HEE from minimal surfaces with/without extrinsic curvature

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Outline

Outline for section 1

Introduction I

- Ryu-Takayanagi formula gives HEE for Einstein gravity; $S_{\text{EE}} \propto \min(A)$ where A is a bulk surface and homologous to the entangling surface.
- *Question*: How can RT formula be generalized to higher order gravitational actions?
- Since RT formula can be interpreted as a minimization of Bekenstein-Hawking entropy, then the first guess is that HEE can in general be obtained by the minimization of "Wald" entropy.
- More studies showed that "Wald" entropy is not a suitable choice, but we still believe that there should be a functional to be extremized.

Introduction II

- One part of this “unknown” functional must be Wald entropy and the rest of it is due to extrinsic curvatures (which do not appear in Wald formula).
- To compute EE in a QFT, one uses the replica trick by introducing a conical singularity. In the context of AdS/CFT, a similar trick may be used by deforming the bulk metric so that the minimal surface contains a conical singularity. Dealing with this singularity is the main problem!

Introduction III

- Some methods were suggested:
  - *Maldacena et al[1]*: Expand the equations of motion near the singularity to obtain an equation for minimal surface.
  - *Fursaev et al[2]*: Consider an especial regularized metric and calculate the curvatures near conical singularity.
  - *Dong[3]*: Use a compact general formula which can be derived by using a different regularization.

The first approach seems not to be applicable for higher derivative terms because it's difficult to find a functional from the eom. We consider the second method known by "FPS".

Outline for section 2

Outline

FPS Method I

I. Consider a geometry $\mathcal{M}$ and construct an orbifold $\mathcal{M}_n$ made by cutting $\mathcal{M}$ along a co-dimension one hyper-surface in $\mathcal{M}$, then glue n identical copies (replica method). The new metric will be

$$ds^2 = r^2 d\tau^2 + dr^2 + \gamma_{ij}(r, \tau; x) dx^i dx^j \,,$$

II. A regularized metric ($\tilde{\mathcal{M}}_n$) is introduced to remove conical and curvature singularities:

$$ds^2 = r^2 d\tau^2 + \frac{r^2 + b^2 n^2}{r^2 + b^2} dr^2 + (a + r^n c^{1-n} \cos \tau)^2 ds^2_\Sigma \,.$$

FPS Method II

III. In the presence of $O(2)$ symmetry,

$$R^{\mu\nu} = \mathcal{R}^{\mu\nu} + 2\pi(1-n) n_i^\mu n_j^\nu \delta_\Sigma$$

IV. In the absence of rotational symmetry, extrinsic curvatures play the role, e. g.

$$\int_{\tilde{\mathcal{M}}_n} d\tau dr d^{d-2}x \sqrt{g} R_{\mu\nu}^2 \to n \int_{\mathcal{M}} d\tau dr d^{d-2}x \sqrt{\mathcal{G}} \mathcal{R}_{\mu\nu}^2 + 4\pi(1-n) \int_\Sigma d^{d-2}x \sqrt{\gamma} (\mathcal{R}_{ii} + A_1 K_i^2 + A_2 K_{2i}) + \cdots ,$$

It has an asymptotic series expansion near $b \to 0$. Sending $n \to 1$ the result may contain a term proportional to $b^0(1-n)$.

FPS Method III

So we compute this integral on two known geometries, then we'll get a system of equations for A_i's which can easily be fixed.

V. HEE is given by $S_{\text{HEE}}(\Sigma) = (n\partial_n - 1)I[\tilde{\mathcal{M}}_n]_{n\to 1}$, therefore, we need to expand $I[\tilde{\mathcal{M}}_n]$ up to $\mathcal{O}(n-1)$. (Other terms do not appear in the HEE.)

- FPS method gives the true formula for general quadratic curvature gravity in any dimension.
- For instance, we can use FPS to derive EE for 3-dim NMG.

$$S_{EE}^{NMG} = \frac{2\pi}{\ell_p}\int dz\sqrt{\gamma}\left[1 + 4\lambda L^2\left(\mathcal{R}_{\mu\nu}n_i^\mu n_i^\nu - \frac{3}{4}\mathcal{R} - \frac{1}{2}K_i^2\right)\right] .$$

Outline

Cubic curvature gravities I

- Two known gravities with cubic curvatures are ENMG in 3-dim and QTG in 5-dim.
- Dimensional analysis shows such terms can exist in anomaly:

$$K^4\,,\quad K_2^2\,,\quad K_2K^2\,,\quad KK_3\,,\quad K_4\,,$$
$$\mathcal{R}_{\mu\nu}K^{\mu\nu}K\,,\quad \mathcal{R}_\mu^\nu K_\nu^\rho K_\rho^\mu\,,\quad \mathcal{R}K_2\,,\quad \mathcal{R}K^2\,,\quad \mathcal{R}_{\mu\nu\rho\sigma}K^{\mu\rho}K^{\nu\sigma}\,,$$

BTW, we can also construct the anomaly with curvatures defined on the surface, not in the bulk.

Cubic curvature gravities II

- We limit ourselves to cylinder and S^n's inside AdS space-time to fix the coefficients of these "ten" terms. In fact, each integral can be written as $\int \sum c_n z^n$ and the equalities must be satisfied at any order of z. However, for these geometries $KK_3 = K_2^2$ and then we consider only "nine" curvature invariants.
- *Bad thing:* Our final formula may not be true for general entangling surfaces. Maybe if we had a surface in which all "ten" terms were available, we could get a better result.

Some regularized invariants I

$$\int_{\tilde{\mathcal{M}}_n} d^d x \sqrt{g} R R_2 = n \int_{\mathcal{M}} d^d x \sqrt{\mathcal{G}} \mathcal{R} \mathcal{R}_2$$
$$+4\pi(1-n) \int_\Sigma d^{d-2} x \sqrt{\gamma} \left(\mathcal{R}_2 + \mathcal{R} \mathcal{R}_{\mu\nu} n_i^\mu n_i^\nu - \frac{1}{4} K^4 + \frac{1}{4} K_2 K^2 - \frac{1}{2} \mathcal{R} K^2 \right),$$

Or in terms of curvatures of the surface, the anomaly part will be

$$\int_{\tilde{\mathcal{M}}_n} d^d x \sqrt{g} R R_2 \to \pi(1-n) \int_\Sigma d^{d-2} x \sqrt{\gamma} \Big(\mathcal{K}^4 + 7 \mathcal{K}_2 \mathcal{K}^2 + 10 \mathcal{K}_2^2$$
$$- 18 \mathcal{K}_4 - 2 r \mathcal{K}^2 - 14 \mathcal{K} \mathcal{K}_{ij} r^{ij} - 18 \mathcal{K}_{im} r^{mj} \mathcal{K}_j^i + 6 r \mathcal{K}_2 \Big)$$

HEE results by using FPS

- Now the action is something like $\mathcal{L} \supset RR_2 + R_3 + \cdots$. Each term must be regularized according to FPS method and then put in "replica" formula. So our functional will be ready to minimize on the surface.
- This kind of regularization leads to correct universal terms in EE for both ENMG and QTG (both Z_1 and Z_2), i. e.,

$$S_{\rm EE}^{\rm enmg} = \frac{2\pi L}{\ell_p \sqrt{f_\infty}}(\mu f_\infty^2 + 2\lambda f_\infty + 1)\ln\frac{z_0}{\epsilon},$$

$$S^2 \text{ geo:} \quad S_{\rm EE}^{\rm qtg} = \frac{4\pi^2 L^3}{\ell_p^3 f_\infty^{3/2}}(6\lambda f_\infty - 9\mu f_\infty^2 - 1)\ln\frac{z_0}{\epsilon},$$

$$\text{Cyl geo:} \quad S_{\rm EE}^{\rm qtg} = \frac{\pi^2 L^3}{2\ell_p^3 f_\infty^{3/2} z_0}(3\mu f_\infty^2 + 2\lambda f_\infty - 1)\ln\frac{z_0}{\epsilon}$$

Outline for section 3

Is everything normal?! I

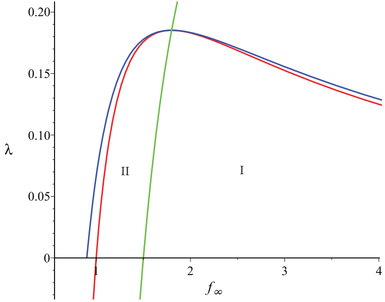
There are at least 2 unusual things in this work:

I. In each case, besides the usual function ($f(z)$) that gives the "normal" EE, we can also find another function which does not lead to a central charge in dual CFT theory, but it can minimize the functional; e. g. for QTG and S^2 entangling surface:

$$f_{\text{usu}} = \sqrt{z_0^2 - z^2}$$
$$f_{\text{unu}} = \sqrt{z_0^2 + 2z_0 q z - z^2}, \quad q^2 = -1 + (\lambda \pm \sqrt{\lambda^2 + 3\mu}) f_\infty$$

Is everything normal?! II

In region (I), the leading term of $S_{\mathrm{EE}}^{\mathrm{usu}}$ is greater than the leading term of $S_{\mathrm{EE}}^{\mathrm{unu}}$, so the minimal surface may be correspond to f_{unu}.



Is everything normal?! III

Again in NMG case, two surfaces minimize the functional. They differ in the value of their extrinsic curvatures:

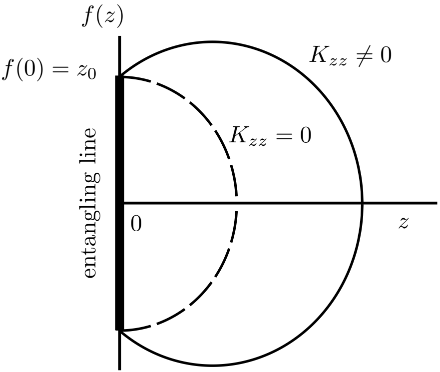
$$f_{\text{usu}} = \sqrt{z_0^2 - z^2}\,, \quad f_{\text{unu}} = \sqrt{z_0^2 + 2z_0 q z - z^2}; q^2 = 2\lambda f_\infty - 1$$

$$K_{\text{usu}} = 0\,, \quad K_{\text{unu}} = \frac{\tilde{L} q (z_0^2 + 2z_0 q z - z^2)}{z^2 z_0^2 (q^2+1)^{3/2}}$$

$$S^{\text{EE}}_{\text{usu}} = \frac{2\pi L}{\ell_p} \frac{3 f_\infty - 2}{f_\infty^{3/2}}\,, \quad S^{\text{EE}}_{\text{unu}} = \frac{2\pi L}{\ell_p} \sqrt{8 \frac{f_\infty - 1}{f_\infty^2}}$$

If $f_\infty > 2$, then $S^{\text{EE}}_{\text{unu}} < S^{\text{EE}}_{\text{usu}}$. Therefore f_{unu} describes HEE.

Is everything normal?! IV



Is everything normal?! V

II. Dong gives a compact formula for HEE. In that approach, the "anomaly" part is:

$$S_{EE}^{anom} = 2\pi \int d^d x \sqrt{g} \left(\frac{\partial^2 \mathcal{L}}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right)_{\alpha} \frac{8 K_{zij} K_{\bar{z}kl}}{q_\alpha + 1}$$

and this method also leads to correct EE for QTG. Now, we expect that for each cubed term (RR_2, R_3 etc.) FPS and Dong method give the same result, but they don't! Although for the whole action (QTG or ENMG) the final result is true, but these methods do not lead the same result for each one of cubed terms.

Is everything normal?! VI

This inconsistency doesn't occur for ENMG, but for QTG and cylindrical entangling surface, we have

$$S^{\mathrm{FPS}}_{\mathrm{EE}} - S^{\mathrm{D}}_{\mathrm{EE}} = \frac{2\pi^2 H L^3 \mu\nu f_\infty (12\mu_0 + 3\mu_1 - 8\mu_2)}{a_0 z \sqrt{f_\infty}}$$

which "strangely" vanishes for QTG actions (Z_1, Z_2):

$$Z_1 \to \mu_0 = 0\,, \quad \mu_1 = 1\,, \quad \mu_2 = \frac{3}{8}$$

$$Z_2 \to \mu_0 = 1\,, \quad \mu_1 = 0\,, \quad \mu_2 = \frac{3}{2}$$

Outline for section 4

Summary

- FPS method gives the true universal term of EE for general quadratic curvature, ENMG and QTG theories. Although it seems that in general the results are not equivalent with what are found by Dong method.
- Besides the usual minimal surfaces for spherical and cylindrical entangling surfaces, we found another surface which can minimize the functional and has its own domain of validity. This surface has a non-vanishing extrinsic curvature.

Thank You

For Further Reading I

- A. Lewkowycz and J. Maldacena, [arXiv:1304.4926 [hep-th]].
- D. V. Fursaev, A. Patrushev and S. N. Solodukhin, [arXiv:1306.4000 [hep-th]].
- X. Dong, [arXiv:1310.5713 [hep-th]].
- A. Ghodsi and M. Moghadassi, [arXiv:1508.02527 [hep-th]].