# On Exact Solutions
# and
# the Consistency of 3D Minimal Massive Gravity

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Introduction

Research Problem/Open Questions

1) Consistency of Field Equations

    A)Source Free Case

    B)Matter Coupled Case

2) Constant Scalar Curvature Solutions

    A)Type-N

    B)Type-D

Conclusion

References

# INTRODUCTION

TMG [1] modifies the field equations of general relativity by adding a new term with three derivatives.

The TMG action is:

$$S = \int d^3x\sqrt{-g}(R - 2\Lambda) + \frac{1}{4\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}(\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\nu\rho})$$

The TMG field equations derived from this action as:

$$\frac{1}{\mu}C_{\mu\nu} + \sigma G_{\mu\nu} + \Lambda_0 g_{\mu\nu} = 0$$

for generic values of the parameters the corresponding equations of motion have several solutions including AdS, BTZ .

<u>For TMG, the central charge of a holographically dual conformal field theory (CFT) is negative whenever the bulk spin-2 mode has positive energy, implying a non-unitary CFT.</u>

[1] S. Deser, R. Jackiw and S. Templeton;”Topologically Massive Gauge Theories”, Ann. Phys. (N.Y.) 140 372 (1982)

The field equation of MMG [2] includes the additional,curvature squared, symmetric tensor-J.

$$E_{\mu\nu} \equiv \bar{\sigma} G_{\mu\nu} + \bar{\Lambda}_0 \, g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} = 0$$

The MMG field equation is:

$$J^{\mu\nu} \equiv -\frac{1}{2 \det g} \, \varepsilon^{\mu\rho\sigma} \varepsilon^{\nu\tau\eta} S_{\rho\tau} S_{\sigma\eta} = G^{\rho}_{\mu} G_{\rho\nu} - \frac{1}{2} g_{\mu\nu} G_{\rho\sigma} G^{\rho\sigma} + \frac{1}{4} G_{\mu\nu} R + \frac{1}{16} g_{\mu\nu} R^2$$

$$J^{\mu\nu} \equiv \frac{1}{2} \eta^{\mu\rho\sigma} \eta^{\nu\alpha\beta} S_{\rho\alpha} S_{\sigma\beta}$$

MMG field equations can not be obtain from metric formalism.

In contrast to TMG, MMG has both a positive energy graviton and positive central charges for the asymptotic AdS-boundary conformal algebra.

[2] E. Bergshoeff, O. Hohm, W. Merbis, A. J. Routh and P. K. Townsend, “Minimal Massive 3D Gravity,” Class. Quant. Grav. 31, 145008 (2014).

Consistency of the field equations requires that the first divergence vanishes but from direct substitution we get

$$\nabla_\mu E^{\mu\nu} \equiv \nabla_\mu(\bar{\sigma} G^{\mu\nu} + \bar{\Lambda}_0\, g^{\mu\nu} + \tfrac{1}{\mu} C^{\mu\nu} + \tfrac{\gamma}{\mu^2} J^{\mu\nu}) = \nabla_\mu J^{\mu\nu}$$

$$\nabla_\mu J^{\mu\nu} = \eta^{\nu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau},$$

which means the MMG field equations does not obey the Bianchi Identity and therefore cannot be obtained from an action with the metric being the only variable.

But the covariant divergence vanishes for metrics that are solutions to the full MMG equations.

$$\nabla_\mu E^{\mu\nu} = 0$$

Therefore, one has an ”on-shell Bianchi Identity”.

# RESEARCH PROBLEM / OPEN QUESTIONS

The questions we aim to research are as follows:

Are the field equations of MMG consistent for both source-free and matter-coupled cases?

Are there any constant scalar curvature solutions for the theory?

Is it possible to find a relation between constant scalar curvature solutions of MMG and TMG?

# 1) CONSISTENCY OF MMG FIELD EQUATIONS

## A)SOURCE FREE CASE

$$\nabla_\mu J^{\mu\nu} = \eta^{\nu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau} = 0$$

This is necessary condition for the consistency of the classical field equations but not a sufficient condition, since the the rank- two tensor equations are susceptible to double-divergence.

When we calculate double divergence we get,

$$\nabla_\nu \nabla_\mu J^{\mu\nu} = \nabla_\nu \left( \eta^{\nu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau} \right) = 0$$

We show that for the source-free case the double-divergence of the field equations vanish for the solutions of the field  equation.

$$\nabla_\nu \nabla_\mu J^{\mu\nu} = \nabla_\nu \left(\eta^{\nu\rho\sigma} S^\tau{}_\rho C_{\sigma\tau}\right) = \eta^{\nu\rho\sigma} S^\tau{}_\rho \nabla_\nu C_{\sigma\tau} + \eta^{\nu\rho\sigma} C_{\sigma\tau} \nabla_\nu S^\tau{}_\rho$$

Einstein tensor can written in terms of Schouten tensor as

$$G_{\sigma\tau} = S_{\sigma\tau} - \frac{1}{4} g_{\sigma\tau} R$$

And we can write Cotton tensor in terms of Einstein tensor J-tensor, R and metric tensor by using MMG field equation.

$$\nabla_\nu \nabla_\mu J^{\mu\nu} = \eta^{\nu\rho\sigma} S^\tau{}_\rho \nabla_\nu (-\mu\bar{\sigma} S_{\sigma\tau} - \frac{1}{4} g_{\sigma\tau} R - \mu\bar{\Lambda}_0 g_{\sigma\tau} - \frac{\gamma}{\mu} J_{\sigma\tau})$$

$$+\eta^{\nu\rho\sigma} (-\mu\bar{\sigma} S_{\sigma\tau} - \frac{1}{4} g_{\sigma\tau} R - \mu\bar{\Lambda}_0 g_{\sigma\tau} - \frac{\gamma}{\mu} J_{\sigma\tau}) \nabla_\nu S^\tau{}_\rho$$

By using,

$$\eta_\alpha{}^{\beta\gamma} \eta^{\alpha\lambda\delta} = -g^{\beta\lambda} g^{\gamma\delta} + g^{\beta\delta} g^{\gamma\lambda}$$

$$\nabla_\nu \nabla_\mu J^{\mu\nu} = 0$$

**For the source free case MMG field equations are consistent.**

# B)MATTER-COUPLED CASE

Let us now consider the consistency of matter-coupled MMG equations [3].

Given a covariantly conserved energy-momentum tensor ,the field equations become :

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\frac{1}{\mu}C_{\mu\nu} + \frac{\gamma}{\mu^2}J_{\mu\nu} + \eta G_{\mu\nu} = \Theta_{\mu\nu}(T)$$

where the source term reads,

$$\Theta^{\mu\nu}(T) = \frac{b}{\gamma}T^{\mu\nu} + \frac{b^2}{\mu\gamma}\eta^{\mu\rho\sigma}\nabla_\rho \hat{T}^\nu_\sigma - \frac{b^2}{\mu^2}\eta^{\mu\rho\sigma}\eta^{\nu\lambda k}S_{\rho\lambda}\hat{T}_{\sigma k} + \frac{b^4}{2\mu^2\gamma}\eta^{\mu\rho\sigma}\eta^{\nu\lambda k}\hat{T}_{\rho\lambda}\hat{T}_{\sigma k}$$

$$b = \gamma/(1+\gamma\eta)$$

$$T_{\mu\nu} = -\bar{\rho}g_{\mu\nu} + \theta_{\mu\nu}$$

$$\hat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$$

[3] A. S. Arvanitakis, A. J. Routh and P. K. Townsend, "Matter coupling in 3D minimal massive gravity," Class. Quant. Grav. 31, no. 23, 235012 (2014)

For the consistency of the matter-coupled MMG, one should require the covariant divergence of the left-hand side and the right-hand side to be equal to each other when the field equations are used.

$$\nabla_\mu \Theta^{\mu\nu}(T) = \nabla_\mu\left(\frac{1}{\mu}C_{\mu\nu} + \frac{\gamma}{\mu^2}J_{\mu\nu} + \eta G_{\mu\nu}\right) = \frac{\gamma}{\mu}\varepsilon^{\nu\rho\sigma}S^\lambda_\rho \Theta_{\sigma\lambda}(T)$$

So first condition for the consistency of the matter coupled case is satisfied ,this is necessary but not sufficient and one should also check the double divergence.

$$\nabla_\nu \nabla_\mu \Theta^{\mu\nu}(T) = \nabla_\nu \nabla_\mu (\tfrac{1}{\mu} C_{\mu\nu} + \tfrac{\gamma}{\mu^2} J_{\mu\nu} + \eta G_{\mu\nu})$$

$$\text{RHS} = \tfrac{\gamma}{\mu} \varepsilon^{\nu\rho\sigma} S^\lambda_\rho \nabla_\nu \Theta_{\sigma\lambda}(T) + \tfrac{\gamma}{\mu} \Theta_{\sigma\lambda}(T) C^{\sigma\lambda}$$

$$\text{LHS} = \frac{b^2}{\gamma\mu} \eta^{\mu\rho\sigma} \nabla_\nu \nabla_\mu \nabla_\rho \hat{T}_\sigma{}^\nu - \frac{b^2}{\mu^2} \eta^{\mu\rho\sigma} \eta^{\nu\lambda k} \nabla_\nu \nabla_\mu \left(S_{\rho\lambda} \hat{T}_{\sigma k}\right)$$
$$+ \frac{b^4}{2\gamma\mu^2} \eta^{\mu\rho\sigma} \eta^{\nu\lambda k} \nabla_\nu \nabla_\mu \left(\hat{T}_{\rho\lambda} \hat{T}_{\sigma k}\right).$$

By using the equalities

$$\eta^{\lambda\mu k} \eta^{\nu\rho\sigma} \eta_\sigma{}^{\alpha\beta} S_{\mu\alpha} \hat{T}_{k\beta} S_{\rho\lambda} = -\frac{1}{2} \eta^{\lambda\mu k} \eta^{\nu\rho\sigma} \eta_\sigma{}^{\alpha\beta} S_{\mu\alpha} S_{k\beta} \hat{T}_{\rho\lambda} = \eta^{\nu\rho\sigma} \hat{T}_{\rho\lambda} J_\sigma{}^\lambda.$$

$$\eta^{\nu\rho\sigma} \eta^{\lambda\mu k} \eta_\sigma{}^{\alpha\beta} \hat{T}_{\beta k} (\hat{T}_{\rho\lambda} S_{\mu\alpha} + \frac{1}{2} \hat{T}_{\mu\alpha} S_{\rho\lambda}) = 0,$$

$$\eta^{\rho\mu\sigma} \nabla_\mu \hat{T}_{\sigma k} = \eta_k{}^{\mu\sigma} \nabla_\mu \hat{T}_\sigma{}^\rho.$$

The double divergence of the left hand-side and the right hand side of the field equations are equal to each other on shell.

$$\nabla_\nu \nabla_\mu \Theta^{\mu\nu}(T) = \nabla_\nu \nabla_\mu (\frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} + \eta G_{\mu\nu})$$

$$= \frac{\gamma}{\mu} \varepsilon^{\nu\rho\sigma} S_\rho^\lambda \nabla_\nu \Theta_{\sigma\lambda}(T) + \frac{\gamma}{\mu} \Theta_{\sigma\lambda}(T) C^{\sigma\lambda}$$

**For the matter-coupled case MMG field equations are consistent.**

# 2)CONSTANT SCALAR CURVATURE SOLUTIONS

In three dimensions, classification of space-times can be done either using the

Cotton-tensor (analogous to the four dimensional Petrov classification )

or using the;

Traceless Ricci tensor (analogous to the four dimensional Segre classification ).

For Segre classification, one needs the following two scalar curvature invariants:

$$I_1 = \widetilde{R}^{\mu}_{\nu}\widetilde{R}^{\nu}_{\mu} \ , \qquad\qquad I_2 = \widetilde{R}^{\mu}_{\nu}\widetilde{R}^{\nu}_{\rho}\widetilde{R}^{\rho}_{\mu}$$

To search for solutions , let us rewrite the source-free field equations as a trace part and a traceless part.

The trace part of TMG equations simply says that:

$$\mathrm{R} = 6\Lambda$$

while the traceless part reads :

$$\frac{1}{\mu}C_{\mu\nu} + \bar{\sigma}\widetilde{R}_{\mu\nu} = 0$$

trace part of MMG equations

$$I_1 - \frac{1}{24}R^2 + \frac{\mu^2}{\gamma}\bar{\sigma}R - \frac{6\mu^2}{\gamma}\bar{\Lambda}_0 = 0,$$

and traceless part

$$\frac{1}{\mu}C_{\mu\nu} + \bar{\sigma}\widetilde{R}_{\mu\nu} + \frac{\gamma}{\mu^2}\widetilde{J}_{\mu\nu} = 0$$

where the traceless part of the J-tensor is

$$\widetilde{J}_{\mu\nu} = \widetilde{R}_{\mu\rho}\widetilde{R}^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}I_1 - \frac{1}{12}R\widetilde{R}_{\mu\nu}$$

$$\widetilde{J}_{\mu\nu} = J_{\mu\nu} - \frac{1}{3}g_{\mu\nu}J$$

$$J = g^{\mu\nu}J_{\mu\nu}$$

## A)TYPE-N

For Type-N space-times traceless Ricci tensor can be written as

$$\widetilde{R}_{\mu\nu} = \rho\xi_\mu\xi_\nu$$

where ρ is a scalar function which will not play a role and $\xi_\mu$ is a null vector: $\xi_\mu\xi^\mu = 0$ For Type-N space-times, since $I_1 = 0$ and, $\widetilde{J}_{\mu\nu} = -\frac{1}{12}R\widetilde{R}_{\mu\nu}$ ,from the trace part of the MMG field equations, Ricci scalar is constant with two possible values.

$$R_\pm = \frac{12\mu}{\gamma}(\mu\bar{\sigma} \pm m), \qquad m \equiv \sqrt{\mu^2\bar{\sigma}^2 - \gamma\bar{\Lambda}_0}.$$

And the traceless part of the field equation is:

$$\frac{1}{\mu}C_{\mu\nu} + (\bar{\sigma} - \frac{\gamma R}{12\mu^2})\widetilde{R}_{\mu\nu} = 0$$

which is nothing but the field equations of TMG with the simple replacement of the parameters as

$$\mu\bar{\sigma} \rightarrow \mu\bar{\sigma} - \frac{\gamma R}{12\mu}$$

**Type-N solutions of TMG are also solutions of the MMG with the change in the parameters.**

Let us give an example of Type-N solution [4] which is locally equivalent to most Type-N solutions of TMG, including the AdS-pp wave solutions [5]

$$ds^2 = d\rho^2 + e^{2\rho/\ell} du dv + \left(e^{(1/\ell + \mu\bar{\sigma})\rho} f_1(u) + e^{2\rho/\ell} f_2(u) + f_3(u)\right) du^2$$

this is a solution to TMG for arbitrary functions fi(u). This solution also solves MMG after the replacement

$$\mu\bar{\sigma} \to \mu\bar{\sigma} - \frac{\gamma R}{12\mu}$$

[4] S. Olmez, O. Sarioglu and B. Tekin, “Mass and angular momentum of asymptotically ads or flat solutions in the topologically massive gravity,” Class. Quant. Grav. 22, 4355 (2005)

[5] D. D. K. Chow, C. N. Pope and E. Sezgin, “Classification of solutions in topologically massive gravity,” Class. Quant. Grav. 27, 105001 (2010).

# B)TYPE -D

Type-D solutions split into two as Type-Dt and Type-Ds and both types have the traceless Ricci tensor as

$$\widetilde{R}_{\mu\nu} = p\left(g_{\mu\nu} - \tfrac{3}{a}\xi_\mu \xi_\nu\right)$$

Where $\xi_\mu \xi^\mu = \pm 1$ and p is a scalar function. The traceless part of the J-tensor becomes $\widetilde{J}_{\mu\nu} = -(p + \frac{R}{12})\widetilde{R}_{\mu\nu}$ and since $I_1 = 6p^2$, from the trace part of field equations again we have constant curvature scalar with two possible solutions:

$$R_\pm = \frac{12\mu}{\gamma}(\mu\bar{\sigma} \pm M), \qquad M \equiv \sqrt{\mu^2\bar{\sigma}^2 - \gamma\bar{\Lambda}_0 + \frac{\gamma^2 p^2}{\mu^2}}.$$

reducing the MMG equation to the TMG equation as,

$$\frac{1}{\mu}C_{\mu\nu} + \left(\bar{\sigma} - \frac{\gamma}{\mu^2}\left(p + \frac{R}{12}\right)\right)\widetilde{R}_{\mu\nu} = 0$$

$$\mu\bar{\sigma} \rightarrow \mu\bar{\sigma} - \frac{\gamma}{\mu}\left(p + \frac{R}{12}\right)$$

**Type-D solutions of TMG are also solutions of the MMG with the change in the parameters.**

Let us now give two examples of such solutions (time-like squashed $AdS_3$ and space-like squashed $AdS_3$)

$$ds^2 = \frac{\lambda^2 - 4}{2R}\left(-\lambda^2(d\tau + \cosh\theta d\varphi)^2 + d\theta^2 + \sinh^2\theta d\varphi^2\right)$$

$$ds^2 = \frac{\lambda^2 - 4}{2R}\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \lambda^2(dz + \sinh\rho d\tau)^2\right)$$

$$\lambda^2 = \frac{8\bar{\sigma}^2\mu^2}{2\bar{\sigma}^2\mu^2 - 9R}.$$

For these two solutions of TMG also solve MMG with the replacement of the parameters:

$$\mu\bar{\sigma} \rightarrow \mu\bar{\sigma} - \frac{\gamma}{\mu}(p + \frac{R}{12}) = \mu\bar{\sigma} - \frac{\gamma}{\mu}\left(\frac{1}{9}\mu^2\bar{\sigma}^2 + \frac{\bar{\Lambda}_0}{\bar{\sigma}} + \frac{R}{12}\right)$$

Also the general Kundt solution of TMG reported in [6] also solves MMG.

$$ds^2 = 2dudv + \left(\frac{1}{2}R - \frac{1}{9}\mu^2\bar{\sigma}^2\right)v^2du^2 + \left(d\rho + \frac{2}{3}\mu\bar{\sigma}vdu\right)^2 + du^2$$

[6] D. D. K. Chow, C. N. Pope and E. Sezgin, “Kundt spacetimes as solutions of topologically massive gravity,” Class. Quant. Grav. 27, 105002 (2010)

# CONCLUSION

We checked consistency of the field equations of MMG for both source free and matter coupled cases. We show that for the source-free case the double-divergence of the field equations vanish for the solutions of the field equation. In the matter-coupled case, we show that the double-divergence of the left-hand side and the right-hand side are equal to each other for the solutions of the theory. This construction proves the consistency of the field equations.

We have also found a large class of solutions to MMG equations that are also solutions to the TMG equations. These solutions have constant scalar curvature. Type- N solutions and Type-D solutions in the Segre classification. We provided some explicit metrics that are called squashed $AdS_3$ .

THANK YOU

FOR YOUR LISTENING!