Many-body localization: Entanglement and efficient numerical simulations

Part II

Frank Pollmann

Max Planck Institute for the Physics of Complex Systems, Dresden

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Many-body localization: Entanglement and efficient numerical simulations

Part II

Overview

(1) Brief introduction to many-body localisation
(2) Dynamics of entanglement entropy
(3) Finding highly excited eigenstates as matrix-product states
Many-body localization

\[ E \]

\[ \Delta \xi \]

\[ t \quad t \]

\[ \Delta \mu \]

Interactions

\[ c_1^+ c_2^+ c_3^+ c_4^+ |0\rangle \]

Anderson (1958)
Oganesyan and Huse (2007)
Pal and Huse (2010)
Bauer and Nayak (2013)

...
Many-body localization

\[ \epsilon > \epsilon_0 \]

\[ \sigma > 0 \]

Extended

\[ \sigma = 0 \]

Localized

Anderson (1958)
Gornyi, Mirlin, Polyakov (2005)
Oganesyan and Huse (2007)
Pal and Huse (2010)
Bardarson, FP, Moore (2012)
Bauer and Nayak (2013)
Huse and Oganesyan (2013)
Serbyn, Papic, Abanin (2013)
+many more
Many-body localization

MBL systems are perfect insulators: \( \sigma = 0 \)

Band insulator only insulating at \( T = 0 \)
\[ \sigma \sim \exp(-\Delta/kT) \]

Localized systems have \( \sigma = 0 \) at \( T > 0 \)
Many-body localization

\[ \epsilon > \epsilon_0 \]

**Extended**
\[ \sigma > 0 \]
Eigenstate thermalization hypothesis (ETH)

**Localized**
\[ \sigma = 0 \]
ETH violated

Anderson (1958)
Gornyi, Mirlin, Polyakov (2005)
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Many-body localization

MBL: Eigenstate thermalization hypothesis (ETH) violated
[Deutsch 91, Srednicki 94]

Eigenstates of thermalizing system appear thermal to all local measurement
$$\rho_B \rightarrow \frac{1}{Z} \exp(-\beta H_B)$$

$$\rho_B = \text{Tr}_A |\psi\rangle \langle \psi|$$

Localization prevents a system to serve as its own bath
Many-body localization

$\varepsilon > \varepsilon_0$

Extended

$\sigma > 0$

Eigenstate thermalization hypothesis (ETH)

Volume law

Localized

$\sigma = 0$

ETH violated

Area law

Anderson (1958)
Gornyi, Mirlin, Polyakov (2005)
Oganesyan and Huse (2007)
Pal and Huse (2010)
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Many-body localization

Eigenstates in MBL systems follow an **area law**: $S \propto L^{d-1}$

[Srednicki 93]

\[ \rho_B = \text{Tr}_A |\psi\rangle \langle \psi| \]

\[ S = -\text{Tr}_B \rho_B \log \rho_B \]

Kjäll, Bárðarson, FP, PRL 113, 107204 (2014)
Many-body localization

\[ \epsilon > \epsilon_0 \]

Extended

\[ \sigma > 0 \]

ETH

Volume law

Localized

\[ \sigma = 0 \]

ETH breaks down

Area law

Anderson (1958)
Gornyi, Mirlin, Polyakov (2005)
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Many-body localization

- Toy model to study MBL phases: disordered XXZ chain

$$H = J_\perp \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \sum_i h_i S_i^z + J_z \sum_i S_i^z S_{i+1}^z$$

hopping \quad \text{random potential} \quad \text{interaction}

with $h_i \in [-W, W]$

- All single particle states localized for $W \neq 0$

[Anderson '58]
Many-body localization

\[ J_\perp = J_z = 1 \]

**Fully MBL for** \( W \gtrsim 3.5 \)

[Pal & Huse '10, Luitz et al.'15]
Dynamics of entanglement entropy
Dynamics of entanglement entropy

- Start from an unentangled product state \( (S = 0) \)
  \[
  |\psi_0\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle
  \]

- Measure the entanglement after quench and the time evolution with \( U(t) = e^{-itH} \)

- Clean system:

\[
S = \begin{cases} 
1 & \text{for } tJ_\perp > x \\
0 & \text{for } tJ_\perp < x
\end{cases}
\]

Lieb and Robinson (1972)
P. Calabrese and J. Cardy (2006)
Dynamics of entanglement entropy

- What about the evolution in the localized systems?

- Entanglement entropy for $J_z = 0$ can be simply calculated using a mapping to chain of free fermions

\[
\rho_A \sim \exp \left[ \sum_{i,j \in A} h_{ij} c_i^\dagger c_j \right]
\]

with $h = \ln \left[ (1 - C')/C \right]$
Dynamics of entanglement entropy

- Evolution of the non-interacting system

\[ L = 300 \]

\[ \eta / J_\perp = 0.00 \]

- Entanglement grows linearly as function of time in clean system

- Saturation of the entanglement for any amount of disorder

\[ e^{-itH} | \cdots \uparrow \downarrow \uparrow \downarrow \uparrow \uparrow \cdots \rangle \]
Dynamics of entanglement entropy

- Time evolution of $S$ in the disordered XXZ model

\[ S \sim \xi \ln(J_z t) \]

Serbyn, Papic, Abanin (2013)

Bárðarson, FP, Moore, PRL 109, 017202 (2012)
M. Znidaric, T. Prosen, and P. Prelovsek (2008)
Obtaining highly excited eigenstates of MBL
Finding ground states

- **Matrix-product states:** \( 2^L \rightarrow Ld\chi^2 \) [M. Fannes et al. 92]

\[
\psi_{j_1, j_2, j_3, j_4, j_5} = A^{[1]} A^{[2]} A^{[3]} A^{[4]} A^{[5]} A^{[j]}_{\alpha\beta} = A
\]

- Find the **ground state** iteratively

\[
E = A^{[1]} A^{[2]} A^{[3]} A^{[4]} A^{[5]} A^{[6]} A^{[7]} |\psi_0\rangle \quad H \quad \langle \psi_0 |
\]

by locally minimizing energy of \( H_{\alpha i \beta; \alpha' i' \beta'} \) (e.g., Lanczos)
Density matrix renormalization group (DMRG)
Many-body localization

\[ \epsilon > \epsilon_0 \]

Extended

Volume law

Localized

Area law

Local integrals of motion

\[ \rho_B = \text{Tr}_A |\psi\rangle\langle\psi| \]
\[ S = -\text{Tr}_B \rho_B \log \rho_B \]

\[ |\psi_{\tau_1, \tau_2, \ldots, \tau_L}\rangle \quad \tau_j = U^\dagger \sigma_j U \]

“p-bits” and “l-bits”

Andersen (1958)
Gornyi, Mirlin, Polyakov (2005)
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Finding excited states

- **DMRG-X algorithm**

1. Initiate MPS $|\psi\rangle$ (close to “l-bit” state)
2. Diagonalize $H_{\alpha i \beta; \alpha' i' \beta'}$ at $m$:
   \[ H |\tilde{\psi}_n\rangle = E_n |\tilde{\psi}_n\rangle \]
3. Pick eigenstate that has largest overlap with $|\psi\rangle$: $A^{[m]i}_{\alpha \beta} \rightarrow \tilde{A}^{[m]i}_{\alpha \beta}$
4. Move to site $m + 1$

- **No individual update step of the MPS matrices results in a global reorganization** $|\psi_{\tau_1, \tau_2, \ldots, \tau_L}\rangle$

V. Khemani, FP, and S. L. Sondhi, PRL 116, 247204 (2016)
DMRG-X: Results

- Finding eigenstates starting from different product states

\[ L = 12, \quad W = 8 \]

- Algorithm finds resonances!

V. Khemani, FP, and S. L. Sondhi, PRL 116, 247204 (2016)
DMRG-X: Results

- Convergence criteria: Energy variance

![Graph showing energy variance convergence](image-url)