Effective (string) field theory tomography

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This talk is about:

- Exact conserved correlators
- UV and IR limit
- Effective action tomography
- Higher spin fields
- Tensionless strings
- SFT tomography (?)

Why are CFTs important?

- at very high energies masses should become unimportant, thus at such energies the relevant field theory should be scale invariant $\rightarrow$ conformal invariant.
- CFTs are the interface of gravity in AdS/CFT correspondence
- CFTs are relevant to strongly correlated systems
- from a theoretical point of view, CFTs can be solved (even if they are not supersymmetric and or not Lagrangian)
- CFTs say a lot about gravity

In particular why are e.m. tensor correlators so important?

Because the source of the e.m. tensor is the metric (or, better, the metric fluctuations). Thus the e.m. tensor correlators can be interpreted as scattering amplitudes for gravitons in the framework of AdS/CFT.

(C.Closset,D.Dumitrescu,G.Festuccia, Z.Komargodski,N.Seiberg,X.Camanho,J.Edelstein, J.Maldacena, G.Pimentel, A.Zhiboedov,…)

…but this consideration can be generalized to any current!

Not only that. Not only conformal correlators are important and interesting. See below

The conformal Lie algebra

The Lie algebra generators of the conformal group

$$P_\mu = -i\partial_\mu$$
$$D = -ix^\mu\,\partial_\mu$$
$$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$$
$$K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$$

Commutators (extra Poincaré)

$$[P^\mu, D] = iP^\mu$$
$$[K^\mu, D] = -iK^\mu$$
$$[P^\mu, K^\nu] = 2i\eta^{\mu\nu}D + 2iL^{\mu\nu}$$
$$[K^\mu, K^\nu] = 0$$
$$[L^{\mu\nu}, D] = 0$$
$$[L^{\mu\nu}, K^\lambda] = i\eta^{\lambda\mu}K^\nu - i\eta^{\lambda\nu}K^\mu$$

Representations of the conformal Lie algebra

For a generic tensor field $O(x)$ of weight Δ

$$
\begin{aligned}
i[P_\mu, O(x)] &= \partial_\mu O(x) \\
i[L_{\mu\nu}, O(x)] &= (x_\mu \partial_\nu - x_\nu \partial_\mu) O(x) + i\Sigma_{\mu\nu} O(x) \\
i[D, O(x)] &= (\Delta + x^\mu \partial_\mu) O(x) \\
i[K_\mu, O(x)] &= (2\Delta x_\mu + 2x_\mu x^\lambda \partial_\lambda - x^2 \partial_\mu - 2i x^\lambda \Sigma_{\lambda\mu}) O(x)
\end{aligned}
$$

For the e.m. tensor, in particolar,

$$
\begin{aligned}
i[D, T_{\mu\nu}(x)] &= (d + x^\lambda \partial_\lambda) T_{\mu\nu}(x) \\
i[K_\lambda, T_{\mu\nu}] &= \left(2\Delta x_\lambda + 2x_\lambda\, x\cdot\partial - x^2 \partial_\lambda\right) T_{\mu\nu} \\
&\quad + 2\left(x^\alpha T_{\alpha\nu} \eta_{\lambda\mu} + x^\alpha T_{\mu\alpha} \eta_{\lambda\nu} - x_\mu T_{\lambda\nu} - x_\nu T_{\mu\lambda}\right)
\end{aligned}
$$

Free massive fermion model in 3d

Action

$$S = \int d^3x \, \left[i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi \right], \quad D_\mu = \partial_\mu + A_\mu$$

where $A_\mu = A_\mu^a(x) T^a$ and T^a are the generators of a gauge algebra. The generators are antihermitean, $[T^a, T^b] = f^{abc} T^c$, with normalization $\mathrm{tr}(T^a T^b) = \mathrm{n}\delta^{ab}$. The current

$$J_\mu^a(x) = \bar{\psi}\gamma_\mu T^a \psi$$

is (classically) covariantly conserved on shell

$$(DJ)^a = (\partial^\mu \delta^{ac} + f^{abc} A^{b\mu}) J_\mu^c = 0$$

(see also Dunne,Babu,Das, Panigrahi)

Also Vuorio, Giombi, Minwalla, Prakash, Trivedi, Yin, Wadia

Free massive fermion model in 3d (cont.)

The effective action is given by

$$W[A] \;=\; \sum_{n=1}^{\infty} \frac{i^{n+1}}{n!} \int \prod_{i=1}^{n} d^3x_i A^{a_1\mu_1}(x_1)\dots A^{a_n\mu_n}(x_n)\langle 0|\mathcal{T} J^{a_1}_{\mu_1}(x_1)\dots J^{a_n}_{\mu_n}(x_n)|0\rangle$$

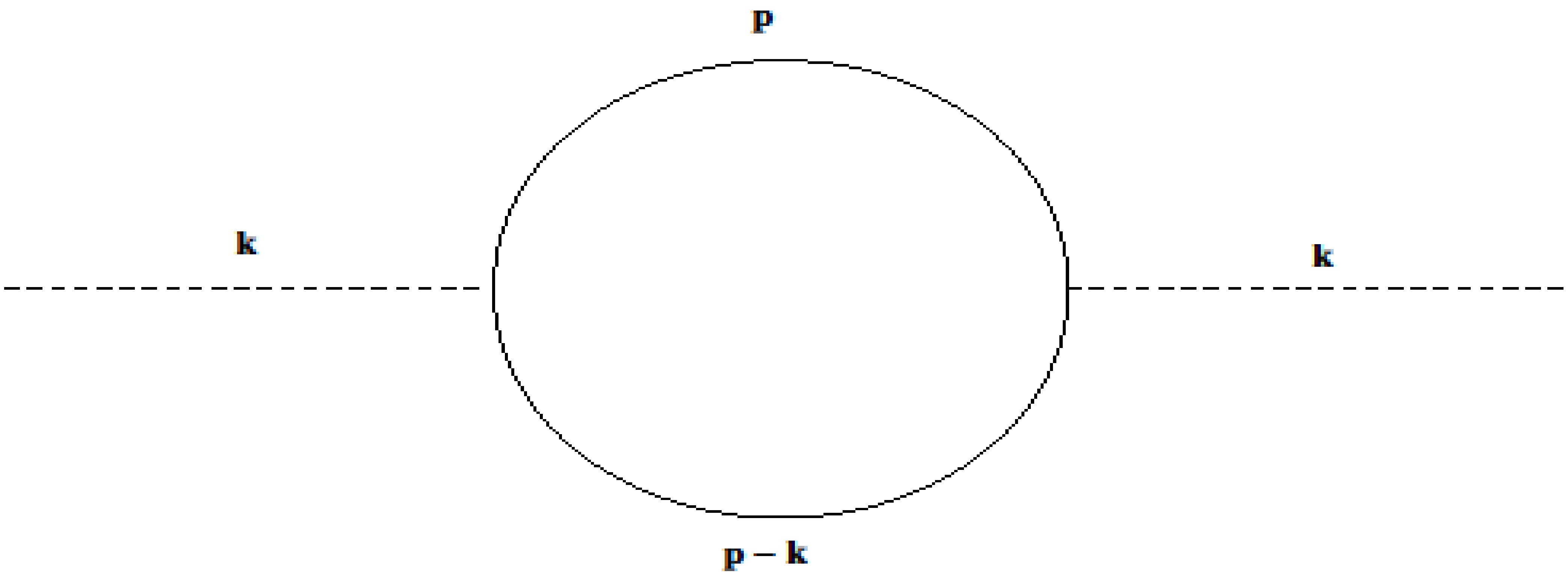
We will consider 2-pt and 3-pt current correlators,

$$\langle 0|\mathcal{T} J^a_\mu(x) J^b_\nu(y)|0\rangle, \quad \text{and} \quad \langle 0|\mathcal{T} J^a_\mu(x) J^b_\nu(y) J^b_\lambda(z)|0\rangle \tag{1}$$

whose Fourier transform are $\tilde{J}^{ab}_{\mu\nu}(k)$ and $\tilde{J}^{abc}_{\mu\nu\lambda}(k_1,k_2)$. The one-loop conservation law in momentum space is

$$\begin{aligned} &k^\mu \tilde{J}^{ab}_{\mu\nu}(k) = 0 \\ &-iq^\mu \tilde{J}^{abc}_{\mu\nu\lambda}(k_1,k_2) + f^{abd}\tilde{J}^{dc}_{\nu\lambda}(k_2) + f^{acd}\tilde{J}^{db}_{\lambda\nu}(k_1) = 0 \end{aligned}$$

where $q = k_1 + k_2$.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Free massive fermion model:2-pt

The 2-pt function is

$$\tilde{J}^{ab(odd)}_{\mu\nu}(k) \quad = \quad \frac{\mathrm{n}}{2\pi}\delta^{ab}\epsilon_{\mu\nu\sigma}k^\sigma \frac{m}{k} \arctan\frac{k}{2m}$$

where $k = \sqrt{k^2} = E$. The IR and UV limit correspond to $\frac{m}{E} \to \infty$ and 0, respectively. We get

$$\tilde{J}^{ab(odd)}_{\mu\nu}(k) = \frac{\mathrm{n}}{2\pi}\delta^{ab}\epsilon_{\mu\nu\sigma}k^\sigma \begin{cases} \frac{1}{2} & \mathrm{IR} \\ \frac{\pi}{2}\frac{m}{k} & \mathrm{UV} \end{cases}$$

Fourier anti-transforming and substituting in $W(A)$ one gets

$$\int d^3x \epsilon^{\mu\nu\lambda} A^a_\mu \partial_\nu A^a_\lambda$$

E.m. tensor correlators

Next come the e.m. tensor correlator. It is naturally coupled to the metric. The action in the massive model is

$$S = \int d^3x\, e \left[i\bar{\psi} E_a^{\mu} \gamma^a \nabla_\mu \psi - m\bar{\psi}\psi \right], \quad \nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}, \quad \Sigma^{bc} = \frac{1}{4}\left[\gamma^b, \gamma^c\right].$$

The mass term breaks parity!

The energy-momentum tensor

$$T^{\mu\nu} = \frac{i}{4}\bar{\psi}\left(E_a^{\mu}\gamma^a \overleftrightarrow{\nabla}^{\nu} + \mu \leftrightarrow \nu \right)\psi.$$

is covariantly conserved (on shell): $\nabla_\mu T^{\mu\nu} = 0$.

At quantum level (the Fourier trasform of) the 2-pt correlator is

$$\tilde{T}^{(odd)}_{\mu\nu\lambda\rho}(k) \quad = \quad \frac{m}{256\pi}\,\epsilon_{\sigma\nu\rho}\, k^{\sigma} \left[2m \left(\eta_{\mu\lambda} - \frac{k_\mu k_\lambda}{k^2} \right) + \left(\eta_{\mu\lambda} + \frac{k_\mu k_\lambda}{k^2} \right) \frac{k^2 + 4m^2}{|k|} \arctan \frac{|k|}{2m} \right].$$

In the effective action the e.m. tensor couples to $h_{\mu\nu}$, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \ldots$.

Gravitational CS

In the IR and UV limit this corresponds to the action term

$$S_{\mathrm{eff}}^{\mathrm{P-odd}} = \frac{\kappa}{192\pi} \int d^3x\, \epsilon_{\sigma\nu\rho}\, h^{\mu\nu}\, \partial^\sigma (\partial_\mu \partial_\lambda - \eta_{\mu\lambda}\Box)\, h^{\lambda\rho} \qquad (1)$$

This is nothing but the lowest order expansion in $h_{\mu\nu}$ of the gravitational Chern-Simons action in 3d.

$$CS \quad = -\tfrac{\kappa}{96\pi} \int d^3x\, \epsilon^{\mu\nu\lambda} \left(\partial_\mu \omega_\nu^{ab} \omega_{\lambda ba} + \tfrac{2}{3} \omega_{\mu a}{}^b \omega_{\nu b}{}^c \omega_{\lambda c}{}^a\right)$$

- In the IR limit we find $\kappa = 1$ (the action is well defined)
- In the UV limit $\kappa = \frac{3}{2}\pi \frac{m}{|k|}$. So again the limit vanishes unless we consider N flavours, in which case we can take the scaling limit that leaves $\lambda = N\frac{m}{|k|}$ fixed.

Higher spin currents

In the massive fermion model in 3d we have other conserved currents. The next after the em tensor is the third order current

$$J_{\mu_1\mu_2\mu_3} = \bar\psi\gamma_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3)}\psi - \frac{5}{3}\partial_{(\mu_1}\bar\psi\gamma_{\mu_2}\partial_{\mu_3)}\psi + \frac{1}{3}\eta_{(\mu_1\mu_2}\partial^\sigma\bar\psi\gamma_{\mu_3)}\partial_\sigma\psi - \frac{m^2}{3}\eta_{(\mu_1\mu_2}\bar\psi\gamma_{\mu_3)}\psi$$

This is conserved (on-shell). We consider the external source $B^{\mu\nu\lambda}$ and couple it to the theory via the action term

$$\int d^3x J_{\mu\nu\lambda}B^{\mu\nu\lambda}$$

Due to current conservation this coupling is invariant under the (infinitesimal) transformations

$$\delta B_{\mu\nu\lambda} = \partial_{(\mu}\Lambda_{\nu\lambda)}$$

In the limit $m \to 0$ we have also invariance under the transformation

$$\delta B_{\mu\nu\lambda} = \Lambda_{(\mu}\eta_{\nu\lambda)}$$

which induces the tracelessness of $J_{\mu\nu\lambda}$ in any couple of indices.

2-pt B correlator

We can construct the effective action for $B_{\mu\nu\lambda}$ with

$$W[B] = \sum_{n=1}^{\infty} \frac{i^{n+1}}{n!} \int \prod_{i=1}^{n} d^3x_i B^{\mu_i\nu_i\lambda_i}(x_1)\langle 0|T J_{\mu_1\nu_1\lambda_1}(x_1)\dots J_{\mu_n\nu_n\lambda_n}(x_n)|0\rangle .$$

by computing the n-pt functions. For instance, the 2-pt correlator (after subtraction), in the IR is

$$\tilde{J}^{(odd,IR)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = \frac{1}{4\pi}\epsilon_{\mu_1\nu_1\sigma}k^{\sigma}\Big[\frac{1}{60}k^4\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3} - \frac{8}{135}k^4\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}$$
$$- \frac{1}{60}k^2\left(k_{\nu_2}k_{\nu_3}\eta_{\mu_2\mu_3} + k_{\mu_2}k_{\mu_3}\eta_{\nu_2\nu_3}\right) + \frac{16}{135}k^2 k_{\mu_2}k_{\nu_2}\eta_{\mu_3\nu_3} - \frac{23}{540}k_{\mu_2}k_{\mu_3}k_{\nu_2}k_{\nu_3}\Big]$$

and in the UV

$$\tilde{J}^{(odd,UV)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = \frac{1}{4}\frac{m}{|k|}\,\epsilon_{\mu_1\nu_1\sigma}k^{\sigma}\Big[\frac{1}{12}k_{\mu_2}k_{\mu_3}k_{\nu_2}k_{\nu_3} - \frac{2}{9}k^2 k_{\mu_3}k_{\nu_3}\eta_{\mu_2\nu_2}$$
$$+\frac{k^2}{36}\left(k_{\nu_2}k_{\nu_3}\eta_{\mu_2\mu_3} + k_{\mu_2}k_{\mu_3}\eta_{\nu_2\nu_3}\right) + \frac{1}{9}k^4\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} - \frac{1}{36}k^4\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}\Big].$$

They are both conserved.

Odd effective action

The UV expression corresponds to the effective action term

$$\sim \int d^3x \quad \epsilon_{\mu_1\nu_1\sigma}\Big[3\partial^\sigma B^{\mu_1\mu_2\mu_3}\partial_{\mu_2}\partial_{\mu_3}\partial_{\nu_2}\partial_{\nu_3}B^{\nu_1\nu_2\nu_3} - 8\partial^\sigma B^{\mu_1\mu_2\mu_3}\Box\partial_{\mu_3}\partial_{\nu_3}B^{\nu_1\nu_3}{}_{\mu_2}$$
$$+2\partial^\sigma B^{\mu_1\lambda}{}_\lambda\Box\partial_{\nu_2}\partial_{\nu_3}B^{\nu_1\nu_2\nu_3} + 4\partial^\sigma B^{\mu_1\mu_2\mu_3}\Box^2 B^{\nu_1}{}_{\mu_2\mu_3}$$
$$-\partial^\sigma B^{\mu_1\lambda}{}_\lambda\Box^2 B^{\nu_1\rho}{}_\rho\Big]$$

where $b_{\mu\nu\lambda} = B_{\mu\nu\lambda} + \ldots$.

This is a slight generalization of an action proposed by Pope and Townsend (1989)

$$\sim \int d^3x \qquad \epsilon_{\mu_1\nu_1\sigma}\Big[\frac{3}{2}\partial^\sigma h^{\mu_1\mu_2\mu_3}\partial_{\mu_2}\partial_{\mu_3}\partial_{\nu_2}\partial_{\nu_3}h^{\nu_1\nu_2\nu_3} - 4\partial^\sigma h^{\mu_1\mu_2\mu_3}\Box\partial_{\mu_3}\partial_{\nu_3}h^{\nu_1\nu_3}{}_{\mu_2}$$
$$+2\partial^\sigma h^{\mu_1\mu_2\mu_3}\Box^2 h^{\nu_1}{}_{\mu_2\mu_3}\Big]$$

one can see that they are equal if we set $B^{\mu\lambda}{}_\lambda = 0$

YM in effective action

The IR limit of the 2pt current correlator is given by

$$\tilde{J}^{ab(even)}_{\mu\nu}(k) = \frac{i}{4\pi}\frac{1}{3|m|}\delta^{ab}(k_\mu k_\nu - k^2\eta_{\mu\nu})$$

which is local. Fourier anti-transforming it and inserting it in the formula for the EA

$$S \sim \frac{1}{m}\int d^3x \; (A^a_\mu \partial^\mu \partial^\nu A^a_\nu - A^a_\nu \Box A^{a\nu})$$

which is the lowest term in the expansion of the YM action

$$S_{YM} = \frac{1}{g}\int d^3x \,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu})$$

where $g \sim m$.

EH in the effective action

Now let us go to the IR limit of the even part of the 2pt e.m. tensor correlator.

$$\langle T_{\mu\nu}(k) T_{\lambda\rho}(-k)\rangle^{IR}_{even} = \frac{i|m|}{96\pi}\Big[\frac{1}{2}\left((k_\mu k_\lambda \eta_{\nu\rho} + \lambda \leftrightarrow \rho) + \mu \leftrightarrow \nu\right) - \\ - (k_\mu k_\nu \eta_{\lambda\rho} + k_\lambda k_\rho \eta_{\mu\nu}) - \frac{k^2}{2}\left(\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda}\right) + k^2 \eta_{\mu\nu}\eta_{\lambda\rho}\Big].$$

This is a local expression multiplied by $|m|$. Fourier anti-transforming it gives rise to the action

$$S \sim |m| \int d^3x \left(-2\partial_\mu h^{\mu\lambda}\partial_\nu h^\nu_\lambda - 2h\, \partial_\mu \partial_\nu h^{\mu\nu} - h^{\mu\nu}\Box h_{\mu\nu} + h\Box h\right)$$

This is the lowest order term in the expansion of the EH action:

$$S_{EH} = \frac{1}{2\kappa}\int d^3x \sqrt{g}\, R$$

where $\kappa \sim \frac{1}{|m|}$.

...and for spin 3

For instance, for the field B one can extract from the even 2-pt function the following action

$$S_B \sim \int \Big((\partial_\mu B_{\mu_1\mu_2\mu_3})^2 - 3(\partial\cdot B_{\mu_1\mu_2})^2 - 3\partial\cdot\partial\cdot B^\mu \frac{1}{\Box}\partial\cdot\partial\cdot B_\mu - \partial\cdot\partial\cdot\partial\cdot B \frac{1}{\Box^2}\partial\cdot\partial\cdot\partial\cdot B \Big)$$

This gives the equation of motion

$$\Box B_{\mu\nu\lambda} - \partial_{\underline{\mu}}\partial\cdot B_{\underline{\nu\lambda}} + \frac{1}{\Box}\partial_{\underline{\mu}}\partial_{\underline{\nu}}\partial\cdot\partial\cdot B_{\underline{\lambda}} - \frac{1}{\Box^2}\partial_\mu\partial_\nu\partial_\lambda\partial\cdot\partial\cdot\partial\cdot B = 0$$

Now take the trace with respect to any two indices of this equation and you will get

$$\partial\cdot\partial\cdot B_\lambda = \Box B'_\lambda - \partial_\lambda\partial\cdot B' + \frac{1}{\Box}\partial_\lambda\partial\cdot\partial\cdot\partial\cdot B$$

Upon replacing this into the third term above one gets precisely the nonlocal Fronsdal equation for massless spin 3.

Fronsdal eqs for higher spins

The Fronsdal equation of motion for a (completely symmetric) spin 3 field $\varphi_{\mu\nu\lambda}$ is the following:

$$\mathcal{F}_{\mu\nu\lambda} \equiv \Box\varphi_{\mu\nu\lambda} - \partial_{\underline{\mu}}\partial\cdot\varphi_{\underline{\nu\lambda}} + \partial_{\underline{\mu}}\partial_{\underline{\nu}}\varphi'_{\underline{\lambda}} = 0$$

where $\partial_{\underline{\mu}}\partial\cdot\varphi_{\underline{\nu\lambda}} = \partial_\mu\partial\cdot\varphi_{\nu\lambda} + \mathrm{perm.},$. Under $\delta\varphi_{\mu\nu\lambda} = \partial_\mu\Lambda_{\nu\lambda} + \mathrm{perm.}$

$$\delta\mathcal{F}_{\mu\nu\lambda} = 3\partial_\mu\partial_\nu\partial_\lambda\Lambda'$$

So covariance requires tracelessness $\Lambda' = 0$. Unnatural!
D.Francia and A.Sagnotti (2002) proposed a way out via nonlocality.

$$\mathcal{F}_{\mu\nu\lambda} - \frac{1}{\Box^2}\partial_\mu\partial_\nu\partial_\lambda\partial\cdot\mathcal{F}' = 0$$

This is invariant, but nonlocal. However nonlocality is irrelevant (a gauge artifact). This can be seen via a compensator.

The compensator

We can rewrite the non-local Fronsdal equation as

$$\mathcal{F}_{\mu\nu\lambda} \equiv \Box\varphi_{\mu\nu\lambda} - \partial_{\underline{\mu}}\partial\cdot\varphi_{\underline{\nu\lambda}} + \partial_{\underline{\mu}}\partial_{\underline{\nu}}\varphi'_{\underline{\lambda}} = \partial_\mu\partial_\nu\partial_\lambda\alpha$$

where

$$\alpha = \frac{3}{\Box}\partial\cdot\varphi' - \frac{2}{\Box^2}\partial\cdot\partial\cdot\partial\cdot\varphi$$

The field α is called compensator, because its transformation property under $\delta\varphi = 3\partial\Lambda$ is

$$\delta\alpha = 3\Lambda' \qquad \longrightarrow \qquad \delta\mathcal{F} = 3\partial^3\Lambda'$$

It allows to write a local Lagrangian. So, the nonlocality of the initial equation is only a gauge tail which serves to gaurantee covariance.

For HS theories, see:

Vasiliev, Prokushkin, Metsaev,...
Bekaert, Young, Mourad, Francia,
Iazeolla, Sagnotti, Campoleoni,
Fredenhagen, Fotopoulos, Tsulaia,
Taronna,...

So what happens?

In the effective action of a massive 3d fermion we have found all the local action for spin 1, 2, 3: YM, CS, EH, Fronsdal, Pope-Townsend,….

Is there more?

Yes. Fortunately we can compute the effective action of a 3d fermion exactly for any current.

Davydychev et al. method

For 2pt functions, we have to compute

$$J^{(2)}_{\mu_1\dots\mu_M}(d;\alpha,\beta;m) = \int \frac{d^d p}{(2\pi)^d} \frac{p_{\mu_1}\cdots p_{\mu_M}}{(p^2-m^2)^\alpha((p-k)^2-m^2)^\beta}.$$

The first step is to reduce it to scalar integrals (Davydychev, Boos-Davydychev, 1991-92)

$$J^{(2)}_{\mu_1\dots\mu_M}(d;\alpha,\beta;m) = \sum_{\substack{\lambda,\kappa_1,\kappa_2 \\ 2\lambda+\sum\kappa_i=M}} \left(-\frac{1}{2}\right)^\lambda (4\pi)^{M-\lambda} \left\{[\eta]^\lambda [q_1]^{\kappa_1} [q_2]^{\kappa_2}\right\}_{\mu_1\dots\mu_M}$$
$$\times (\alpha)_{\kappa_1}(\beta)_{\kappa_2} I^{(2)}(d+2(M-\lambda);\alpha+\kappa_1,\beta+\kappa_2;m).$$

Davydychev et al. method (cont.)

Let us call

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^\alpha ((p-k)^2 - m^2)^\beta} \equiv I^{(2)}(d; \alpha, \beta)$$

then, for instance, the 2pt function for 1-currents is

$$\begin{aligned} \tilde{J}_{\mu\nu}(k) \quad = \quad & -\frac{8\pi}{(2\pi)^{6+\delta}} \eta_{\mu\nu}\, I^{(2)}(6+\delta; 1, 1) + 8 \frac{(4\pi)^2}{(2\pi)^{8+\delta}} k_\mu k_\nu\, I^{(2)}(8+\delta; 1, 3) \\ & + \frac{16\pi}{(2\pi)^{6+\delta}} k_\mu k_\nu\, I^{(2)}(6+\delta; 1, 2) + \frac{1}{(2\pi)^{4+\delta}} k_\mu k_\nu\, I^{(2)}(4+\delta; 1, 1) \end{aligned}$$

If we want to know the behaviour in the IR these integrals are given by:

$$\begin{aligned} & I^{(2)}(d; \alpha, \beta) \\ & = \pi^{\frac{d}{2}} i^{1-d} (-m^2)^{d/2-\alpha-\beta} \frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha+\beta)}\, {}_3F_2 \left(\left. \begin{matrix} \alpha, \beta, \alpha+\beta-d/2 \\ (\alpha+\beta)/2, (\alpha+\beta+1)/2 \end{matrix} \right| \frac{k^2}{4m^2} \right) \end{aligned}$$

Spin s currents

The spin s current has the form

$$J^{(s)}_{\mu_1\ldots\mu_s} = \bar{\psi}\gamma_{(\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_s)}\psi + \ldots\ldots$$

We couple it to an external source $a^{\mu_1\cdots\mu_s}$ through the term $\int d^3x\, a^{\mu_1\cdots\mu_s}(x)J^{(s)}_{\mu_1\ldots\mu_s}(x)$, compute the 2-pt function $\langle|\mathcal{T}J^{(s)}_{\mu_1\ldots\mu_s}J^{(s)}_{\nu_1\ldots\nu_s}|0\rangle$, and insert into the generating function

$$W[a,s] = \sum_{n=1}^{\infty}\frac{i^{n+1}}{n!}\int\prod_{i=1}^{n}d^3x_i\, a^{\mu_{11}\cdots\mu_{1p}}(x_1)\ldots a^{\mu_{1n}\cdots\mu_{sn}}(x_n)$$
$$\times\langle 0|\mathcal{T}J^{(s)}_{\mu_{11}\ldots\mu_{1s}}(x_1)\ldots J^{(s)}_{\mu_{n1}\ldots\mu_{ns}}(x_n)|0\rangle.$$

to obtain the effective action. In particular $a_\mu = A_\mu$, $a_{\mu\nu} = h_{\mu\nu}$ and $a_{\mu\nu\lambda} = b_{\mu\nu\lambda}$.

Exact 2pt correlator for the e.m. tensor

The correlator is

$$\frac{i}{192\pi k}\Bigg(\left(96m^4\coth^{-1}\left(\frac{2m}{k}\right)-48km^3-4k^3m-6k^4\coth^{-1}\left(\frac{2m}{k}\right)\right)\left(n_1\cdot\pi^{(k)}\cdot n_2\right)$$

$$+\left(48m^4\coth^{-1}\left(\frac{2m}{k}\right)-24km^3-24k^2m^2\coth^{-1}\left(\frac{2m}{k}\right)+10k^3m\right.$$

$$\left.+3k^4\coth^{-1}\left(\frac{2m}{k}\right)\right)(n_1\cdot\pi^{(k)}\cdot n_1)(n_2\cdot\pi^{(k)}\cdot n_2)\Bigg)$$

where we use the projector $\pi^{(k)}$ and the compact notation:

$$\pi^{(k)}_{\mu\nu}=\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2},\qquad (n_1\cdot\pi^{(k)}\cdot n_2)=\pi^{(k)}_{\mu\nu}n_1^\mu n_2^\nu$$

after subtracting

$$\mathcal{O}(m^3):\ \frac{im^3}{3\pi}\left((n_1\cdot n_2)^2+(n_1\cdot n_1)(n_2\cdot n_2)\right)$$

which is not conserved, but local.

Correlator tomography

Expanding in powers of m

$$\mathcal{O}(m^2): \quad 0$$

$$\mathcal{O}(m): \quad \frac{im}{12\pi}k^2\left(\left(n_1\cdot\pi^{(k)}\cdot n_2\right)^2 - (n_1\cdot\pi^{(k)}\cdot n_1)(n_2\cdot\pi^{(k)}\cdot n_2)\right)$$

$$\mathcal{O}(m^0): \quad 0$$

$$\mathcal{O}(m^{-1}): \quad -\frac{i}{80\pi m}k^4\left(\left(n_1\cdot\pi^{(k)}\cdot n_2\right)^2 - \frac{1}{3}(n_1\cdot\pi^{(k)}\cdot n_1)(n_2\cdot\pi^{(k)}\cdot r$$

$$\mathcal{O}(m^{-2}): \quad 0$$

These terms are all conserved. The $\mathcal{O}(m)$ term is the linearized version of the EH equation of motion. All the other terms differ only by pure gauge parts.

Correlator tomography for spin s

For spin s the 2-pt correlators can be calculated exactly. After subtracting some local terms, their structure is a generalization of the one for the e.m. tensor. It is a superposition with k, m-dependent coefficient of

$$\tilde{E}^{(s)}(k, n_1, n_2) = \sum_{l=0}^{[s/2]} a_l \tilde{A}_l^{(s)}(k, n_1, n_2)$$

where a_l are numbers, and

$$\begin{aligned}
\tilde{A}_0^{(s)}(k, n_1, n_2) &= \frac{1}{(s!)^2}(n_1 \cdot \pi^{(k)} \cdot n_2)^s \\
\tilde{A}_1^{(s)}(k, n_1, n_2) &= \frac{1}{(s!)^2}(n_1 \cdot \pi^{(k)} \cdot n_2)^{s-2}(n_1 \cdot \pi^{(k)} \cdot n_1)(n_2 \cdot \pi^{(k)} \cdot n_2) \\
\ldots &\qquad \ldots\ldots \\
\tilde{A}_l^{(s)}(k, n_1, n_2) &= \frac{1}{(s!)^2}(n_1 \cdot \pi^{(k)} \cdot n_2)^{s-2l}(n_1 \cdot \pi^{(k)} \cdot n_1)^l(n_2 \cdot \pi^{(k)} \cdot n_2)^l \\
\ldots &\qquad \ldots\ldots
\end{aligned}$$

General eom's

We can represent the eom symbolically as

$$k^2 \sum_{l=0}^{[s/2]} a_l \tilde{A}_l^{(s)}(k, n_1, n_2) = 0 \qquad (1)$$

But one can show that only the term $l = s/2]$ matters, so in fact the eom can be reduced to

$$k^2 (n_1 \cdot \pi^{(k)} \cdot n_2)^s = 0 \qquad (2)$$

This is the nonlocal Fronsdal equation for spin s.

Therefore the 2-pt function of spin s currents contains in particular the information of the spin s Fronsdal equation, or, correspondingly, its Lagrangian.

Example in 4d

In 4d (in any even dimension) one has an additional problem of regularization. The way out is to do the calculations for $d = 4 + \delta$. For instance in the case of the e.m. tensor the 2pt function in the IR takes the form

$$\begin{aligned}
\tilde{T}_{\mu\nu\lambda\rho}(k) \quad = \quad & -\frac{i}{16(2\pi)^2} m^4 (2\eta_{\mu\nu}\eta_{\lambda\rho} + \eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda}) \left(-\frac{1}{4\delta} + \frac{3}{16} - \frac{1}{8} \left(\gamma + \log \frac{4\pi}{m^2} \right) \right. \\
& -\frac{i}{16(2\pi)^2} m^2 \left[\frac{1}{2} \left((k_\mu k_\lambda \eta_{\nu\rho} + \lambda \leftrightarrow \rho) + \mu \leftrightarrow \nu \right) - \right. \\
& \quad - (k_\mu k_\nu \eta_{\lambda\rho} + k_\lambda k_\rho \eta_{\mu\nu}) - \frac{k^2}{2} (\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda}) + k^2 \eta_{\mu\nu}\eta_{\lambda\rho} \Big] \\
& \cdot \left(\frac{1}{6\delta} + \frac{1}{12} - \frac{1}{12} (\gamma + \log 4\pi - \log m^2) \right)
\end{aligned}$$

The term proportional to m^4 is clearly not conserved, but is local and can be subtracted. The term proportional to m^2 corresponds to the lowest order term of the Einstein-Hilbert action, with a coupling

$$\frac{i}{16(2\pi)^2} m^2 \left(\frac{1}{6\delta} + \frac{1}{12} - \frac{1}{12} (\gamma + \log 4\pi - \log m^2) \right)$$

Temporary conclusion:

Free field theories generate one-loop effective actions which contain information (action, eom,…) about a very large spectrum of (if not all) local field theories physicists have been able to invent.

This may be thought of as a form of duality:

Free field theory $\leftrightarrow$ Higher spin field theories

But the higher spin theories so far are in linearized form. Is the correspondence only valid for free higher spin theories?

The answer is: no! The correspondence extends also to interactions.

Let us consider some examples

Free massive fermion model:3-pt

The 3-pt function is more complicated

$$\tilde{J}^{1,abc}_{\mu\nu\lambda}(k_1,k_2) = i\int \frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left(\gamma_\mu T^a \frac{1}{\not p - m}\gamma_\nu T^b \frac{1}{\not p - \not k_1 - m}\gamma_\lambda T^c \frac{1}{\not p - \not q - m}\right)$$

The result is a generalized Lauricella function (Boos,Davydychev). In the IR we find

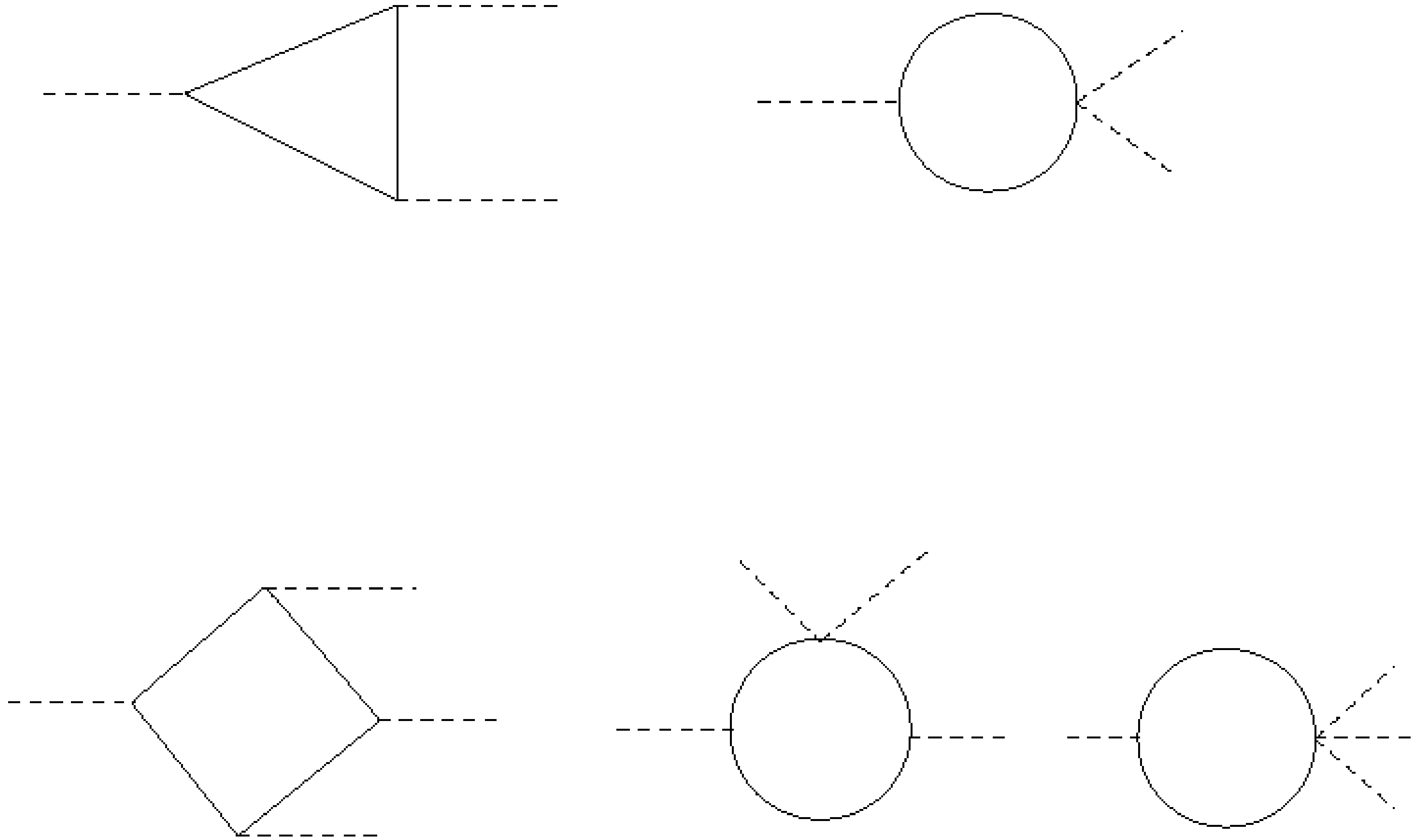
$$\tilde{J}^{1,abc(odd)}_{\mu\nu\lambda}(k_1,k_2) \approx i\frac{\mathrm{n}}{32\pi}\sum_{n=0}^{\infty}\left(\frac{\sqrt{E}}{m}\right)^{2n} f^{abc}\tilde{I}^{(2n)}_{\mu\nu\lambda}(k_1,k_2)$$

and, in particular,

$$I^{(0)}_{\mu\nu\lambda}(k_1,k_2) = -6\epsilon_{\mu\nu\lambda}$$

which corresponds to the action term

$$\sim \int d^3x\, \epsilon^{\mu\nu\lambda} f^{abc} A^a_\mu A^b_\nu A^c_\lambda$$



Lauricella hypergeometric function

Basic integral

$$J_3(\alpha,\beta,\gamma;m) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2-m^2)^\alpha((p-k_1)^2-m^2)^\beta((p-q)^2-m^2)^\gamma}$$

This can be transformed into

$$\begin{aligned} J_3(\alpha,\beta,\gamma;m) \quad &= \quad \frac{i^{1-d}}{(4\pi)^{\frac{d}{2}}}(-m^2)^{\frac{d}{2}-\alpha-\beta-\gamma}\frac{\Gamma(\alpha+\beta+\gamma-\frac{d}{2})}{\Gamma(\alpha+\beta+\gamma)} \\ &\quad \Phi_3\left[\begin{matrix}\alpha+\beta+\gamma-\frac{d}{2},\alpha,\beta,\gamma \\ \alpha+\beta+\gamma\end{matrix}\middle| \frac{k_1^2}{m^2},\frac{q^2}{m^2},\frac{k_2^2}{m^2}\right]\end{aligned}$$

where Φ_3 is a generalized Lauricella function $\Big((a)_n = \Gamma(a+n)/\Gamma(n)\Big)$:

$$\begin{aligned}&\Phi_3\left[\begin{matrix}a_1,a_2,a_3,a_4 \\ c\end{matrix}\middle| z_1,z_2,z_3\right] \\ &= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{(a_1)_{j_1+j_2+j_3}(a_2)_{j_1+j_2}(a_3)_{j_1+j_3}(a_4)_{j_2+j_3}}{(c)_{2j_1+2j_2+2j_3}}\end{aligned}$$

Free massive fermion model:3-pt (cnt.)

Putting things together we find the effective CS action

$$CS = \frac{\kappa}{4\pi}\int d^3x \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

- In the IR $\kappa = 1$, so the CS action is invariant also under large gauge transformation.
- - In the UV things are more complicated. Eventually we get the same action with $\kappa = \pi\frac{m}{k}$. So, the UV limit is 0.
  - If ψ carries a flavour index $i = 1,\ldots,N$, the previous result is multiplied by N, and $\kappa = \pi N\frac{m}{k}$. So we can consider the scaling limit $N\to\infty$, $\frac{m}{k}\to 0$ and κ fixed and finite.

Important! Both 2-pt and 3-pt correlator satisfy the WI's of CFT (and they are pure contact term)!

A note on conservation

For the 3-pt correlator of the massive theory we have

$$
\begin{aligned}
q^{\mu}\,\tilde{J}^{abc}_{\mu\nu\lambda}(k_1,k_2) \quad &= \quad -\frac{i}{4\pi} f^{abc}\epsilon_{\lambda\nu\sigma}k_1^{\sigma}\,\frac{2m}{k_1}\mathrm{arccot}\left(\frac{2m}{k_1}\right) \\
&\quad -\frac{i}{4\pi} f^{abc}\epsilon_{\lambda\nu\sigma}k_2^{\sigma}\,\frac{2m}{k_2}\mathrm{arccot}\left(\frac{2m}{k_2}\right) \neq 0
\end{aligned}
$$

but

$$
\begin{aligned}
&-iq^{\mu}\,\tilde{J}^{(odd)abc}_{\mu\nu\lambda}(k_1,k_2) + f^{abd}\,\tilde{J}^{(odd)dc}_{\nu\lambda}(k_2) + f^{acd}\,\tilde{J}^{(odd)db}_{\lambda\nu}(k_1) \\
&= -\frac{1}{4\pi} f^{abc}\epsilon_{\lambda\nu\sigma}\left(k_1^{\sigma}\,\frac{2m}{k_1}\mathrm{arccot}\left(\frac{2m}{k_1}\right) + k_2^{\sigma}\,\frac{2m}{k_2}\mathrm{arccot}\left(\frac{2m}{k_2}\right)\right) \\
&+\frac{1}{4\pi} f^{abc}\epsilon_{\lambda\nu\sigma}\left(k_1^{\sigma}\,\frac{2m}{k_1}\mathrm{arccot}\left(\frac{2m}{k_1}\right) + k_2^{\sigma}\,\frac{2m}{k_2}\mathrm{arccot}\left(\frac{2m}{k_2}\right)\right) = 0
\end{aligned}
$$

In the IR and UV limit the last equality is not conserved and, to preserve covariance, one has to subtract counterterms.

Gravitational CS (cont.)

The expansion for CS is

$$CS_g = \kappa \int d^3x \, \epsilon^{\mu\nu\lambda} \left(\partial_\mu \omega_\nu^{ab} \omega_{\lambda ba} + \frac{2}{3} \omega_{\mu a}{}^b \omega_{\nu b}{}^c \omega_{\lambda c}{}^a \right) = CS_g^{(2)} + CS_g^{(3)} + \dots ,$$

where

$$CS_g^{(2)} = \frac{\kappa}{2} \int d^3x \, \epsilon_{\sigma\nu\rho} \, h^{\lambda\rho} \left(\partial^\sigma \partial_\lambda \partial_b h^{b\nu} - \partial^\sigma \Box h_\lambda^\nu \right)$$

and

$$\begin{aligned} CS_g^{(3)} \quad = \quad & \frac{\kappa}{4} \int d^3x \, \epsilon^{\mu\nu\lambda} \Big(2\partial_a h_{\nu b} \partial_\lambda h_\sigma^b \partial_\mu h^{\sigma a} - 2\partial_a h_\mu^b \partial^c h_{b\nu} \partial^a h_{c\lambda} - \frac{2}{3} \partial_a h_\mu^b \partial_b h_\nu^c \partial_c h_\lambda^a \\ & -2\partial_\mu \partial^b h_\nu^a (h_a^c \partial_c h_{b\lambda} - h_b^c \partial_c h_{a\lambda}) + \partial_\mu \partial^b h_\nu^a (h_\lambda^c \partial_a h_{bc} - \partial_a h_\lambda^c h_{bc}) \\ & + \partial_\mu \partial^b h_\nu^a \, (\partial_b h_\lambda^c h_{ac} - h_\lambda^c \partial_b h_{ac}) - h_\lambda^\rho h_\rho^a \partial_\mu \left(\Box h_{a\nu} - \partial_a \partial_b h_\nu^b \right) \Big). \end{aligned}$$

This term is understandably more complicated, but it can be reconstructed from the 3pt function of the e.m. tensor.

Interaction in HS theories

In general higher spin theories the reconstruction of the full action kinetic term + interaction is more complicated. In particular we have to introduce the generalized Christoffel symbols (de Wit-Freedman), whose lowest term is, for instance,

$$\Gamma_{\alpha_1\alpha_2;\beta_1\beta_2\beta_3} = \frac{1}{3}\Big\{\partial_{\alpha_1}\partial_{\alpha_2}B_{\beta_1\beta_2\beta_3} - \frac{1}{2}\left(\partial_{\alpha_1}\partial_{\beta_1}B_{\alpha_2\beta_2\beta_3} + \partial_{\alpha_1}\partial_{\beta_2}B_{\alpha_2\beta_1\beta_3}\right.$$
$$+\partial_{\alpha_1}\partial_{\beta_3}B_{\alpha_2\beta_1\beta_2} + \partial_{\alpha_2}\partial_{\beta_1}B_{\alpha_1\beta_2\beta_3} + \partial_{\alpha_2}\partial_{\beta_2}B_{\alpha_1\beta_1\beta_3} + \left.\partial_{\alpha_2}\partial_{\beta_3}B_{\alpha_1\beta_1\beta_2}\right)$$
$$+\partial_{\beta_1}\partial_{\beta_2}B_{\alpha_1\alpha_2\beta_3} + \partial_{\beta_1}\partial_{\beta_3}B_{\alpha_1\alpha_2\beta_2} + \partial_{\beta_2}\partial_{\beta_3}B_{\alpha_1\alpha_2\beta_1}\Big\}. \qquad (1)$$

This may be a very important test for the consistency of HS theories.

(work in progress)

In any case we seem to be able to reconstruct not only the linearized part of the actions, but also (perturbatively) the interactions.

If this is true the previous correspondence will be:

Free field theory $\leftrightarrow$ Interacting AS field theories

AS= any spin

(also interacting?)

We have indications that in order to describe quantum gravitational effect we need a theory with infinite many fields. (Camanho-Edelstein-Maldacena-Zhiboedov, 2014)

But what is this theory: string theory, HS theories, or what else? We don't know.

It would be interesting to apply the previous duality also to string theory.

Tensionless strings

In the tensionless limit ($\alpha' \to \infty$) of the free string spectrum becomes massless. This limit is well defined. Recalling that $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ and $p_\mu = -i\frac{\partial}{\partial x^\mu}$ and redefining the Virasoro generators as follows:

$$L_0 \to \frac{1}{\alpha'}L_0, \qquad L_k \to \frac{1}{\sqrt{2\alpha'}}L_k, \qquad k \neq 0$$

one obtains the reduced generators

$$L_0 \to l_0 = p^2, \qquad L_k \to l_k = p \cdot \alpha_k, \qquad k \neq 0$$

which satisfy the reduced Virasoro algebra

$$[l_k, l_n] = k\, l_0\, \delta_{k+n,0}$$

Tensionless strings (cont.)

Rescaling also the ghost oscillators C_k, B_k as follows

$$C_k \to c_k = \sqrt{2\alpha'} C_k, \qquad B_k \to b_k = \frac{1}{2\alpha'} B_k, \qquad k \neq 0$$
$$C_0 \to c_0 = \alpha' C_0, \qquad B_0 \to b_0 = \frac{1}{\alpha'} B_0$$

in the $\alpha' \to \infty$ one finds a reduced BRST charge

$$Q = \sum_{k=-\infty}^{\infty} \left(c_k l_k - \frac{k}{2} b_0 c_{-k} c_k \right)$$

There is no central charge. $Q^2 = 0$ in any dimension and $Q^\dagger = Q$.
The SFT action reduces to the free action (Bonelli, 2003)

$$S = \langle \Phi | Q | \Phi \rangle$$

It is invariant under the BRST transformation $\delta|\Phi\rangle = Q|\Lambda\rangle$.

Tensionless string (cont.)

The states of this theory have been analyzed by Francia-Sagnotti-Tsulaia. The simplest states are the so-called triplets:

$$|\Phi\rangle = |\phi_0\rangle + c_0|\phi_1\rangle, \qquad |\Lambda\rangle = |\Lambda_0\rangle + c_0|\Lambda_1\rangle,$$

where

$$\begin{aligned} |\phi_0\rangle \quad &= \quad \sum_{s=0}^{\infty} \frac{1}{s!} \varphi_{\mu_1 \ldots \mu_s}(x) \alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_s} c_1 |0\rangle \\ &\quad + \sum_{s=0}^{\infty} \frac{1}{(s-2)!} D_{\mu_1 \ldots \mu_{s-2}}(x) \alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s-2}} c_{-1} |0\rangle \end{aligned}$$

and

$$|\phi_1\rangle \quad = \quad \sum_{s=0}^{\infty} \frac{1}{(s-1)!} C_{\mu_1 \ldots \mu_{s-1}}(x) \alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s-1}} |0\rangle$$

where $|k\rangle = |0\rangle e^{ikx}$. Satisfying the eom $Q|\Phi\rangle = 0$ implies

Tensionless strings (cont.)

$$\Box\varphi = \partial C,$$
$$\partial\cdot\varphi - \partial D = C$$
$$\Box D = \partial\cdot C$$

Now take the gradient of the second and replace into the first. Then take the trace of the first, and insert the result into the previous equation. The end result is (in unnormalized notation)

$$\Box\varphi_{\mu\nu\lambda} - \partial\partial\cdot\varphi + \partial\partial\varphi' = 3\frac{\partial^3}{\Box}C' \qquad (1)$$

This is the Fronsdal equation written in terms of a 'compensator' α field

$$\alpha = \frac{C'}{\Box}$$

This is actually general for all the tensionless strings. This reduces tensionless string theory to a particular HS theory.

Effective SFT?

The tensionless SFT should be the UV limit of SFT. But we have learnt that the UV limit tells us a lot about the whole theory.
This suggests us to treat free SFT as the previous free theories and try to compute the corresponding effective action. The appropriate action is

$$S = \frac{1}{2g_o} \int \left(\Phi * Q\Phi + \frac{2}{3} \Phi * \Phi * \Psi \right) \tag{1}$$

where Φ is the original string field of SFT and Ψ represents the string field of external sources.
The resulting effective theory in terms of Ψ should tell us a lot about SFT itself and its relation to HS theories. (*work in progress*)

THANKS