A Schrödinger Approach to Non-relativistic Gravity

Recent Trends in String Theory and Related Topics

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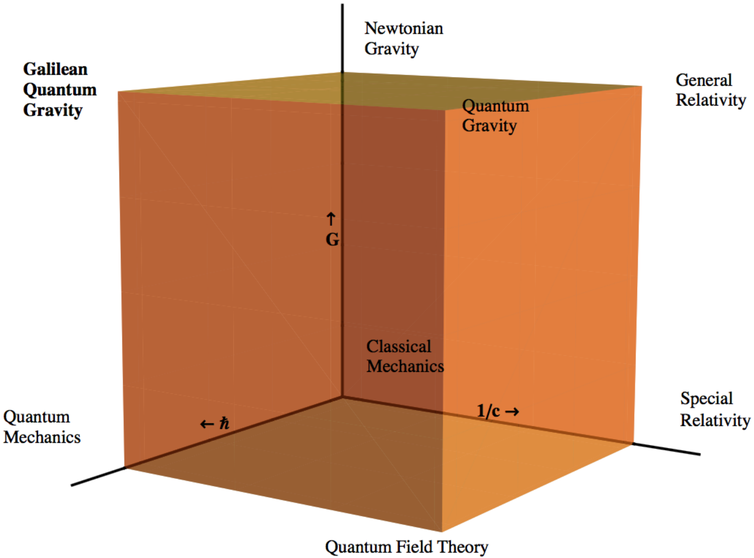
Introduction



Introduction

No gravitational force in free falling frames.

- Newtonian mechanics; **Galilei group**,
- Special relativity; **Lorenz group**,

Arbitray frame formulation;

- General relativity; **coordinate transformations**,
- Frame-independence in Newtonian mechanics?!

"Newton-Cartan Gravity"

Cartan 1923

Why Non-relativistic gravity?

The space and time coordinates are distinct, the idea of scale transformation is generalized to

$$(t, \mathbf{x}) \to (\lambda^z t, \lambda \mathbf{x}), \qquad z \neq 1$$

- Dynamics of non-equilibrium physics.
- Condensed matter applications.
- Non-relativistic holography.

Lifshitz holography; field theory is coupled to NC geometry.

Gravity as a gauge theory

Poincaré algebra,

$$[J_{AB}, J_{CD}] = 4\eta_{[A[C} J_{B]D]} , \qquad [J_{AB}, P_C] = 2\eta_{C[A} P_{B]} .$$

- Gauge theory formulation of gravity;

$$A_\mu = e_\mu{}^A P_A + \omega_\mu{}^{AB} J_{AB}$$
$$F_{\mu\nu} = T_{\mu\nu}{}^A P_A + R_{\mu\nu}{}^{AB} J_{AB}$$
$$T_{\mu\nu}{}^A = 2\partial_{[\mu} e_{\nu]}{}^A + 2e_{[\mu}{}^b \omega_{\nu]}{}^A{}_B = 0 .$$

Diffeo's are generated by gauge transformations $\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$, with parameter, $\epsilon^A = e_\mu{}^A \xi^\mu$.

Galilei algebra and gauge fields

$$P_0 = H \qquad P_i = P_a \qquad J_{0i} = G_a \qquad J_{ij} = J_{ab}$$

The centrally extended Galilei $Gal(d)$ algebra in $d+1$ dimensions:

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}\, J_{b]d]}$$

$$[J_{ab}, P_c] = 2\delta_{c[a} P_{b]}\,, \qquad [P_a, G_b] = \delta_{ab} N$$

$$[J_{ab}, G_c] = 2\delta_{c[a} G_{b]}\,, \qquad [H, G_a] = P_a$$

H	P_a	G_a	J_{ab}	N
ξ^0	ξ^a	Λ^a	Λ^{ab}	σ
τ_μ	$e_\mu{}^a$	$\omega_\mu{}^a$	$\omega_\mu{}^{ab}$	m_μ

Newton-Cartan gravity

Transformation rules;

$$\delta\tau_\mu = 0\,,$$
$$\delta e_\mu{}^a = \Lambda^a{}_b e_\mu{}^b + \Lambda^a \tau_\mu\,,$$
$$\delta m_\mu = \partial_\mu \sigma + \Lambda^a e_{\mu a} \tag{1}$$

Equations of motion;

$$(\tau^\mu)^{\mathrm{G}}(e^\nu{}_a)^{\mathrm{G}}\mathcal{R}_{\mu\nu}{}^a(G) = 0\,, \tag{2}$$
$$(e^\nu{}_c)^{\mathrm{G}}\mathcal{R}_{\mu\nu}{}^{ca}(J) = 0\,, \tag{3}$$

The Poisson equation $\triangle\Phi = 0$ is found after gauge fixing to flat space where $\Phi = M_0$.

Conformal method

Kaku, Townsend, van Nieuwenhuizen; 1977-1978

$$\text{Conformal} = \text{Poincaré} \oplus \text{Dilataion } (D) \oplus \text{SCT } (K_\mu)$$

$$[D, P_A] = -P_A\,, \qquad [P_A, K_B] = -2(\eta_{AB} D + J_{AB})\,,$$
$$[D, K_A] = K_A\,, \qquad [K_A, J_{BC}] = 2\eta_{A[B} K_{C]}\,.$$

Einstein-Hilbert Lagrangian can be related to the conformal field theory of a real scalar field;

$$\mathcal{L}_{\mathrm{P}} = \sqrt{-g^{\mathrm{P}}}\,\mathcal{R}^{\mathrm{P}}\,, \qquad (g_{\mu\nu})^{\mathrm{P}} = \varphi^2\,(g_{\mu\nu})^{\mathrm{C}}\,,$$
$$\delta(g_{\mu\nu})^{\mathrm{C}} = \Lambda_D (g_{\mu\nu})^{\mathrm{C}}\,, \qquad \delta\varphi = -\Lambda_D \varphi\,,$$

Conformal (Stükelberg) method

Guage fixing $(g_{\mu\nu})^{\mathsf{C}} = \eta_{\mu\nu}$;

$$e^{-1}\mathcal{L}_{\mathsf{P}} = \mathcal{R} \longrightarrow \tfrac{1}{2}\varphi\Box^{\mathsf{C}}\varphi$$

This relation also works the other way around;

Scalar conformal field theories $\longrightarrow$ Poincaré invariants
Scalar Schrödinger field theories $\longrightarrow$ Galilean invariants

Conformal method is a Stückelberg mechanism for a compensating scalar multiplet.

Schrödinger algebra and gauge fields

The Schrödinger algebra in $d+1$ dimensions:

$$Sch(d) = Gal(d) \oplus Dilataion(D) \oplus SCT(K)$$

D	K
Λ_D	Λ_K
b_μ	f_μ

$$[D, P_a] = -P_a\,, \qquad [D, H] = -2H\,, \qquad [H, K] = -D\,,$$
$$[D, G_a] = G_a\,, \qquad [D, K] = 2K\,, \qquad [K, P_a] = -G_a\,.$$

Hagen, Niedere, Jackiw-Pi, Gauntlett-Gomis-Townsend, Duval-Horvathy, Henkel, ...

Gauge transformation

$$A_\mu = \tau_\mu H + e_\mu{}^a P_a + \omega_\mu{}^a G_a + \omega_\mu{}^{ab} J_{ab} + m_\mu N + \cdots$$
$$\epsilon = \xi^0 H + \xi^a P_a + \Lambda^a G_a + \Lambda^{ab} J_{ab} + \sigma N + \cdots$$

Their standard transformation rules for independent fields;

$$\delta\tau_\mu = 2\Lambda_D \tau_\mu\,,$$
$$\delta e_\mu{}^a = \Lambda^a{}_b e_\mu{}^b + \Lambda^a \tau_\mu + \Lambda_D e_\mu{}^a\,,$$
$$\delta m_\mu = \partial_\mu \sigma + \Lambda^a e_{\mu a}$$
$$\delta b_0 = \partial_0 \Lambda_D - \Lambda^a b_a - 2\Lambda_D b_0 + \Lambda_K\,.$$

Non-relativistic conformal method

The gauge fixed theory should be invariant under dilatation D and the central charge symmetry N. Two scalars;

$$(\tau_\mu)^{\mathrm{G}} = \varphi^2 (\tau_\mu)^{\mathrm{Sch}}\,, \quad (e_\mu{}^a)^{\mathrm{G}} = \varphi (e_\mu{}^a)^{\mathrm{Sch}}\,, \quad M_\mu = m_\mu - \tfrac{1}{\mathrm{M}}\partial_\mu \chi\,.$$

They form a complex scalar,

$$\Psi(t, \mathbf{x}) = \varphi\, e^{i\chi} \qquad \text{with} \qquad \delta\varphi = w\Lambda_D \varphi\,, \qquad \delta\chi = \mathrm{M}\,\sigma\,.$$

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The canonical form of the invariant action under Schrödinger, in the rigid case, $e_\mu{}^a = \delta_\mu{}^a$, $\tau_\mu = \delta_\mu{}^0$ and $m_\mu = 0$, is

$$S^{(n)}_{\mathrm{Sch}} = \int dt d^d\mathbf{x}\, \Psi^\star \Big(i\partial_0 - \frac{1}{2\mathrm{M}}\partial_a^2\Big)^n \Psi\,, \qquad w_\Psi = -d/2 + n - 1$$

Gauging the symmetries

Under dilatation and central charge transformation,

$$\delta\Psi = (w\Lambda_D + i\mathrm{M}\sigma)\Psi$$

The covariant derivatives are naturally defined as,

$$D_0\Psi = \tau^\mu(\partial_\mu - w\, b_\mu - i\mathrm{M}\, m_\mu)\Psi\,,$$
$$D_a\Psi = e^\mu{}_a(\partial_\mu - w\, b_\mu - i\mathrm{M}\, m_\mu)\Psi\,.$$

The variation of these covariant derivatives,

$$\delta\triangle\Psi = [(w-2)\Lambda_D + i\mathrm{M}\sigma]\triangle\Psi - i\mathrm{M}(2\Lambda^a D_a\Psi - d\Lambda_K\Psi)$$
$$\delta D_0\Psi = [(w-2)\Lambda_D + i\mathrm{M}\sigma]D_0\Psi - \Lambda^a D_a\Psi - w\Lambda_K\Psi$$

Invariance

$$\delta\Box_{\rm Sch}\Psi = -i\left(w+\tfrac{d}{2}\right)\Lambda_K\Psi\,,$$
$$\delta\Box^2_{\rm Sch}\Psi = -i\left(2w-2+d\right)\Lambda_K\Box_{\rm Sch}\Psi\,.$$

The invariance under Schrödinger symmetries, fixes the weight of Ψ to $w=-\frac{d}{2}$ and $w=-\frac{d-2}{2}$. Where

$$\Box_{\rm Sch} = iD_0 - \frac{1}{2\mathrm{M}}\triangle \quad \text{and} \quad \triangle = D^a D_a\,.$$

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Real case ($\delta\phi = w\Lambda_D\phi$);

$$\delta D_0^2\phi \;= -2\Lambda^a D_0 D_a\phi - 2(w-1)\Lambda_K D_0\phi\,.$$

Newton-Cartan equations of motions

When $w_\phi = 1$, the constraint $D_0^2\phi = 0$ is invariant under Schrödinger symmetries if $D_a\phi = 0$. Relax this condition by adding the pseudo scalar field χ transforming $\delta\chi = \mathrm{M}\sigma$,

$$D_0^2\phi - \frac{2}{\mathrm{M}}(D_0 D_a\phi) D_a\chi + \frac{1}{\mathrm{M}^2}(D_a D_b\phi) D_a\chi D_b\chi = 0\,.$$

Imposing the gauge-fixing condition $\phi = 1$ and $\chi = 0$,

$$\hat{\tau}^\mu \partial_\mu K + K^{ab}K_{ab} - \triangle\Phi - 8\,\Phi\, b\cdot b - 2\,\Phi\,\mathcal{D}\cdot b - 6\, b^a\mathcal{D}_a\Phi = 0$$

In the torsionless case and earth-based gauge we reproduce the Poisson equation in the vacuum.

Schrödinger Kinetic terms

Schrödinger invariant complex scalar theories with two time derivatives, $w = -\frac{d-2}{2}$.

I. $$\int dt d^d\mathbf{x}\, e\, \Psi^\star \Big(iD_0 - \tfrac{1}{2\mathrm{M}}\triangle\Big)^2 \Psi\,,$$

II. $$\int dt d^d\mathbf{x}\, e\, \Big| iD_0\Psi + \frac{1}{\mathrm{M}d}\left(w\triangle\Psi - \Psi^{-1}D_a\Psi D_a\Psi\right)\Big|^2 .$$

After gauge-fixing $\Psi = 1$, they lead to the Hořava-Lifshtz Kinetic terms;

$$S = \frac{1}{\kappa^2}\int dt d^d\mathbf{x}\, e\left(K_{ab}K^{ab} - \lambda K^2 + \mathcal{V}\right)$$

$K_{ab}(\tau, e, m) = \mathcal{D}_a M_b + 2M_{[a}b_{b]}$ is the Galilean boost invariant version of $\mathcal{D}_a M_b$.

Potential terms

Terms Made of only spatial derivative and rotation curvature are also independently invariant under Schrödinger symmetries,

$$D_a\varphi D_a\varphi\,,\qquad \varphi\triangle\varphi\,,\qquad \varphi^2 R(J)\,,\qquad D_a\varphi D_b\varphi R^{ab}(J)\,,$$
$$\varphi D_a D_b\varphi R^{ab}(J),\quad \varphi^2 R_{ab}(J)R_{ab}(J)\,,\quad \varphi^2 R_{abcd}(J)R_{abcd}(J)\,.$$

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$$\varphi D_a D_b\varphi R^{ab}(J), \quad \varphi^2 R_{ab}(J)R_{ab}(J)\,, \quad \varphi^2 R_{abcd}(J)R_{abcd}(J)\,.$$

After the gauge fixing $\varphi = 1$,

$$\begin{aligned}
&b\cdot b\,, \quad \mathcal{R}(J)\,, && 2-\text{derivative}\,,\\
&(b\cdot b)^2\,, \quad (\mathcal{D}\cdot b)^2\,, \quad (\mathcal{D}\cdot b)b\cdot b\,, && 4-\text{derivative}\,,\\
&\mathcal{R}(J)^2\,, \quad \mathcal{R}(J)b\cdot b\,, \quad \mathcal{R}(J)\mathcal{D}\cdot b\,, && 4-\text{derivative}\,,\\
&\mathcal{R}_{ab}^2(J)\,, \quad \mathcal{R}_{ab}(J)b^a b^b\,, \quad \mathcal{R}_{ab}(J)\mathcal{D}^a b^b\,, && 4-\text{derivative}\,,\\
&\mathcal{R}_{abcd}^2(J)\,, && 4-\text{derivative}\,.
\end{aligned}$$

Summary

- Schrödinger invariant complex/real scalar field theories with at most two time, and four spatial derivatives lead to Galilean invariants (2nd order in time) theory.
- We showed how the Hořava-Lifshitz gravity at $z = 2$ and the Newton-Cartan gravity can be embedded into this construction.
- Classical vacuum and (Black-hole) solutions.
- Non-relativistic holography.
- Non-relativistic hydrodynamics.
- Non-relativistic supergravity.

Summary and remarks

- Schrödinger invariant complex/real scalar field theories with at most two time, and four spatial derivatives lead to Galilean invariants (2nd order in time) theory.
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Thanks — متشكرم