Quantum corrections to Hawking radiation

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Motivation

- To understand the effect of interactions on Hawking radiation
- Usually neglected because "quantum corrections are small" as they come with $\hbar$
- This is equilibrium physics or stationary physics intuition
- This situation is non-stationary
- What is the final state of Hawking radiation?

Introduction: What is Hawking radiation?

- Change in vacuum because of gravitational effects
- Assume a stationary spacetime becomes non-stationary then stationary again
- E.g. take a real scalar and canonically quantise

$$\phi(x) = \int d\omega (f(\omega)\, a_\omega + h.c.)$$

- $f(\omega)$ solves the Klein-Gordon equation
- Let $f(\omega)$ be orthonormal with respect to inner product

$$(f(\omega), f(\omega')) = i \int f^*(\omega') \overleftrightarrow{\partial_\mu} f(\omega) d\Sigma^\mu$$

- Vacuum is

$$a_\omega |0\rangle_{\text{in}} = 0$$

- We can again quantise the field in the last stationary stage

$$\phi(x) = \int d\omega (g(\omega)\, b_\omega + h.c.)$$

- f and g both solve the Klein-Gordon with different asymptotics

$$f(\omega) \propto e^{-i\omega t} \quad \text{+ve frequency mode in the first region}$$
$$g(\omega) \propto e^{-i\omega t'} \quad \text{+ve frequency mode in the last region}$$

- f and g both provide a complete basis for ϕ

$$\phi(x) = \int d\omega (f(\omega)\, a_\omega + h.c.) = \int d\omega (g(\omega)\, b_\omega + h.c.)$$

- Also f can be expanded in g

$$f(\omega) = \int d\omega' \left(\alpha_{\omega\omega'}\, g(\omega') + \beta_{\omega\omega'}\, g^*(\omega')\right)$$
$$\implies \qquad b_\omega = \int d\omega' \left(\alpha_{\omega'\omega}\, a_{\omega'} + \beta^*_{\omega'\omega}\, a^\dagger_{\omega'}\right)$$

Particle production

- Start in in-vacuum $|0\rangle_{\rm in}$

$$ {}_{\rm in}\langle 0|a^\dagger_\omega a_\omega|0\rangle_{\rm in} = 0 $$

- In Heisenberg representation the state that we're in will remain the same

- But this is not the vacuum at the last stage

$$ \langle N_\omega\rangle = {}_{\rm in}\langle 0|b^\dagger_\omega b_\omega|0\rangle_{\rm in} = \int d\omega' |\beta_{\omega'\omega}|^2 $$

- If Bogolubov coefficients $\beta_{\omega'\omega} \neq 0$ then we have particles produced

- Or can instead phrase in terms of flux. In some vacuum state

$$ \langle T_{\mu\nu}\rangle = \lim_{x'\to x}\left[\left(\partial_\mu\partial_{\nu'} - \frac{1}{2}g_{\mu\nu}\partial_\rho\partial^{\rho'}\right)G(x,x')\right] $$
$$ G(x,x') = \langle T\phi(x)\phi(x')\rangle $$

- In a non-stationary background $\longrightarrow$ Schwinger-Keldysh [Kamenev '04]

- Can rewrite correlation function as

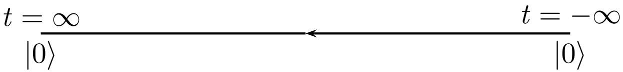
$$\langle\Omega|\,T\phi(x)\phi(y)|\Omega\rangle = \langle 0|S^{\dagger}(\infty,-\infty)\,T\left[\phi_I(x)\phi_I(y)S(\infty,-\infty)\right]|0\rangle$$

  where $S(t,t') = T\exp(\int_{t'}^{t} H_I)$, H_I and ϕ_I are interaction Hamiltonian and ϕ in the interaction picture, and $|\Omega\rangle$ and $|0\rangle$ are the vacua of the full and free theory, respectively.

- The free Hamiltonian H_0 time-independent $\Longrightarrow\ |\langle 0|S(\infty,-\infty)|0\rangle| = 1.$

- Use Feynman

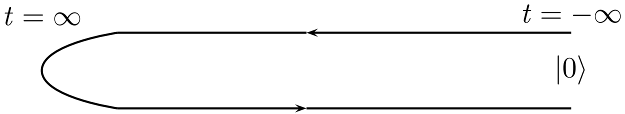
$$\langle\Omega|\,T\phi(x)\phi(y)|\Omega\rangle = \frac{\langle 0|\,T\left[\phi_I(x)\phi_I(y)S(\infty,-\infty)\right]|0\rangle}{\langle 0|S(\infty,-\infty)|0\rangle}$$



Schwinger-Keldysh Formalism

- If $H_0(t) \longrightarrow$ the vacuum of the free theory at $t = -\infty$ and $t = \infty$ different
- Use Schwinger-Keldysh

$$\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle = \langle 0 | S^\dagger(\infty, -\infty)\, T\left[\phi_I(x)\phi_I(y)S(\infty,-\infty)\right] |0\rangle$$



- Four propagators $D_{\pm\pm}$ satisfying

$$D_{++} + D_{--} = D_{-+} + D_{+-}$$

- After Keldysh rotation

$$D^R(x_1, x_2) = \Theta(t_1 - t_2)[\phi(x_1), \phi(x_2)], \quad D^A(x_1, x_2) = \Theta(t_2 - t_1)[\phi(x_2), \phi(x_1)]$$

$$D^K = 1/2\langle\{\phi(x_1), \phi(x_2)\}\rangle$$

- Keldysh propagator sensitive to the state

Setup

- Want to calculate the Keldysh function for scalars on a collapsing shell background
- Background: A thin shell that collapses from radius R_0 under its own weight

$$ds^2 = \begin{cases} dt_-^2 - dr^2 - r^2 d\Omega^2, & r \leq R(t) \\ (1 - \frac{r_g}{r}) dt^2 - \frac{dr^2}{1-\frac{r_g}{r}} - r^2 d\Omega^2, & r \geq R(t) \end{cases}$$

- Relate t_- and t using Israel matching [Israel, '66, '67]
- Before collapse

$$t_- = \sqrt{1 - \frac{r_g}{R_0}} t, \qquad R(t) = R_0$$

- After collapse as $t \to \infty$

$$t_- = (R_0 - r_g)\left(1 - e^{-t/r_g}\right), \quad R(t) = r_g \left(1 + \frac{R_0 - r_g}{r_g} e^{-t/r_g}\right)$$

$$R_* = R_0^* - R_0 + r_g - t$$

Assumptions

To make analytic headway we assume:

- Collapse is spherically symmetric
- No backreaction
- $|R_0 - r_g| \ll r_g \longrightarrow$ shell starts off close to its Schwarzschild radius r_g
- $\frac{dR}{dt_-} \approx 1 \longrightarrow$ shell has high binding energy

These assumptions are needed in order to find the modes

These are no extra assumptions for the loop computation

We need these in order to reproduce Hawking's result

Theory

- Real scalar field theory with ϕ^4 interaction

$$S = \int d^4x \sqrt{|g|} \left[(\partial_\mu \phi)^2 - m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right]$$

- Find modes for the scalar field on the collapsing shell background

- Expand ϕ in spherical harmonics

$$\phi(t, r, \theta, \varphi) = \sum_{l,n} Y_{l,n}(\theta, \varphi)\ \phi_l(t, r),$$

Strategy and preview of results

- Find $\phi_l(t, r)$ on the collapse geometry and calculate the Keldysh function

- At tree-level we calculate the Keldysh function

$$D^K(x_1, x_2) = \sum_{l,n} \delta(\Omega_1 - \Omega_2) \int_m^\infty \frac{d\omega}{2\pi}\, \phi^*_{\omega,l}(t_1, r_1)\phi_{\omega,l}(t_2, r_2) + h.c.$$

  and show that the energy flux is thermal

$$J \equiv \int_{S^2} r^2\; d\Omega\, \langle T^r{}_t \rangle = 2\left(1 - \frac{r_g}{R_0}\right)^{\frac{1}{2}} \sum_l (2\,l + 1) \int_m^\infty \frac{d\omega}{2\pi} \frac{\omega}{e^{4\pi r_g \omega} - 1}$$

- One-loop correction does not give anything interesting

- At two-loops the Keldysh function is

$$G_K^{(2)}(x_1, x_2) \propto \lambda^2 t$$

  where $t = (t_1 + t_2)/2$

Solution for modes

- In the rest of the talk, I will describe how the calculation is done

- First we need to solve for the modes

- Equations of motion of the free theory

$$\begin{cases} \left[\partial_{t_-}^2 - \partial_r^2 + m^2 + \frac{l(l+1)}{r^2}\right](r\phi_l) = 0, & r \leq R(t) \\ \left[\partial_t^2 - \partial_{r^*}^2 + \left(1 - \frac{r_g}{r}\right)\left(m^2 + \frac{l(l+1)}{r^2} + \frac{r_g}{r^3}\right)\right](r\phi_l) = 0, & r \geq R(t) \end{cases}$$

- Boundary conditions: continuity of ϕ and

$$\left[\frac{\partial t}{\partial t_-}\left|\frac{dR}{dt}\right|\partial_t\phi_l - \frac{\partial t_-}{\partial t}\partial_r\phi_l\right]_{r=R(t)_-} = \left[\frac{\partial_t\phi_l}{1-\frac{r_g}{r}}\left|\frac{dR}{dt}\right| - \partial_{r_*}\phi_l\right]_{r=R(t)_+}$$

Modes: Before collapse

- Equations of motion

$$\begin{cases} \left[\partial_{t_-}^2 - \partial_r^2 + m^2 + \frac{l(l+1)}{r^2}\right] (r\phi_l) = 0, & r \le R_0 \\ \left[\partial_t^2 - \partial_{r*}^2 + \left(1 - \frac{r_g}{r}\right)\left(m^2 + \frac{l(l+1)}{r^2} + \frac{r_g}{r^3}\right)\right] (r\phi_l) = 0, & r \ge R_0 \end{cases}$$

- Solve the Klein-Gordon equation on static shell background
- Inside the shell, $r \le R_0$

$$\phi_{\omega,l}(t,r) = \sqrt{\frac{\pi}{r}} J_{l+\frac{1}{2}}\left(\sqrt{\omega_-^2 - m^2}\, r\right) e^{-i\,\omega_-\,t_-}$$

where

$$\omega_- = \omega\left(1 - \frac{r_g}{R_0}\right)^{-1/2}$$

- $r \ge R_0$

$$\phi_{\omega,l}(t,r) = \frac{1}{r} e^{-i\omega t} \begin{cases} A_\omega e^{-i\omega r_*} + B_\omega e^{i\omega r_*}, & \text{near shell,} \\ C_\omega\, e^{-ikr_*} + D_\omega\, e^{ikr_*}, & \text{at infinity} \end{cases}$$

where

$$k = \sqrt{\omega^2 - m^2}$$

In-state

- Expand ϕ in terms of modes

$$\phi\left(\underline{x}, t\right) = \sum_{l,n} Y_{l,n}(\theta, \varphi) \int_m^\infty \frac{d\omega}{2\pi} \left[a_{\omega,l,n}\ \phi_{\omega,l}(r,t) + \text{h.c.}\right]$$

- In-vacuum given by

$$a_{\omega,l,n}|0\rangle = 0$$

- Assume these modes can be completed (defined for all $r \in \mathbb{R}$) to an orthonormal basis of modes

- Free Hamiltonian

$$H_0(t \leq 0) = \sum_{l,n} \int_m^\infty \frac{d\omega}{2\pi} \omega \left[a_{\omega,l,n} a^\dagger_{\omega,l,n} + a^\dagger_{\omega,l,n} a_{\omega,l,n}\right]$$

Inmodes at late time

- As $t \to \infty$ relation between t_- and t more complicated
- In order to be able to match the modes at shell we let the modes outside the shell be more general

$$\phi_{\omega,l}(t,r) = \frac{1}{r} \begin{cases} \sqrt{\pi\, r} J_{l+\frac{1}{2}} \left(\sqrt{\omega_-^2 - m^2}\, r \right) e^{-i\omega_- t_-}, & r \leq R(t), \\ f_{\omega,l}(u) + g_{\omega,l}(v), & r \geq R(t),\ |r - R(t)| \ll r_g \end{cases}$$

$$u = t - r_* \qquad \text{and} \qquad v = t + r_*$$

- Shell described by $v =$ const as $t \to \infty$
- v-dependent part unaffected

$$g_{\omega,l}(v) = A_\omega\, e^{-i\omega v}$$

Solution for $f_{\omega,l}$

- $f_{\omega,l}$ is found by imposing the boundary conditions:

$$\phi_{\omega,l}(R(t)_-) = \phi_{\omega,l}(R(t)_+)$$
$$\left(\frac{\partial t_-}{\partial t}\right)\left[\partial_{t_-}\phi_{\omega,l} - \partial_r \phi_{\omega,l}\right]_{r=R(t)_-} = 2\left[\partial_u \phi_{\omega,l}\right]_{r=R(t)_+}$$

- As $t \to \infty$,

$$u|_{r=R} = 2t - R_0^* - R_0 + r_g, \qquad t_- = (R_0 - r_g)\left(1 - e^{-t/r_g}\right)$$

- Continuity of ϕ gives

$$f_{\omega,l}\left[2t - (R_0^* + r_g - R_0)\right] = \sqrt{\pi\, R} J_{l+\frac{1}{2}}\left(\sqrt{\omega_-^2 - m^2}\; R\right) e^{-i\omega_- t_-}$$

- Using behaviour of t_- at late times

$$f_{\omega,l}(u) = \sqrt{\pi\, R(u)} J_{l+\frac{1}{2}}\left[\sqrt{\omega_-^2 - m^2}\; R(u)\right] e^{-i\omega_-(R_0 - r_g)\left(1 - e^{-\frac{u + R_0^* + r_g - R_0}{2r_g}}\right)}$$

Hawking radiation

- Because of gravitational collapse

$$e^{-i\omega u} \longrightarrow e^{-i\omega(1-e^{-u/r_g})}$$

- It is this that gives rise to Hawking radiation
- Bogolubov coefficient is

$$\alpha\left(\omega,\omega'\right) = \sqrt{2|\omega'|}\int_{u_*}^{\infty} du\, f_{\omega,l}(u)\, e^{i\omega' u}$$

- Flux near horizon

$$J \equiv \int_{S^2} r^2\, d\Omega\, \langle T^r{}_t\rangle = 2\left(1-\frac{r_g}{R_0}\right)^{\frac{1}{2}} \sum_l (2\,l+1) \int_m^{\infty} \frac{d\omega}{2\pi}\, \frac{\omega}{e^{4\pi r_g \omega}-1}$$

- For flux at infinity include a grey-body factor to take into account the potential

$$\left[\partial_t^2 - \partial_{r^*}^2 + \left(1-\frac{r_g}{r}\right)\left(m^2 + \frac{l(l+1)}{r^2} + \frac{r_g}{r^3}\right)\right](r\phi_l) = 0$$

2-loop correction

- At tree-level Hawking flux can be found using the Keldysh propagator

$$D^K(x_1,x_2) = \sum_{l,n} \delta(\Omega_1 - \Omega_2) \int_m^\infty \frac{d\omega}{2\pi}\, \phi^*_{\omega,l}(t_1,r_1)\phi_{\omega,l}(t_2,r_2) + h.c.$$

- Recall

$$\langle T_{\mu\nu} \rangle = \lim_{x'\to x} \left[\left(\partial_\mu \partial_{\nu'} - \frac{1}{2} g_{\mu\nu} \partial_\rho \partial^{\rho'} \right) G(x,x') \right]$$

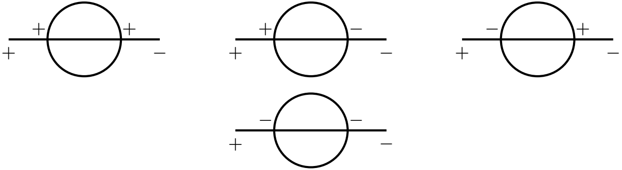
- What are higher loop corrections?

- One-loop correction uninteresting

- Calculate the 2-loop correction to the Keldysh function

$$G_K^{(2)}(x_1,x_2) = \frac{1}{2} \left[G^{(2)}_{+-}(x_1,x_2) + G^{(2)}_{-+}(x_1,x_2) \right]$$

- 8 diagrams
- 4 diagrams for each $G^{(2)}_{+-}(x_1, x_2)$ and $G^{(2)}_{-+}(x_1, x_2)$



- Keldysh rules:

$$D_{+-}(x_1, x_2) = \sum_{l,n} Y_{l,n}(1)\, Y_{l,n}(2) \int_m^\infty \frac{d\omega}{2\pi}\, \phi^*_{\omega,l}(1)\phi_{\omega,l}(2),$$

$$D_{-+}(x_1, x_2) = D_{+-}(x_2, x_1),$$

$$D_{++}(x_1, x_2) = \Theta(t_2 - t_1) D_{+-}(x_1, x_2) + \Theta(t_1 - t_2) D_{-+}(x_1, x_2),$$

$$D_{--}(x_1, x_2) = \Theta(t_1 - t_2) D_{+-}(x_1, x_2) + \Theta(t_2 - t_1) D_{-+}(x_1, x_2)$$

- Summing the diagrams we get

$$G_K^{(2)}(1,2) = \sum_{l_1,m_1,l_2,m_2} Y_{l_1,m_1}(\Omega_1)\, Y_{l_2,m_2}(\Omega_2) \int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}$$

$$\Big\{ N_{\omega_1,l_1,n_1|\omega_2,l_2,n_2}(t_1,t_2)\phi^*_{\omega_1,l_1}(t_1,r_1)\,\phi_{\omega_2,l_2}(t_2,r_2)$$

$$+ K_{\omega_1,l_1,n_1|\omega_2,l_2,n_2}(t_1,t_2)\,\phi_{\omega_1,l_1}(t_1,r_1)\,\phi_{\omega_2,l_2}(t_2,r_2) + \text{h.c.} \Big\}$$

- Recall at tree-level the Keldysh function is

$$D^K(x_1,x_2) = \sum_{l,n} \delta(\Omega_1 - \Omega_2) \int_m^\infty \frac{d\omega}{2\pi}\, \phi^*_{\omega,l}(t_1,r_1)\phi_{\omega,l}(t_2,r_2) + h.c.$$

- $N_{\omega_1,l_1,n_1|\omega_2,l_2,n_2}(t_1,t_2)$ is contributing to the particle number density

- $K_{\omega_1,l_1,n_1|\omega_2,l_2,n_2}(t_1,t_2)$ is showing that the ground state is changing because of interactions

- Both are non-diagonal $\longrightarrow$ non-thermal contribution

Quantum corrections negligible?

- We approximate $N_{\omega_1,l_1,n_1|\omega_2,l_2,n_2}(t_1,t_2)$ at late times
- Neglect difference between t_1 and t_2 and let $t=(t_1+t_2)/2$

$$N_{\omega,l,n|\omega',l',n'}(t) = \frac{\lambda^2}{3} \iint_{t_0}^{t} dt_3\, dt_4 \iint (r_3 r_4)^2\, dr_3\, dr_4\, \phi_{\omega,l}(r_3,t_3)\, \phi^*_{\omega',l'}(r_4,t_4)$$

$$Y(l,n,l',n') \prod_{j=1}^{3} \int_m^\infty \frac{d\omega_j}{2\pi}\, \phi_{\omega_j,l_j}(r_3,t_3)\, \phi^*_{\omega_j,l_j}(r_4,t_4)$$

- $Y(l,n,l',n')$ is an integral over spherical harmonics
- $\phi_{\omega,l}(r,t)$ are the modes on the collapse geometry

Leading order contribution

- Interaction negligible inside shell: At late times

$$V(t) = \frac{\lambda}{4!} \int_{R(t)}^{\infty} \phi^4 r^2 dr d\Omega + \frac{\lambda}{4!} e^{-t/r_g} \int_{0}^{R(t)} \phi^4 r^2 dr d\Omega.$$

- At large radius, modes are unaffected by collapse $\longrightarrow$ behaviour of modes near horizon important
- Contribution from integral for large radius negligible
- Use harmonics found near shell/horizon for $\phi_{\omega,l}(r,t)$

Quantum corrections not suppressed

- Use these approximations and let

$$T = (t_3 + t_4)/2 \quad \text{and} \quad \tau = t_3 - t_4$$

- We find that $N_{\omega,l,n|\omega',l',n'}(t)$ is proportional to

$$\frac{\lambda^2}{r_g^8\,\sqrt{\omega\,\omega'}}\,\left(1-\frac{r_g}{R_0}\right)^2 \int_0^t dT \int_{-\infty}^{\infty} d\tau \int_{r_g}^{\infty} r_3^2\,dr_3 \int_{r_g}^{\infty} r_4^2\,dr_4$$

$$\prod_{j=1}^{3}\int_{\omega_j>m}\frac{d\omega_j}{4\pi\omega_j}\left\{\left[n(-\omega_j)\,e^{-i\omega_j(\tau-\Delta r)} + n(\omega_j)\,e^{i\omega_j(\tau-\Delta r)}\right] + \,e^{-i\omega_j(\tau+\Delta r)}\right\},$$

where

$$n(\omega) = \frac{sign(\omega)}{e^{4\pi r_g\omega}-1}.$$

- In particular

$$N_{\omega,l,n|\omega',l',n'}(t) \propto \lambda^2 t$$

- Constant of proportionality not zero because of collapse
- If no collapse then

$$N_{\omega,l,n|\omega',l',n'}(t) \propto \lambda^2 t\,\delta(\omega_1+\omega_2+\omega_3) = 0$$

Conclusions

- We have shown secular growth in loop corrections
- Breakdown of perturbation theory
- Need to resum to find out if this has a physical affect
- Could be a sign that the picture in which the black hole emits thermally until it evaporates is incorrect and the correct state is something else!

Thank you!

Details of 2-loop calculation

- We approximate $N_{\omega_1,l_1,n_1|\omega_2,l_2,n_2}(t)$ at late times
- Interaction negligible inside shell: At late times

$$V(t) = \frac{\lambda}{4!}\int_{R(t)}^{\infty} \phi^4 r^2 \, dr d\Omega + \frac{\lambda}{4!} e^{-t/r_g} \int_0^{R(t)} \phi^4 r^2 \, dr d\Omega.$$

- Neglect difference between t_1 and t_2 and let $t = (t_1 + t_2)/2$

$$N_{\omega,l,n|\omega',l',n'}(t) = \frac{\lambda^2}{3} \iint_{t_0}^{t} dt_3 \, dt_4 \iint_{r_g}^{\infty} (r_3 r_4)^2 \, dr_3 \, dr_4 \, \phi_{\omega,l}(r_3,t_3) \, \phi^*_{\omega',l'}(r_4,t_4)$$

$$Y(l,n,l',n') \prod_{j=1}^{3} \int_m^{\infty} \frac{d\omega_j}{2\pi} \, \phi_{\omega_j,l_j}(r_3,t_3) \, \phi^*_{\omega_j,l_j}(r_4,t_4)$$

- $Y(l,n,l',n')$ is an integral over spherical harmonics
- $\phi_{\omega,l}(r,t)$ are the modes on the collapse geometry

- At large radius, modes are unaffected by collapse $\longrightarrow$ behaviour of modes near horizon important
- Contribution from integral for large radius negligible
- Use harmonics found near shell/horizon for $\phi_{\omega,l}(r,t)$

$$\phi_{\omega,l}(r,t) = f_{\omega,l}(u) + g_{\omega,l}(v)$$

$$f_{\omega,l}(u) = \frac{1}{r_g}\left(1 - \frac{r_g}{R_0}\right)^{\frac{1}{4}} \sqrt{\frac{2}{\omega}} \cos\left(\frac{\pi l}{2} - \omega_- r_g\right) e^{i\,\omega r_g\, e^{-\frac{u-u_0}{2r_g}}}$$

- Largest contribution to $N_{\omega,l,n|\omega',l',n'}(t)$ when $t_3 \gg r_g \log r_g\omega$ and $t_4 \gg r_g \log r_g\omega'$

$$N_{\omega,l,n|\omega',l',n'}(t) = \frac{\lambda^2}{3} \iint_{t_0}^{t} dt_3\, dt_4 \iint_{r_g}^{\infty} (r_3 r_4)^2\, dr_3\, dr_4\, \phi_{\omega,l}(r_3,t_3)\, \phi^*_{\omega',l'}(r_4,t_4)$$

$$Y(l,n,l',n') \prod_{j=1}^{3} \int_m^\infty \frac{d\omega_j}{2\pi}\, \phi_{\omega_j,l_j}(r_3,t_3)\, \phi^*_{\omega_j,l_j}(r_4,t_4)$$

- In which case $\phi_{\omega,l}(r_3,t_3)$ and $\phi_{\omega',l}(r_4,t_4)$ independent of u_3, u_4. Also v-dependent contribution of $\phi_{\omega,l}(r_3,t_3)$ and $\phi_{\omega',l}(r_4,t_4)$ negligible

- With these approximations

$$N_{\omega,l,n|\omega',l',n'}(t) \propto \frac{\lambda^2}{r_g^2\,\sqrt{\omega\,\omega'}}\,\left(1-\frac{r_g}{R_0}\right)^{\frac{1}{2}}\,Y(l,n,l',n')\,\int_{r_g\,\log\left(r_g\,\omega\right)}^{t} dt_3$$

$$\int_{r_g\,\log\left(r_g\,\omega'\right)}^{t} dt_4 \int_{r_g}^{\infty} r_3^2\,dr_3 \int_{r_g}^{\infty} r_4^2\,dr_4$$

$$\prod_{j=1}^{3}\int_{m}^{\infty}\frac{d\omega_j}{2\pi}\,\phi_{\omega_j,l_j}(r_3,t_3)\,\phi^*_{\omega_j,l_j}(r_4,t_4).$$

- Approximate $\phi_{\omega,l}(r,t)$ by modes near shell/horizon and integral becomes

$$\left(1-\frac{r_g}{R_0}\right)^{\frac{1}{2}}\frac{1}{r_g^2}\int_{\omega_j>m}\frac{d\omega_j}{4\pi\omega_j}\,\Big\{\Big[n(-\omega_j)\,e^{-i\omega_j(\tau-\Delta r)}+n(\omega_j)\,e^{i\omega_j(\tau-\Delta r)}\Big]$$

$$+\,e^{-i\omega_j(\tau+\Delta r)}\Big\}$$

where $T=(t_3+t_4)/2$, $\tau=t_3-t_4$ and

$$n(\omega)=\frac{sign(\omega)}{e^{4\pi r_g\omega}-1}.$$