Example 3: QCD and Symmetries

Elementary degrees of freedom are

- Fermionic fields: $\psi^a_{\alpha,f}$ Quarks in fundamental repr.
- Gauge fields: A^a_μ Gluons in adjoint repr.

with the indices

- $\alpha = 1, \cdots 4$ labels the spinor indices
- $f = 1, \cdots N_f$ labels the quark flavors (u, d, s, c, b, t)
- a labels the color indices

  in fundamental repr. for fermions and in adjoint repr. for gluons
- μ labels the space-time indices

$N_f = 2$ case:

- (u, d) light quarks (approximately massless) ♠
- (s, c, b, t) heavy quarks

$$\mathcal{L} = \bar{\Psi}^a \, (\, i\not{D} \,) \, \Psi^a - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}, \qquad \Psi^a = \begin{pmatrix} \psi^a_1 \\ \psi^a_2 \end{pmatrix}$$

with $$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

Lagrangian density

$$\mathcal{L} = \bar{\Psi}^a \left(\, i \not{D} \right) \Psi^a - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

is invariant under

Global Vector Symmetries $SU_V(2) \times U_V(1)$:

$$\Psi \to \exp\left(-i\alpha^a \tau^a/2\right)\Psi, \qquad \bar{\Psi} \to \bar{\Psi} \exp\left(+i\alpha^a \tau^a/2\right)$$

τ^a, $a = 0, 1, 2, 3$ with $\tau^0 = \mathbf{1}$, and $\vec{\tau} = \vec{\sigma}$ are the Pauli matrices

Conserved global Noether vector currents

$$j^a_\mu = \bar{\Psi} \, \gamma_\mu \frac{\tau^a}{2} \Psi, \qquad a = 0, 1, 2, 3$$

Conserved charges

$$Q^a = \int d^3x \; j^a_0(x) = \int d^3x \; \bar{\Psi} \; \gamma_0 \frac{\tau^a}{2} \Psi$$

- For $a = 0$ $\quad \to Q^0$ is the $U_V(1)$ baryon charge
- For $a = 1, 2, 3 \to Q^a$ is the $SU_V(N_f = 2)$ isospin charge

Lagrangian density

$$\mathcal{L} = \bar{\Psi}^a \left(i \not{D} \right) \Psi^a - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

is invariant under

Global Axial Vector Symmetries $SU_A(2) \times U_A(1)$:

$$\Psi \rightarrow \exp\left(-i\gamma_5 \alpha^a \tau^a/2\right) \Psi, \qquad \bar{\Psi} \rightarrow \bar{\Psi} \exp\left(-i\gamma_5 \alpha^a \tau^a/2\right)$$

Conserved global Noether axial vector currents

$$j^a_{\mu,5} = \bar{\Psi}\, \gamma_\mu \gamma_5 \frac{\tau^a}{2} \Psi, \qquad a = 0, 1, 2, 3$$

Conserved charges

$$Q^a_5 = \int d^3x\, j^a_{0,5}(x) = \int d^3x\, \bar{\Psi}\, \gamma_0 \gamma_5 \frac{\tau^a}{2} \Psi$$

Canonical Commutation Relations:

Canonical ETC relations of the quantum fields $\Longrightarrow$ Charge algebra

$$[Q^a, Q^b] = [Q^a_5, Q^b_5] = i\varepsilon^{abc} Q^c, \qquad \text{and} \qquad [Q^a_5, Q^b] = i\varepsilon^{abc} Q^c_5 \qquad \text{for} \qquad a = 1, 2, 3$$

Chiral Representation:

Aim:

$$SU_V(2) \times SU_A(2) \times U_V(1) \times U_A(1) \longrightarrow SU_L(2) \times SU_R(2) \times U_V(1) \times U_A(1)$$

To do this, define

$$\Psi_{L,R} \equiv P_{L,R}\Psi, \qquad P_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5)$$

The Lagrangian density (in the chiral limit $m_u = m_d = 0$)

$$\mathcal{L} = \bar{\Psi}_L\,(\,i\not{D}\,)\,\Psi_L + \bar{\Psi}_R\,(\,i\not{D}\,)\,\Psi_R$$

which is invariant under $SU_L(2) \times SU_R(2)$ transformation

$$\psi_{L,R} \to U_{L,R}\psi_{L,R}, \qquad \text{with} \qquad U_{L,R} = e^{-i\alpha^a_{L,R}T^a_{L,R}}, \qquad T^a_{L,R} = T^a \otimes P_{L,R}$$

Further define

$$Q^a_{L,R} \equiv \frac{1}{2}(Q^a \mp Q^a_5)\,, \qquad a = 1,2,3$$

with the new chiral charge algebra

$$[Q^a_L, Q^b_L] = i\varepsilon^{abc}Q^c_L, \qquad [Q^a_R, Q^b_R] = i\varepsilon^{abc}Q^c_R, \qquad [Q^a_R, Q^b_L] = 0, \qquad \text{for} \qquad a = 1,2,3$$

(II) Spontaneous Symmetry Breaking (SSB) of Continuous Global Symmetries

Various Symmetry Breaking Mechanisms:

- Spontaneous symmetry breaking: ♠
  Symmetry of the Lagrangian is not shared with the ground state solution
- Anomalous symmetry breaking:
  $\mathcal{L}$ is invariant under certain sym. transformation $\Longrightarrow$
  Classical conservation law $\partial_\mu j^\mu = 0$, but $\langle \partial_\mu j^\mu \rangle = \mathcal{A} \neq 0$
  $\mathcal{A}$ is the so called Quantum Anomaly

Example 1: Abelian $U(1)$ global symmetry

$$\mathcal{L} = \partial_\mu \varphi^\star \, \partial^\mu \varphi - V\left(\varphi, \varphi^\star\right) \qquad \text{with} \qquad V\left(\varphi, \varphi^\star\right) \equiv m^2 \varphi^\star \varphi + \lambda \left(\varphi^\star \varphi\right)^2$$

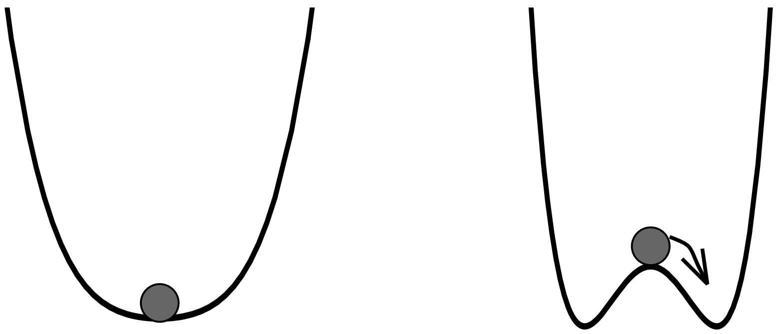
$\mathcal{L}$ is invariant under global $U(1)$ transformation:

$$\varphi(x) \to e^{i\alpha} \varphi(x)$$

Find the Ground state by minimizing the potential

(a) $m^2 > 0$: $\quad \varphi = \varphi^\star = 0$

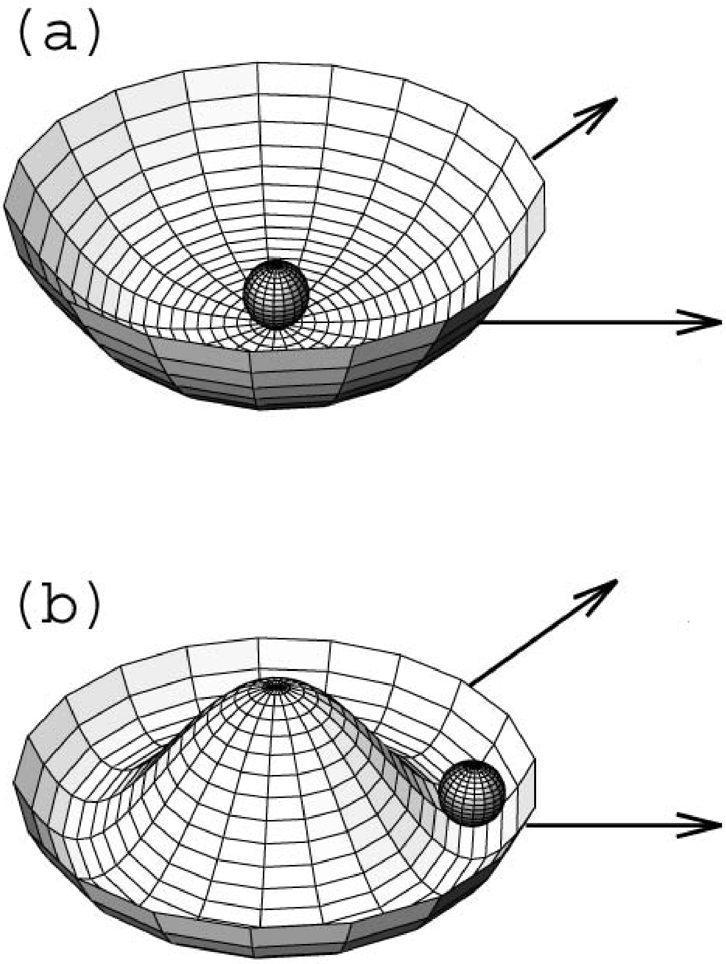
(b) $m^2 < 0$: $\quad |\varphi|^2 = -\frac{m^2}{2\lambda} \equiv a^2$



(a) (b)

Minima of $V(\varphi)$ lie along the circle $|\varphi| = a$ forming a set of degenerate vacua

$$|\langle 0|\varphi|0\rangle|^2 = a^2$$



Polar Coordinates

$$\varphi(x) = \rho(x) e^{i\theta(x)}$$

Then $\langle\varphi\rangle = a$ leads to

$$\langle 0|\rho(x)|0\rangle = a \qquad \text{and} \qquad \langle 0|\vartheta(x)|0\rangle = 0$$

- Choose only one vacuum among the set of degenerate vacua

- Expand $\mathcal{L}$ around $\rho' = \rho - a$ with

$$\langle 0|\rho'(x)|0\rangle = 0 \qquad \text{and} \qquad \langle 0|\vartheta(x)|0\rangle = 0$$

We get

$$\mathcal{L} = \partial_\mu \rho' \partial^\mu \rho' + (\rho' + a)^2 \ \partial_\mu \theta \partial^\mu \theta - \lambda \rho'^4 - 4a\lambda\rho'^3 - 4\lambda a^2 \rho'^2 + \lambda a^4$$

There is no mass term for $\vartheta(x)$

φ and $\varphi^\star$ are massive $\quad \xrightarrow{SSB}$

ϑ is the massless Goldstone boson and ρ' is massive with $m^2_{\rho'} = 4\lambda a^2$

Example 2: Non-Abelian $SO(3)$ global symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{1}{2}m^2 \varphi_i \varphi_i - \lambda \left(\varphi_i \varphi_i\right)^2 , \qquad i = 1, 2, 3$$

$\mathcal{L}$ is invariant under global $SO(3)$ transformation:

$$G = SO(3) : \qquad \varphi_i \to \left(e^{-iT_k \alpha_k}\right)_{ij} \varphi_j = \Big[U(g)\varphi\Big]_i$$

Find the Ground state by minimizing the potential

(a) $m^2 > 0$: $\qquad |\vec{\varphi}| = 0$

(b) $m^2 < 0$: $\qquad |\vec{\varphi}_0|^2 \equiv \langle \sum_{i=1}^{3} \varphi_i^2 \rangle = -\frac{m^2}{4\lambda} \equiv |\vec{a}|^2$

There are infinitely many degenerate vacua

We choose only one of them

$$\vec{\varphi}_0 \equiv \langle \vec{\varphi} \rangle = a\hat{e}_3,$$

and break the symmetry spontaneously

- It exists $g \in G$ under which $\vec{\varphi'}_0 = U(g)\vec{\varphi}_0 \neq \vec{\varphi}_0$ or

$$\vec{\varphi'}_0 = U(h)\vec{\varphi}_0 = \vec{\varphi}_0 \qquad \forall h \in H \subset G$$

- But the potential is invariant under the whole group G

$$V(\vec{\varphi'}) = V(\vec{\varphi}), \qquad \vec{\varphi'} = U(g)\vec{\varphi}, \qquad \forall g \in G$$

The Number of Goldstone Bosons

$\mathcal{L}$ can be evaluated around the new vacuum $\vec{\varphi} = \vec{\varphi}_0 + \vec{\chi}$

$$\begin{aligned} V &= \frac{m^2}{2}\left(\varphi_1^2 + \varphi_2^2 + (\chi + a)^2\right) + \lambda\left(\varphi_1^2 + \varphi_2^2 + (\chi + a)^2\right)^2 \\ &= 4\lambda a^2 \chi^2 + 4a\lambda\chi\left(\varphi_1^2 + \varphi_2^2 + \chi^2\right) + \lambda\left(\varphi_1^2 + \varphi_2^2 + \chi^2\right)^2 - \lambda a^4 \end{aligned}$$

Therefore

$$\begin{cases} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{cases} \quad \text{with} \quad m_{\varphi_i}^2 = -m^2, \quad \xrightarrow{SSB} \begin{cases} \varphi_1 & \text{Goldstone boson} & m_{\varphi_1} = 0, \\ \varphi_2 & \text{Goldstone boson} & m_{\varphi_2} = 0, \\ \chi & \text{massive} & m_\chi^2 = 8\lambda a^2 \end{cases}$$

Goldstone Theorem: The Classical Proof

STEP 1: Expanding $V(\varphi)$ about its minimum, $\frac{\partial V}{\partial \varphi}\big|_{\vec{\varphi}_0} = 0$

$$V(\vec{\varphi}) = V(\vec{\varphi}_0) + \underbrace{\left.\frac{\partial V(\vec{\varphi})}{\partial \varphi_i}\right|_{\vec{\varphi}_0} \chi^i}_{=0} + \frac{1}{2!}\left.\frac{\partial^2 V(\vec{\varphi})}{\partial \varphi_i \partial \varphi_j}\right|_{\vec{\varphi}_0} \chi^i \chi^j$$

Since minimum $\Longrightarrow$ the mass matrix≥ 0

$$M_{ij} \equiv \left.\frac{\partial^2 V(\vec{\varphi})}{\partial \varphi_i \partial \varphi_j}\right|_{\vec{\varphi}_0} \geq 0$$

$\#$ of zeros of $M_{ij} = \#$ of massless Goldstone bosons

STEP 2: Use the invariance of $V(\vec{\varphi})$

$$
\begin{aligned}
V\left(\vec{\varphi}'_0\right) &= V\left(U(g)\vec{\varphi}_0\right) = V\left(\vec{\varphi}_0 + \delta\vec{\varphi}_0\right) \\
&= V(\vec{\varphi}_0) + \underbrace{\left.\frac{\partial V(\vec{\varphi})}{\partial \varphi_i}\right|_{\vec{\varphi}_0}}_{=0} \delta\varphi_0^i + \frac{1}{2!}\left.\frac{\partial^2 V(\vec{\varphi})}{\partial\varphi_i\partial\varphi_j}\right|_{\vec{\varphi}_0} \delta\varphi_0^i\delta\varphi_0^j = V(\vec{\varphi}_0)
\end{aligned}
$$

leading to

$$
\left.\frac{\partial^2 V(\vec{\varphi})}{\partial\varphi_i\partial\varphi_j}\right|_{\vec{\varphi}_0} \delta\varphi_0^i\delta\varphi_0^j = 0
$$

- If $g = h \in H$, with $U(h)\vec{\varphi}_0 = \vec{\varphi}_0$ we have $\delta\varphi_0^i = 0$ and the above relation vanishes trivially. This is only satisfied for one direction (direction which we have chosen for the vacuum, here the $\hat{e}_3$-direction)
- If $g \in G/H$, we have $\delta\varphi_0^i \neq 0$ and to satisfy $\left.\frac{\partial^2 V(\vec{\varphi})}{\partial\varphi_i\partial\varphi_j}\right|_{\vec{\varphi}_0} \delta\varphi_0^i\delta\varphi_0^j = 0$, the mass matrix M_{ij} must vanishes, *i.e.*

$$
\left.\frac{\partial^2 V(\vec{\varphi})}{\partial\varphi_i\partial\varphi_j}\right|_{\vec{\varphi}_0} = 0
$$

We conclude:

# of Goldstone bosons = dimension of the coset space G/H
= dim G – dim H.

What about Quantum Effects?

Effective action and effective potential:

Generating functional for connected Feynman diagrams

$$W[J] \equiv -i \ln \mathcal{Z}[J], \qquad \text{with} \qquad \mathcal{Z}[J] \equiv \frac{1}{\mathcal{Z}[0]} \int \mathcal{D}\varphi \; e^{iS[\varphi]+i\int d^4x \;\; J(x)\varphi(x)}$$

- Define classical field φ_c

$$\varphi_c(x) \equiv \frac{\delta W[J]}{\delta J}, \qquad \text{with} \qquad \lim_{J\to 0} \varphi_c \equiv \langle \varphi \rangle$$

- Perform a Legendre transformation on $W[J]$ $\rightarrow$ Effective action

$$\Gamma[\varphi_c] \equiv W[J] - \int d^4x J(x)\varphi_c(x) \qquad \text{obeying} \qquad \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c(x)} = -J(x)$$

- For $J \rightarrow 0$

$$\left.\frac{d\Gamma[\varphi_c]}{d\varphi_c}\right|_{\langle\varphi\rangle} = 0, \qquad \textit{i.e.} \;\; \langle\varphi\rangle \text{ is min. of } \Gamma[\varphi_c]$$

- Expansion of $\Gamma[\varphi_c]$

$$\Gamma[\varphi_c] = \sum_n \frac{1}{n!} \int dp_1 \cdots dp_n \, \delta\left(p_1 + \cdots + p_n\right) \underbrace{\Gamma^{(n)}\left(p_1, \cdots, p_n\right)}_{\text{1PI-diagrams}} \varphi_c(p_1) \cdots \varphi_c(p_n).$$

- $\Gamma[\varphi_c]$ is the generating functional of connected 1PIs

- Alternatively:

$$\text{Eff. action } [\varphi_c] = \int_\Omega \left(\text{ effective kinetic}[\varphi_c] \; + \; \text{effective potential terms}[\varphi_c] \right)$$

- If vacuum is translational invariant then $\varphi_c = \langle\varphi\rangle \equiv \bar{\varphi} =$const. and eff. kin. term vanishes
- We are left with

$$\Gamma[\bar{\varphi}] = -\Omega \, U(\bar{\varphi})$$

- $U(\bar{\varphi}) =$ the effective potential
- Expansion in $\bar{\varphi}$

$$U(\bar{\varphi}) = -\sum_{n=0}^{\infty} \frac{1}{n!} \, \Gamma^{(n)}(p_i = 0) \, \bar{\varphi}^n$$

Goldstone Theorem: Proof using the effective potential

Assumption: Two properties for the transformation

$$\varphi_n(x) \to \varphi_n(x) + i\varepsilon^\alpha \sum_m (t^\alpha)_{nm}\varphi_m(x), \qquad \text{(with generator } t^\alpha\text{)}$$

- Continuous and global linear transformation
- Path integral measure remains invariant under this transformation (otherwise $\to$ Anomaly)

Use a theorem (which can be proved):

- For linear symmetry transformations the symmetries of the original action $S[\varphi]$ are automatically also symmetries of the effective action $\Gamma[\bar{\varphi}]$

Proof:

- Use the invariance of $\Gamma[\bar{\varphi}]$:

$$0 = \frac{\delta\Gamma[\varphi_c]}{\delta(\delta\varphi_c)} \equiv \sum_n \int d^4x \, \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c^n(x)} \, \delta\varphi_c^n(x) = i\varepsilon^\alpha \sum_{n,m} \int d^4x \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c^n(x)}(t^\alpha)_{nm}\varphi_c^m(x)$$

- Taking the special case of $\bar{\varphi} =$const., we had $\Gamma[\bar{\varphi}] = -\Omega \, U(\bar{\varphi})$ and

$$\sum_n \int d^4x \, \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c^n(x)} \, \delta\varphi_c^n(x) = 0 \qquad \Longrightarrow \qquad \left.\frac{dU\,(\varphi)}{d\varphi}\right|_{\bar{\varphi}} = 0 \tag{1}$$

i.e.

$$\sum_{n,m} \frac{\partial U(\varphi)}{\partial\varphi_n(x)} \, t_{nm}\varphi_m(x)\Big|_{\bar{\varphi}} = 0 \tag{2}$$

- Differentiate (2) with respect to $\bar{\varphi}_\ell$ and use (1), we get

$$\sum_{n,m} \left.\frac{\partial^2 U(\varphi)}{\partial\varphi_n(x)\partial\varphi_\ell(x)}\right|_{\bar{\varphi}} (t^\alpha)_{nm}\bar{\varphi}_m = 0$$

- Question: What is $\frac{\partial^2 U(\varphi)}{\partial\varphi_n(x)\partial\varphi_\ell(x)}$?

Claim:

$\frac{\partial^2 U(\varphi)}{\partial\varphi_n(x)\partial\varphi_\ell(x)}$ = inverse of the full propagator for zero external momentum p

Proof:

- $\Gamma[\varphi]$ is the generating functional of all connected 1PI Graphs
- Defining the full propagator (two-point function) by

$$\Delta(x,y) = -i\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}$$

and the vertex function by

$$\Gamma(x,y) = \frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(y)}$$

- We can verify

$$\int d^4z\ \Delta(x,z)\Gamma(z,y) = i\delta^4(x-y)$$

- Thus

$$\left.\frac{\partial^2 U(\varphi)}{\partial\varphi_n(x)\partial\varphi_\ell(x)}\right|_{\bar{\varphi}} = \widetilde{\Delta}^{-1}_{nm}(p=0)$$

- Using this notation

$$\sum_{n,m} \frac{\partial^2 U(\varphi)}{\partial\varphi_n(x)\partial\varphi_\ell(x)}\bigg|_{\bar{\varphi}} (t^\alpha)_{nm}\bar{\varphi}_m = 0 \qquad \Longrightarrow \qquad \sum_{n,m} \widetilde{\Delta}^{-1}_{\ell n}(0)(t^\alpha)_{nm}\bar{\varphi}_m = 0$$

- If the symmetry is broken (min. of the effective pot. is not invariant), *i.e.*

$$\delta\bar{\varphi}_n = \sum_m (t^\alpha)_{nm}\bar{\varphi}_m \neq 0$$

then the only possibility to satisfy the above equation is

$$\widetilde{\Delta}^{-1}_{\ell n}(0) v_n = 0$$

where v_n are the eigenvectors of $\widetilde{\Delta}^{-1}_{\ell n}(0)$ with zero eigenvalue

- This means that $\widetilde{\Delta}_{\ell n}(p)$ must have a pole at $p^2 = 0$ and these are the massless Goldstone bosons
- We conclude

  - Goldstone bosons are eigenvectors of the mass matrix
  - Goldstone bosons are poles of the full Green's function
  - # Goldstone bosons = the dimensionality of the space of eigenvectors with zero eigenvalues

(III) Spontaneously Broken Approximate Symmetries; Vacuum Alignment

Question:

What happens if we add a small (explicit) symmetry breaking term to the action?

($\longrightarrow$ *pseudo – Goldstone bosons*)

Example: Non-Abelian $SO(3)$ case + additional term

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\varphi}\cdot\partial^\mu\vec{\varphi} - V(\varphi), \qquad V(\vec{\varphi}) \equiv V_0 + V_1 = \frac{m^2}{2}\vec{\varphi}\cdot\vec{\varphi} + \lambda\left(\vec{\varphi}\cdot\vec{\varphi}\right)^2 + \vec{u}\cdot\vec{\varphi}$$

Claim: Vacuum alignment, *i.e.*, it automatically chooses the direction of $\vec{u}$

$\vec{\varphi}_0 || \vec{u}$, where $\vec{\varphi}_0$ was the minimum of $V_0(\varphi)$

Proof: Use perturbation theory

$$U(\varphi) = U_0(\varphi) + U_1(\varphi), \qquad \text{with } U_1 \text{ a small perturbation}$$

- The small perturbation will shift the minimum of U_0, *i.e.* $\bar{\varphi}_0$

$$\left.\frac{\partial U(\varphi)}{\partial \varphi_n}\right|_{\bar{\varphi}=\bar{\varphi}_0+\bar{\varphi}_1} = 0 \qquad i.e. \qquad \bar{\varphi} = \text{min. of } U(\varphi)$$

- Expand $U(\varphi)$ in the first order of $\bar{\varphi}_1$

$$0 = \left.\frac{\partial\left(U_0 + U_1\right)}{\partial \varphi_n}\right|_{\bar{\varphi}\equiv\bar{\varphi}_0+\bar{\varphi}_1} \Longrightarrow \sum_m \left.\frac{\partial^2 U_0(\varphi)}{\partial \varphi_n \varphi_m}\right|_{\bar{\varphi}_0} \bar{\varphi}_{1m} + \left.\frac{\partial U_1(\varphi)}{\partial \varphi_n}\right|_{\bar{\varphi}_0} = 0$$

- Multiply this equation with $(t^\alpha)_{n\ell}\, \bar{\varphi}_\ell^0$ and add over n and ℓ

$$0 = \sum_m \left(\sum_{n,\ell} \left.\frac{\partial^2 U_0(\varphi)}{\partial \varphi_m \partial \varphi_n}\right|_{\bar{\varphi}_0} (t^\alpha)_{n\ell} \bar{\varphi}_\ell^0 \right) \bar{\varphi}_{1m} + \sum_{n,\ell} \left.\frac{\partial U_1(\varphi)}{\partial \varphi_n}\right|_{\bar{\varphi}_0} (t^\alpha_{n\ell}) \bar{\varphi}_\ell^0$$

- The first term vanishes using the invariance of U_0 and we get

$$0 = \sum_{n,\ell} \left.\frac{\partial U_1(\varphi)}{\partial \varphi_n}\right|_{\bar{\varphi}_0} \delta \bar{\varphi}_n^0 \qquad \text{Vacuum Alignment}$$

iii) Spontaneously Broken Approximate Symmetries; Vacuum Alignment

Question:

What happens if we add a small (explicit) symmetry breaking term to the action?

($\longrightarrow$ *pseudo – Goldstone bosons*)

Example: Non-Abelian $SO(3)$ case + additional term

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\varphi}\cdot\partial^\mu\vec{\varphi} - V(\varphi), \qquad V(\vec{\varphi}) \equiv V_0 + V_1 = \frac{m^2}{2}\vec{\varphi}\cdot\vec{\varphi} + \lambda\left(\vec{\varphi}\cdot\vec{\varphi}\right)^2 + \vec{u}\cdot\vec{\varphi}$$

Claim: Vacuum alignment, *i.e.*, it automatically chooses the direction of $\vec{u}$

$$\vec{\varphi}_0 \| \vec{u}, \qquad \text{where} \qquad \vec{\varphi}_0 \text{ was the minimum of } V_0(\varphi)$$

To show this build $\frac{\partial V_1(\varphi)}{\partial\varphi_n} = u_n \Longrightarrow$

$$0 = \sum_{n,\ell} \frac{\partial V_1(\varphi)}{\partial\varphi_n}\bigg|_{\bar{\varphi}_0} \delta\bar{\varphi}^0_n = u_n (t^\alpha)_{n\ell}\bar{\varphi}^0_\ell = u_n \epsilon^{\alpha n\ell}\bar{\varphi}^0_\ell = (\vec{u}\times\bar{\varphi}_0)^\alpha \Longrightarrow \bar{\varphi}_0 \| \vec{u}$$

The vacuum is aligned

(IV) Hypothesis of Spontaneous Chiral Symmetry Breaking in Strong Interaction

- In chiral limit $m_u = m_d = 0$, $\mathcal{L}_{QCD}$ exhibits $SU_L(2) \times SU_R(2)$ symmetry
- Claim: This symmetry must be broken spontaneously, *i.e.*

$$Q_R^a|0\rangle \neq 0, \quad Q_L^a|0\rangle \neq 0, \qquad \text{or} \qquad Q^a|0\rangle = 0, \quad Q_5^a|0\rangle \neq 0, \qquad a = 1, 2, 3$$

- The physical vacuum $|0\rangle$ is defined by minimizing the Hamiltonian H

- To prove: let us assume the invariance of the vacuum and see what goes wrong, *i.e.*

$$Q_R^a|0\rangle = Q_L^a|0\rangle = 0 \qquad \text{or} \qquad Q^a|0\rangle = Q_5^a|0\rangle = 0$$

- On the other hand:

$$[Q_L, H] = [Q_R, H] = 0$$

- Take an eigenstate of the Hamiltonian which is simultaneously eigenstate of the parity operator

$$H|\Psi\rangle = E|\Psi\rangle \qquad \text{and} \qquad P|\Psi\rangle = +|\Psi\rangle$$

- $$[Q^a_{L,R}, H]|\Psi\rangle = 0 \Longrightarrow Q^a_{L,R}H|\Psi\rangle = H\left(Q^a_{L,R}|\Psi\rangle\right) = E\left(Q^a_{L,R}|\Psi\rangle\right).$$

- Define new eigenstate of H and P $|\Psi'\rangle$

$$P|\Psi\rangle = +|\Psi\rangle, \qquad \text{then} \qquad |\Psi'\rangle \equiv \frac{1}{\sqrt{2}}\left(Q_R - Q_L\right)|\Psi\rangle \qquad P|\Psi'\rangle = -|\Psi'\rangle$$

- We conclude: If Hamiltonian and the vacuum are symmetric under global chiral transformation, then two states $|\Psi\rangle$ and $|\Psi'\rangle$ arise which are simultaneous eigenstates of the Hamiltonian (true particles) and parity (with opposite parity)
- But: No such states exist in the spectrum of particles

- Hence: The chiral symmetry of QCD $SU_L(2) \times SU_R(2)$ is spontaneously broken into isospin symmetry $SU(2)$, *i.e.*

$$Q_R^a|0\rangle \neq 0, \quad Q_L^a|0\rangle \neq 0, \qquad \text{or} \qquad Q^a|0\rangle = 0, \quad Q_5^a|0\rangle \neq 0, \qquad a = 1,2,3$$

- Remember: According to Goldstone theorem

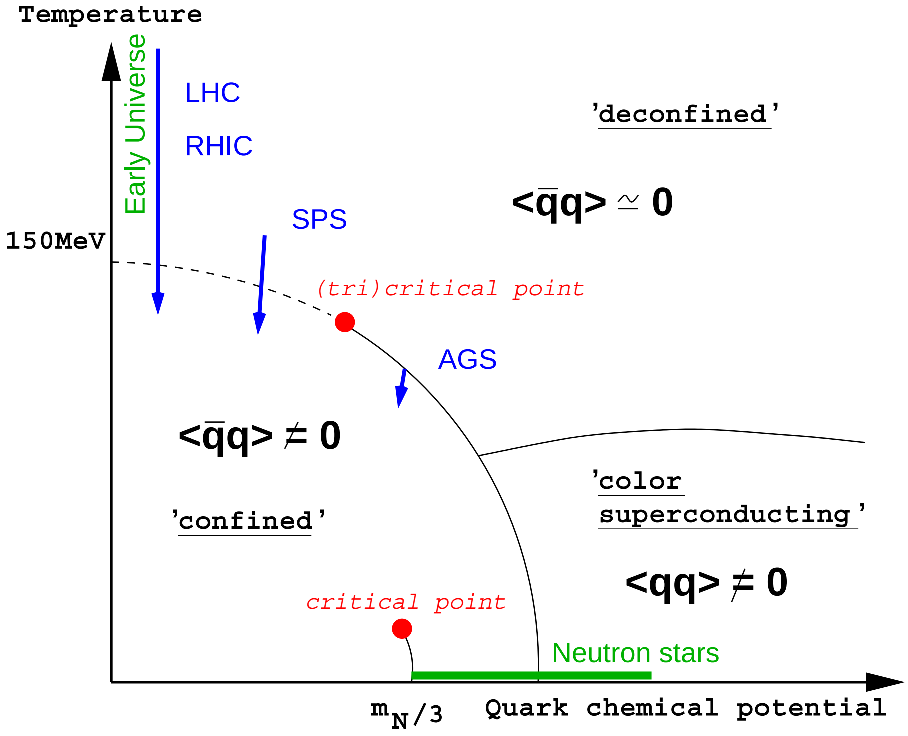
> *If a theory has a global symmetry of the Lagrangian which is not a symmetry of the vacuum, there must be a massless Goldstone boson, scalar or pseudoscalar, corresponding to each generator which does not leave the vacuum invariant*

- Question: What are the Goldstone boson associated with this SSB
- In the $SU_L(2) \times SU_R(2)$ case:
  3 generator of broken invariance $\Longrightarrow$ 3 pseudoscalar mesons: 3 pions

$$\pi^+, \pi^0, \pi^-$$

- In the $SU_L(3) \times SU_R(3)$ case: There are also η mesons and K associated with PCAC
- Question: What about $U_A(1)$ symmetry? $\rightarrow$ Anomalous breaking of symmetry $\Longrightarrow$ leading to pion decay, etc.

- Tomorrow: QCD under extreme conditions:



Phase diagram of QCD