Twistors, Killing spinors and extended superalgebras

Ümit Ertem

Ankara University

Recent Trends in String Theory and Related Topics
IPM, Tehran, 24-27 May 2016

26 May 2016

Contents

Spin geometry

- We start with a manifold (M, g) that admits a **spin structure**

Spin geometry

- We start with a manifold (M,g) that admits a **spin structure**
- Then, we have the **spinor bundle** and its sections are **spinor fields**

Spin geometry

- We start with a manifold (M, g) that admits a **spin structure**
- Then, we have the **spinor bundle** and its sections are **spinor fields**

- Two first-order differential operators can be defined on it

Spin geometry

- We start with a manifold (M, g) that admits a **spin structure**
- Then, we have the **spinor bundle** and its sections are **spinor fields**
- Two first-order differential operators can be defined on it
    - **Dirac operator**

Spin geometry

- We start with a manifold (M, g) that admits a **spin structure**
- Then, we have the **spinor bundle** and its sections are **spinor fields**
- Two first-order differential operators can be defined on it
  - **Dirac operator**
  - **twistor operator**

Dirac operator

- **Dirac operator**

$$\not{D} = e^a . \nabla_{X_a}$$

e^a are co-frame basis, . denotes the Clifford product and ∇_{X_a} is spinor covariant derivative w.r.t frame basis X_a

Dirac operator

- **Dirac operator**

$$\not{D} = e^a . \nabla_{X_a}$$

e^a are co-frame basis, . denotes the Clifford product and ∇_{X_a} is spinor covariant derivative w.r.t frame basis X_a

- Massless Dirac equation - spinors that are in the kernel of the Dirac operator are called **harmonic spinors**

$$\not{D}\psi = 0$$

Dirac operator

- **Dirac operator**

$$\not{D} = e^a . \nabla_{X_a}$$

e^a are co-frame basis, . denotes the Clifford product and ∇_{X_a} is spinor covariant derivative w.r.t frame basis X_a

- Massless Dirac equation - spinors that are in the kernel of the Dirac operator are called **harmonic spinors**

$$\not{D}\psi = 0$$

- Massive Dirac equation

$$\not{D}\psi = m\psi$$

Twistor spinors

- **Twistor operator** (Penrose)

$$\mathcal{P}_X = \nabla_X - \frac{1}{n}\tilde{X}.\not{D}$$

Twistor spinors

- **Twistor operator** (Penrose)

$$\mathcal{P}_X = \nabla_X - \frac{1}{n}\tilde{X}.\not{D}$$

- Spinors that are in the kernel of the twistor operator, namely the solutions of the following twistor equation are called **twistor spinors**

$$\nabla_X \psi = \frac{1}{n}\tilde{X}.\not{D}\psi$$

Killing spinors

- If a spinor is solution of both massive Dirac equation and twistor equation, then it is called a **Killing spinor** (geometric Killing spinor) and it satisfies the following Killing spinor equation

$$\nabla_X \psi = \lambda \tilde{X}.\psi$$

Killing spinors

- If a spinor is solution of both massive Dirac equation and twistor equation, then it is called a **Killing spinor** (geometric Killing spinor) and it satisfies the following Killing spinor equation

$$\nabla_X \psi = \lambda \tilde{X}.\psi$$

- λ is a real or pure imaginary number which is called the **Killing number**

Killing spinors

- If a spinor is solution of both massive Dirac equation and twistor equation, then it is called a **Killing spinor** (geometric Killing spinor) and it satisfies the following Killing spinor equation

$$\nabla_X\psi = \lambda\tilde{X}.\psi$$

- λ is a real or pure imaginary number which is called the **Killing number**
- The existence of Killing spinors confines the geometry of the manifold such that the curvature scalar becomes

$$\mathcal{R} = -4\lambda^2 n(n-1)$$

Cone construction

- Parallel spinors

$$\nabla_X \psi = 0$$

Cone construction

- Parallel spinors

$$\nabla_X \psi = 0$$

- **Cone construction**
  There is 1:1 correspondence between the Killing spinors on (M, g) and the parallel spinors on the metric cone of M that is $(\mathbb{R} \times M, \bar{g})$ with $\bar{g} = dr^2 + r^2 g$
  (Bär 1993)

Supergravity Killing spinors

- Some special spinors play an important role in **supergravity backgrounds** - solutions of bosonic supergravity theories

Supergravity Killing spinors

- Some special spinors play an important role in **supergravity backgrounds** - solutions of bosonic supergravity theories

- The vanishing conditions on the variations of fermionic fields define some restriction equations for spinor parameters

Supergravity Killing spinors

- Some special spinors play an important role in **supergravity backgrounds** - solutions of bosonic supergravity theories

- The vanishing conditions on the variations of fermionic fields define some restriction equations for spinor parameters
- These are called **supergravity Killing spinor** equations

Supergravity Killing spinors

- Some special spinors play an important role in **supergravity backgrounds** - solutions of bosonic supergravity theories
- The vanishing conditions on the variations of fermionic fields define some restriction equations for spinor parameters
- These are called **supergravity Killing spinor** equations
- The variation with respect to gravitino field gives a **differential** equation for the spinor parameteres

Supergravity Killing spinors

- Some special spinors play an important role in **supergravity backgrounds** - solutions of bosonic supergravity theories

- The vanishing conditions on the variations of fermionic fields define some restriction equations for spinor parameters
- These are called **supergravity Killing spinor** equations

- The variation with respect to gravitino field gives a **differential** equation for the spinor parameteres
- The variation with respect to other fermionic fields gives **algebraic** constraints

Supergravity Killing spinors

- Some special spinors play an important role in **supergravity backgrounds** - solutions of bosonic supergravity theories

- The vanishing conditions on the variations of fermionic fields define some restriction equations for spinor parameters
- These are called **supergravity Killing spinor** equations

- The variation with respect to gravitino field gives a **differential** equation for the spinor parameteres
- The variation with respect to other fermionic fields gives **algebraic** constraints

- The number of solutions of supergravity Killing spinor equations determine the number of **preserved supersymmetries** on the supergravity background

11D and 10D sugra Killing spinors

- 11D supergravity Killing spinor equation

$$\nabla_X \epsilon = \frac{1}{12}(\tilde{X} \wedge F).\epsilon - \frac{1}{6} i_X F.\epsilon$$

F 4-form field

11D and 10D sugra Killing spinors

- 11D supergravity Killing spinor equation

$$\nabla_X \epsilon = \frac{1}{12}(\tilde{X} \wedge F).\epsilon - \frac{1}{6} i_X F.\epsilon$$

F 4-form field

- heterotic supergravity Killing spinor equations

$$\nabla_X \epsilon = \frac{1}{4} i_X H_3.\epsilon$$

$$\left(d\phi + \frac{1}{2} H_3 \right).\epsilon = 0$$

$$F_2.\epsilon = 0$$

H_3 3-form field, F_2 2-form field, ϕ dilaton

Supergravity spinor connection

- One can define new supergravity spinor connections in 11D and heterotic cases as

$$\mathcal{D}_X = \nabla_X - \frac{1}{12}\tilde{X} \wedge F + \frac{1}{6} i_X F$$

$$\mathcal{D}_X = \nabla_X - \frac{1}{4} i_X H_3$$

Supergravity spinor connection

- One can define new supergravity spinor connections in 11D and heterotic cases as

$$\mathcal{D}_X = \nabla_X - \frac{1}{12}\tilde{X} \wedge F + \frac{1}{6} i_X F$$

$$\mathcal{D}_X = \nabla_X - \frac{1}{4} i_X H_3$$

- Supergravity Killing spinors correspond to **parallel spinors** w.r.t. supergravity spinor connection

$$\mathcal{D}_X \epsilon = 0$$

Freund-Rubin backgrounds

- In **Freund-Rubin** backgrounds such as

Freund-Rubin backgrounds

- In **Freund-Rubin** backgrounds such as
  - $AdS_5 \times S^5$
  - $AdS_4 \times S^7$
  - $AdS_7 \times S^4$

Freund-Rubin backgrounds

- In **Freund-Rubin** backgrounds such as
  - $AdS_5 \times S^5$
  - $AdS_4 \times S^7$
  - $AdS_7 \times S^4$
- **Supergravity** Killing spinors can be constructed from **geometric** Killing spinors in component spaces

Supergravity twistor spinors

- **Rigid supersymmetric** field theories in conformal supergravity backgrounds

Supergravity twistor spinors

- **Rigid supersymmetric** field theories in conformal supergravity backgrounds
- Supersymmetry parameters satisfy the **gauged twistor equation**

$$\nabla_X^A \epsilon = \frac{1}{n} \widetilde{X}.\not{D}^A \epsilon$$

Supergravity twistor spinors

- **Rigid supersymmetric** field theories in conformal supergravity backgrounds
- Supersymmetry parameters satisfy the **gauged twistor equation**

$$\nabla_X^A \epsilon = \frac{1}{n} \widetilde{X}.\not{D}^A \epsilon$$

- where gauged connection and Dirac operator are defined as

$$\nabla_X^A = \nabla_X + i_X A$$

$$\not{D}^A = \not{D} + A$$

for a 1-form gauge field A

Dirac currents

- Spinor space S, dual spinor space S^*

Dirac currents

- Spinor space S, dual spinor space S^*
- Tensor product of a spinor and a dual spinor can be written as a sum of differential forms

Dirac currents

- Spinor space S, dual spinor space S^*
- Tensor product of a spinor and a dual spinor can be written as a sum of differential forms
- **Fierz identity**

$$\epsilon\bar{\epsilon} = (\epsilon, \epsilon) + (\epsilon, e_a\epsilon)e^a + (\epsilon, e_{ba}\epsilon)e^{ab} + ... + (-1)^{\lfloor \frac{n}{2} \rfloor}(\epsilon, z\epsilon)z$$

where $(\,,\,)$ is the spinor inner product, $e^{ab} = e^a \wedge e^b = e^a e^b$ and z is the volume form

Dirac currents

- Spinor space S, dual spinor space S^*
- Tensor product of a spinor and a dual spinor can be written as a sum of differential forms
- **Fierz identity**

$$\epsilon\overline{\epsilon} = (\epsilon, \epsilon) + (\epsilon, e_a\epsilon)e^a + (\epsilon, e_{ba}\epsilon)e^{ab} + ... + (-1)^{\lfloor \frac{n}{2} \rfloor}(\epsilon, z\epsilon)z$$

where $(,)$ is the spinor inner product, $e^{ab} = e^a \wedge e^b = e^a e^b$ and z is the volume form

- **p-form Dirac current**

$$(\epsilon\overline{\epsilon})_p = (\epsilon, e_{a_p...a_2a_1}\epsilon)e^{a_1a_2...a_p}$$

Dirac currents of Killing and twistor spinors

- Special case: **1-form Dirac current**

$$(\epsilon\overline{\epsilon})_1$$

Dirac currents of Killing and twistor spinors

- Special case: **1-form Dirac current**

$$(\epsilon\overline{\epsilon})_1$$

- For **geometric Killing spinors** and **supergravity Killing spinors**, 1-form Dirac currents correspond to metric duals of **Killing vector fields**

$$\mathcal{L}_K g = 0$$

Dirac currents of Killing and twistor spinors

- Special case: **1-form Dirac current**

$$(\epsilon\bar{\epsilon})_1$$

- For **geometric Killing spinors** and **supergravity Killing spinors**, 1-form Dirac currents correspond to metric duals of **Killing vector fields**

$$\mathcal{L}_K g = 0$$

- For **twistor spinors**, 1-form Dirac currents correspond to metric duals of **conformal Killing vector fields**

$$\mathcal{L}_K g = 2\lambda g$$

where $\mathcal{L}$ is the Lie derivative and λ is a function

Lie derivative of spinors

- **Lie derivative** of a spinor ψ w.r.t a Killing vector field K is defined as

$$\mathcal{L}_K\psi = \nabla_K\psi + \frac{1}{4}d\widetilde{K}.\psi$$

Lie derivative of spinors

- **Lie derivative** of a spinor ψ w.r.t a Killing vector field K is defined as

$$\mathcal{L}_K\psi = \nabla_K\psi + \frac{1}{4}d\widetilde{K}.\psi$$

- which is induced from the Lie derivative of an inhomogeneous differential form α w.r.t a Killing vector

$$\mathcal{L}_K\alpha = \nabla_K\alpha + \frac{1}{4}[d\widetilde{K}, \alpha]_{Cl}$$

Lie derivative of spinors

- **Lie derivative** of a spinor ψ w.r.t a Killing vector field K is defined as

$$\mathcal{L}_K\psi = \nabla_K\psi + \frac{1}{4}d\widetilde{K}.\psi$$

- which is induced from the Lie derivative of an inhomogeneous differential form α w.r.t a Killing vector

$$\mathcal{L}_K\alpha = \nabla_K\alpha + \frac{1}{4}[d\widetilde{K}, \alpha]_{Cl}$$

- where $[.,.]_{Cl}$ is the **Clifford bracket**

Symmetry operators

- **Symmetry operator**: Takes a solution of an equation and gives another solution

Symmetry operators

- **Symmetry operator**: Takes a solution of an equation and gives another solution

- Symmetry operator of a **Killing spinor**

$$\mathcal{L}_K$$

where K is a **Killing vector**

Symmetry operators

- **Symmetry operator**: Takes a solution of an equation and gives another solution

- Symmetry operator of a **Killing spinor**

$$\mathcal{L}_K$$

where K is a **Killing vector**

- Symmetry operator of a **twistor spinor**

$$\mathcal{L}_K - \frac{1}{2}\lambda$$

where K is a **conformal Killing vector**

Superalgebras

- A **Superalgebra** $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ consists of an algebra $\mathfrak{g}_0$ and a $\mathfrak{g}_0$-module $\mathfrak{g}_1$ with a bilinear multiplication

$$[,] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$$

Superalgebras

- A **Superalgebra** $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ consists of an algebra $\mathfrak{g}_0$ and a $\mathfrak{g}_0$-module $\mathfrak{g}_1$ with a bilinear multiplication

$$[,] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$$

- $(i, j = 0, 1)$ such that

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0 \qquad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1 \qquad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$$

Superalgebras

- A **Superalgebra** $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ consists of an algebra $\mathfrak{g}_0$ and a $\mathfrak{g}_0$-module $\mathfrak{g}_1$ with a bilinear multiplication

$$[,] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$$

- $(i, j = 0, 1)$ such that

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0 \qquad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1 \qquad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$$

- which satisfies the skew-supersymmetry identiites

$$[a, b] = -(-1)^{|a||b|}[b, a]$$

$|a|$ denotes the degree of a, which corresponds to 0 or 1 depending on that a is in $\mathfrak{g}_0$ or $\mathfrak{g}_1$

Lie superalgebras

- If $\mathfrak{g}_0$ is a Lie algebra

Lie superalgebras

- If $\mathfrak{g}_0$ is a Lie algebra
- and the bracket $[,]$ satisfies the graded-Jacobi identities

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]$$

Lie superalgebras

- If $\mathfrak{g}_0$ is a Lie algebra
- and the bracket $[,]$ satisfies the graded-Jacobi identities

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]$$

- Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called a **Lie superalgebra**

Killing superalgebras

- **Killing superalgebras**
  $\mathfrak{g}_0$: Lie algebra of Killing vector fields
  $\mathfrak{g}_1$: space of Killing spinors

Killing superalgebras

- **Killing superalgebras**
  $\mathfrak{g}_0$: Lie algebra of Killing vector fields
  $\mathfrak{g}_1$: space of Killing spinors
- Lie bracket of vector fields

$$[,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$$

Killing superalgebras

- **Killing superalgebras**
  $\mathfrak{g}_0$: Lie algebra of Killing vector fields
  $\mathfrak{g}_1$: space of Killing spinors
- Lie bracket of vector fields

$$[,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$$

- Lie derivative on spinors w.r.t. Killing vectors

$$\mathcal{L} : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$$

Killing superalgebras

- **Killing superalgebras**
  $\mathfrak{g}_0$: Lie algebra of Killing vector fields
  $\mathfrak{g}_1$: space of Killing spinors
- Lie bracket of vector fields

$$[,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$$

- Lie derivative on spinors w.r.t. Killing vectors

$$\mathcal{L} : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$$

- Squaring map (1-form Dirac current)

$$(\quad)_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$$

Killing superalgebras

- graded-Jacobi identities correspond to the Jacobi identity of the Lie bracket of vector fields ($X, Y, Z \in \mathfrak{g}_0$)

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

Killing superalgebras

- graded-Jacobi identities correspond to the Jacobi identity of the Lie bracket of vector fields ($X, Y, Z \in \mathfrak{g}_0$)

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

- and the properties of the Lie derivative ($\epsilon, \kappa \in S$)

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$$
$$\mathcal{L}_X(\epsilon\kappa) = (\mathcal{L}_X\epsilon)\kappa + \epsilon(\mathcal{L}_X\kappa)$$

Killing superalgebras

- graded-Jacobi identities correspond to the Jacobi identity of the Lie bracket of vector fields ($X, Y, Z \in \mathfrak{g}_0$)

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

- and the properties of the Lie derivative ($\epsilon, \kappa \in S$)

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$$
$$\mathcal{L}_X(\epsilon\kappa) = (\mathcal{L}_X\epsilon)\kappa + \epsilon(\mathcal{L}_X\kappa)$$

- and the identity

$$\mathcal{L}_{\widetilde{(\epsilon\overline{\epsilon})_1}}\epsilon = 0$$

Killing superalgebras

- graded-Jacobi identities correspond to the Jacobi identity of the Lie bracket of vector fields ($X, Y, Z \in \mathfrak{g}_0$)

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

- and the properties of the Lie derivative ($\epsilon, \kappa \in S$)

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$$
$$\mathcal{L}_X(\epsilon\kappa) = (\mathcal{L}_X\epsilon)\kappa + \epsilon(\mathcal{L}_X\kappa)$$

- and the identity

$$\mathcal{L}_{\widetilde{(\epsilon\bar{\epsilon})_1}}\epsilon = 0$$

- The last identity is not automatically satisfied and have to be checked for different cases to obtain a Lie superalgebra structure

Conformal superalgebras

- **Conformal superalgebras**
  $\mathfrak{g}_0$: Lie algebra of **conformal Killing** vector fields
  $\mathfrak{g}_1$: space of **twistor** spinors

Conformal superalgebras

- **Conformal superalgebras**
  $\mathfrak{g}_0$: Lie algebra of **conformal Killing** vector fields
  $\mathfrak{g}_1$: space of **twistor** spinors
- Lie bracket of vector fields

$$[,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$$

Conformal superalgebras

- **Conformal superalgebras**
  $\mathfrak{g}_0$: Lie algebra of **conformal Killing** vector fields
  $\mathfrak{g}_1$: space of **twistor** spinors
- Lie bracket of vector fields

$$[,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$$

- Lie derivative on spinors w.r.t. conformal Killing vectors

$$\mathcal{L} - \frac{1}{2}\lambda : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$$

Conformal superalgebras

- **Conformal superalgebras**
  $\mathfrak{g}_0$: Lie algebra of **conformal Killing** vector fields
  $\mathfrak{g}_1$: space of **twistor** spinors
- Lie bracket of vector fields

$$[\,,\,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$$

- Lie derivative on spinors w.r.t. conformal Killing vectors

$$\mathcal{L} - \frac{1}{2}\lambda : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$$

- Squaring map (1-form Dirac current)

$$(\quad)_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$$

Conformal superalgebras

- In general, the conformal superalgebra of conformal Killing vectors and twistor spinors does not correspond to a Lie superalgebra

Conformal superalgebras

- In general, the conformal superalgebra of conformal Killing vectors and twistor spinors does not correspond to a Lie superalgebra
- To obtain a **Lie superalgebra**, one should add extra $\mathcal{R}$-**symmetries** which correspond to an extra gauge symmetry that is a Lie algebra with constant parameters (de Medeiros, Hollands 2013)

Conformal superalgebras

- In general, the conformal superalgebra of conformal Killing vectors and twistor spinors does not correspond to a Lie superalgebra
- To obtain a **Lie superalgebra**, one should add extra $\mathcal{R}$-**symmetries** which correspond to an extra gauge symmetry that is a Lie algebra with constant parameters (de Medeiros, Hollands 2013)
- Then, the **even part** of the Lie superalgebra corresponds to conformal Killing vectors and $\mathcal{R}$-symmetries
  the **odd part** consists of $\mathcal{R}$-module valued twistor spinors
  and the graded Jacobi identites are satisfied

Classification of backgrounds

- Classification of supersymmetric backgrounds can be done through Killing and conformal superalgebras

Classification of backgrounds

- Classification of supersymmetric backgrounds can be done through Killing and conformal superalgebras
- **Killing superalgebras:**
  Classification of supergravity backgrounds in various dimensions
  (Figueroa O'Farrill, Papadopoulos, Meessen, Philip, Hacket-Jones, Moutsopoulos 2005-2007)

Classification of backgrounds

- Classification of supersymmetric backgrounds can be done through Killing and conformal superalgebras

- **Killing superalgebras:**
  Classification of supergravity backgrounds in various dimensions
  (Figueroa O'Farrill, Papadopoulos, Meessen, Philip, Hacket-Jones, Moutsopoulos 2005-2007)

- **Conformal superalgebras:**
  Classification of conformal backgrounds for supersymmetric field theories
  (de Medeiros, Hollands 2013)

p-form Dirac currents

- However, this is not the whole story!

p-form Dirac currents

- However, this is not the whole story!
- Remember that the Fierz identity contains different form degree components

$$\epsilon\bar{\epsilon} = (\epsilon, \epsilon) + (\epsilon, e_a\epsilon)e^a + (\epsilon, e_{ba}\epsilon)e^{ab} + ... + (-1)^{\lfloor \frac{n}{2} \rfloor}(\epsilon, z\epsilon)z$$

p-form Dirac currents

- However, this is not the whole story!
- Remember that the Fierz identity contains different form degree components

$$\epsilon\overline{\epsilon} = (\epsilon, \epsilon) + (\epsilon, e_a\epsilon)e^a + (\epsilon, e_{ba}\epsilon)e^{ab} + ... + (-1)^{\lfloor \frac{n}{2} \rfloor}(\epsilon, z\epsilon)z$$

- and we have used only the 1-form Dirac currents $(\epsilon\overline{\epsilon})_1$

p-form Dirac currents

- However, this is not the whole story!
- Remember that the Fierz identity contains different form degree components

$$\epsilon\overline{\epsilon} = (\epsilon, \epsilon) + (\epsilon, e_a\epsilon)e^a + (\epsilon, e_{ba}\epsilon)e^{ab} + ... + (-1)^{\lfloor \frac{n}{2} \rfloor}(\epsilon, z\epsilon)z$$

- and we have used only the 1-form Dirac currents $(\epsilon\overline{\epsilon})_1$
- but we still have **p-form Dirac currents** for $p > 1$

$$(\epsilon\overline{\epsilon})_p = (\epsilon, e_{a_p...a_2a_1}\epsilon)e^{a_1a_2...a_p}$$

Killing-Yano forms

- If ϵ is a **geometric Killing spinor**

Killing-Yano forms

- If ϵ is a **geometric Killing spinor**
- Then, the p-form Dirac currents $(\epsilon\bar{\epsilon})_p$ correspond to **Killing-Yano** (KY) forms or their Hodge duals

Killing-Yano forms

- If ϵ is a **geometric Killing spinor**
- Then, the p-form Dirac currents $(\epsilon\bar{\epsilon})_p$ correspond to **Killing-Yano** (KY) forms or their Hodge duals
- KY forms are generalizations of Killing vectors to higher degree forms

Killing-Yano forms

- If ϵ is a **geometric Killing spinor**
- Then, the p-form Dirac currents $(\epsilon\bar{\epsilon})_p$ correspond to **Killing-Yano** (KY) forms or their Hodge duals
- KY forms are generalizations of Killing vectors to higher degree forms
- A KY p-form ω satisfies the following equation

$$\nabla_X \omega = \frac{1}{p+1} i_X d\omega$$

for every vector field X

conformal Killing-Yano forms

- If ϵ is a **twistor spinor**

conformal Killing-Yano forms

- If ϵ is a **twistor spinor**
- Then, the p-form Dirac currents $(\epsilon\bar{\epsilon})_p$ correspond to **conformal Killing-Yano** (CKY) forms

conformal Killing-Yano forms

- If ϵ is a **twistor spinor**
- Then, the p-form Dirac currents $(\epsilon\bar{\epsilon})_p$ correspond to **conformal Killing-Yano** (CKY) forms
- CKY forms are generalizations of conformal Killing vectors to higher degree forms

conformal Killing-Yano forms

- If ϵ is a **twistor spinor**
- Then, the p-form Dirac currents $(\epsilon\bar{\epsilon})_p$ correspond to **conformal Killing-Yano** (CKY) forms
- CKY forms are generalizations of conformal Killing vectors to higher degree forms
- A CKY p-form ω satisfies the following equation

$$\nabla_X \omega = \frac{1}{p+1} i_X d\omega - \frac{1}{n-p+1} \widetilde{X} \wedge \delta\omega$$

for every vector field X
KY forms are co-closed CKY forms ($\delta\omega = 0$)

Symmetry operators of Killing spinors

- We can define more general **symmetry operators** for geometric Killing spinors and twistor spinors

Symmetry operators of Killing spinors

- We can define more general **symmetry operators** for geometric Killing spinors and twistor spinors
- For a geometric Killing spinor ϵ with Killing number λ the symmetry operator is defined from an **odd KY p-form** ω as

$$L_\omega = -(-1)^p \lambda p \omega + \frac{p}{2(p+1)} d\omega$$

Symmetry operators of Killing spinors

- We can define more general **symmetry operators** for geometric Killing spinors and twistor spinors
- For a geometric Killing spinor ϵ with Killing number λ the symmetry operator is defined from an **odd KY p-form** ω as

$$L_\omega = -(-1)^p \lambda p \omega + \frac{p}{2(p+1)} d\omega$$

- in **constant curvature backgrounds**

Symmetry operators of twistor spinors

- For a twistor spinor ϵ
  the symmetry operator is defined from a **CKY p-form** ω as

$$L_\omega = -(-1)^p \tfrac{p}{n}\omega.\not{D} + \frac{p}{2(p+1)}d\omega + \frac{p}{2(n-p+1)}\delta\omega$$

Symmetry operators of twistor spinors

- For a twistor spinor ϵ
  the symmetry operator is defined from a **CKY p-form** ω as

$$L_\omega = -(-1)^p \frac{p}{n}\omega.\displaystyle{\not}D + \frac{p}{2(p+1)}d\omega + \frac{p}{2(n-p+1)}\delta\omega$$

- in **constant curvature backgrounds**

Symmetry operators of twistor spinors

- For a twistor spinor ϵ
  the symmetry operator is defined from a **CKY p-form** ω as

$$L_\omega = -(-1)^p \tfrac{p}{n}\omega.\not{D} + \frac{p}{2(p+1)}d\omega + \frac{p}{2(n-p+1)}\delta\omega$$

- in **constant curvature backgrounds**

- and from a **normal CKY p-form** ω
  in **Einstein manifolds**

Normal CKY forms

- A **normal** CKY p-form satisfies the CKY equation and the following **integrability condition**

$$\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1}d\delta\omega = -2(n-p)K_a \wedge i_{X^a}\omega$$

Normal CKY forms

- A **normal** CKY p-form satisfies the CKY equation and the following **integrability condition**

$$\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1}d\delta\omega = -2(n-p)K_a \wedge i_{X^a}\omega$$

- which is a special case of the general integrability condition of CKY forms (Leitner 2004)

Normal CKY forms

- A **normal** CKY p-form satisfies the CKY equation and the following **integrability condition**

$$\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1}d\delta\omega = -2(n-p)K_a \wedge i_{X^a}\omega$$

- which is a special case of the general integrability condition of CKY forms (Leitner 2004)

- Here K_a is defined in terms of Ricci 1-forms P_a and the curvature scalar $\mathcal{R}$ as

$$K_a = \frac{1}{n-2}\left(\frac{\mathcal{R}}{2(n-1)}e_a - P_a\right)$$

Normal CKY forms

- A **normal** CKY p-form satisfies the CKY equation and the following **integrability condition**

$$\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1}d\delta\omega = -2(n-p)K_a \wedge i_{X^a}\omega$$

- which is a special case of the general integrability condition of CKY forms (Leitner 2004)

- Here K_a is defined in terms of Ricci 1-forms P_a and the curvature scalar $\mathcal{R}$ as

$$K_a = \frac{1}{n-2}\left(\frac{\mathcal{R}}{2(n-1)}e_a - P_a\right)$$

- In constant curvature spacetimes, all CKY forms are normal CKY forms

Graded Lie algebra of KY forms

- Lie algebra of Killing vectors can be extended to **KY forms**

Graded Lie algebra of KY forms

- Lie algebra of Killing vectors can be extended to **KY forms**
- KY forms satisfy a graded Lie algebra with respect to **Schouten-Nijenhuis** (SN) bracket in **constant curvature** backgrounds

Graded Lie algebra of KY forms

- Lie algebra of Killing vectors can be extended to **KY forms**
- KY forms satisfy a graded Lie algebra with respect to **Schouten-Nijenhuis** (SN) bracket in **constant curvature** backgrounds
- For a KY p-form ω_1 and a KY q-form ω_2, the SN bracket of them gives a KY $(p+q-1)$-form

$$[\omega_1, \omega_2]_{SN} = i_{X^a}\omega_1 \wedge \nabla_{X_a}\omega_2 + (-1)^{pq} i_{X^a}\omega_2 \wedge \nabla_{X_a}\omega_1$$

(Kastor, Ray, Traschen 2007)

Graded Lie algebra of CKY forms

- Similarly, Lie algebra of conformal Killing vectors can be extended to **CKY forms**

Graded Lie algebra of CKY forms

- Similarly, Lie algebra of conformal Killing vectors can be extended to **CKY forms**
- CKY forms satisfy a graded Lie algebra with respect to a bracket that corresponds to the slightly modified SN bracket in **constant curvature** backgrounds

Graded Lie algebra of CKY forms

- Similarly, Lie algebra of conformal Killing vectors can be extended to **CKY forms**
- CKY forms satisfy a graded Lie algebra with respect to a bracket that corresponds to the slightly modified SN bracket in **constant curvature** backgrounds
- For a CKY p-form ω_1 and a CKY q-form ω_2, the following bracket gives a CKY $(p+q-1)$-form

$$[\omega_1, \omega_2]_{CKY} = \frac{1}{q+1} i_{X_a}\omega_1 \wedge i_{X^a} d\omega_2 + \frac{(-1)^p}{p+1} i_{X_a} d\omega_1 \wedge i_{X^a}\omega_2 \\ + \frac{(-1)^p}{n-q+1}\omega_1 \wedge \delta\omega_2 + \frac{1}{n-p+1}\delta\omega_1 \wedge \omega_2$$

Graded Lie algebra of CKY forms

- Similarly, Lie algebra of conformal Killing vectors can be extended to **CKY forms**
- CKY forms satisfy a graded Lie algebra with respect to a bracket that corresponds to the slightly modified SN bracket in **constant curvature** backgrounds
- For a CKY p-form ω_1 and a CKY q-form ω_2, the following bracket gives a CKY $(p+q-1)$-form

$$[\omega_1, \omega_2]_{CKY} = \frac{1}{q+1} i_{X_a}\omega_1 \wedge i_{X^a} d\omega_2 + \frac{(-1)^p}{p+1} i_{X_a} d\omega_1 \wedge i_{X^a}\omega_2 + \frac{(-1)^p}{n-q+1}\omega_1 \wedge \delta\omega_2 + \frac{1}{n-p+1}\delta\omega_1 \wedge \omega_2$$

- and it is also relevant for **normal** CKY forms in **Einstein** manifolds (Ertem 2016)

Extended Killing superalgebras

- By using the graded Lie algebra of KY forms and symmetry operators of Killing spinors,
  we can extend the Killing superalgebras to include KY forms in constant curvature backgrounds

Extended Killing superalgebras

- By using the graded Lie algebra of KY forms and symmetry operators of Killing spinors,
  we can extend the Killing superalgebras to include KY forms in constant curvature backgrounds

- **Extended** Killing superalgebra $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$
  $\bar{\mathfrak{g}}_0$: Lie algebra of odd KY forms
  $\mathfrak{g}_1$: space of geometric Killing spinors

Extended Killing superalgebras

- By using the graded Lie algebra of KY forms and symmetry operators of Killing spinors,
  we can extend the Killing superalgebras to include KY forms in constant curvature backgrounds

- **Extended** Killing superalgebra $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$
  $\bar{\mathfrak{g}}_0$: Lie algebra of odd KY forms
  $\mathfrak{g}_1$: space of geometric Killing spinors

- The **even-even** bracket corresponds to the SN bracket of KY forms

$$[,]_{SN} : \bar{\mathfrak{g}}_0 \times \bar{\mathfrak{g}}_0 \to \bar{\mathfrak{g}}_0$$

Extended Killing superalgebras

- The **even-odd** bracket corresponds to the symmetry operator of geometric Killing spinors

$$L : \bar{\mathfrak{g}}_0 \times \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_1$$

$$L_\omega = -(-1)^p \lambda p \omega + \frac{p}{2(p+1)} d\omega$$

Extended Killing superalgebras

- The **even-odd** bracket corresponds to the symmetry operator of geometric Killing spinors

$$L : \bar{\mathfrak{g}}_0 \times \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_1$$

$$L_\omega = -(-1)^p \lambda p \omega + \frac{p}{2(p+1)} d\omega$$

- The **odd-odd** bracket corresponds to the p-form Dirac currents of Killing spinors

$$(\quad)_p : \bar{\mathfrak{g}}_1 \times \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_0$$

Extended Killing superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra

Extended Killing superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra
- **Dimension** of the superalgebra $(\alpha|\beta)$

Extended Killing superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra

- **Dimension** of the superalgebra $(\alpha|\beta)$
- α is the dimesion of the space of **KY forms** (which is maximal in constant curvature backgrounds)

$$\binom{n}{p} + \binom{n}{p+1}$$

Extended Killing superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra
- **Dimension** of the superalgebra $(\alpha|\beta)$
- α is the dimesion of the space of **KY forms** (which is maximal in constant curvature backgrounds)

$$\binom{n}{p} + \binom{n}{p+1}$$

- β is the dimension of the space of **Killing spinors**

Extended conformal superalgebras

- By using the graded Lie algebra of CKY forms and symmetry operators of twistor spinors,
  we can extend the conformal superalgebras to include CKY forms in constant curvature backgrounds and Einstein manifolds

Extended conformal superalgebras

- By using the graded Lie algebra of CKY forms and symmetry operators of twistor spinors,
  we can extend the conformal superalgebras to include CKY forms in constant curvature backgrounds and Einstein manifolds

- **Extended** conformal superalgebra $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$
  $\bar{\mathfrak{g}}_0$: Graded Lie algebra of CKY forms
  $\mathfrak{g}_1$: space of twistor spinors

Extended conformal superalgebras

- By using the graded Lie algebra of CKY forms and symmetry operators of twistor spinors,
  we can extend the conformal superalgebras to include CKY forms in constant curvature backgrounds and Einstein manifolds

- **Extended** conformal superalgebra $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$
  $\bar{\mathfrak{g}}_0$: Graded Lie algebra of CKY forms
  $\mathfrak{g}_1$: space of twistor spinors

- The **even-even** bracket corresponds to the Lie bracket of CKY forms

$$[\,,\,]_{CKY} : \bar{\mathfrak{g}}_0 \times \bar{\mathfrak{g}}_0 \to \bar{\mathfrak{g}}_0$$

Extended conformal superalgebras

- The **even-odd** bracket corresponds to the symmetry operator of twistor spinors

$$L : \bar{\mathfrak{g}}_0 \times \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_1$$

$$L_\omega = -(-1)^p \tfrac{p}{n} \omega . \not{D} + \frac{p}{2(p+1)} d\omega + \frac{p}{2(n-p+1)} \delta\omega$$

Extended conformal superalgebras

- The **even-odd** bracket corresponds to the symmetry operator of twistor spinors

$$L : \bar{\mathfrak{g}}_0 \times \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_1$$

$$L_\omega = -(-1)^p \tfrac{p}{n} \omega . \not{D} + \frac{p}{2(p+1)} d\omega + \frac{p}{2(n-p+1)} \delta\omega$$

- The **odd-odd** bracket corresponds to the p-form Dirac currents of twistor spinors

$$(\quad)_p : \bar{\mathfrak{g}}_1 \times \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_0$$

Extended conformal superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra

Extended conformal superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra

- **Dimension** of the superalgebra $(\alpha|\beta)$

Extended conformal superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra

- **Dimension** of the superalgebra $(\alpha|\beta)$
- α is the dimesion of the space of **CKY forms** (which is maximal in constant curvature backgrounds)

$$\binom{n}{p-1} + 2\binom{n}{p} + \binom{n}{p+1}$$

Extended conformal superalgebras

- However, these brackets do not satisfy the graded Jacobi identities, so, we do not have a Lie superalgebra

- **Dimension** of the superalgebra $(\alpha|\beta)$
- α is the dimesion of the space of **CKY forms** (which is maximal in constant curvature backgrounds)

$$\begin{pmatrix} n \\ p-1 \end{pmatrix} + 2\begin{pmatrix} n \\ p \end{pmatrix} + \begin{pmatrix} n \\ p+1 \end{pmatrix}$$

- β is the dimension of the space of **twistor spinors** (which is maximal in constant curvature backgrounds)

$$2^{\lfloor n/2 \rfloor} + 1$$

Conclusion

- KY forms and CKY forms satisfy graded Lie algebra structures in conformally flat backgrounds

Conclusion

- KY forms and CKY forms satisfy graded Lie algebra structures in conformally flat backgrounds
- Symmetry operators of Killing spinors and twistor spinors can be constructed from KY forms and CKY forms respectively in conformally flat backgrounds

Conclusion

- KY forms and CKY forms satisfy graded Lie algebra structures in conformally flat backgrounds
- Symmetry operators of Killing spinors and twistor spinors can be constructed from KY forms and CKY forms respectively in conformally flat backgrounds
- So, Killing superalgebras and conformal superalgebras can be extended to include KY forms and CKY forms respectively

Conclusion

- KY forms and CKY forms satisfy graded Lie algebra structures in conformally flat backgrounds
- Symmetry operators of Killing spinors and twistor spinors can be constructed from KY forms and CKY forms respectively in conformally flat backgrounds
- So, Killing superalgebras and conformal superalgebras can be extended to include KY forms and CKY forms respectively
- This may give hints about the classification problems of supergravity backgrounds and supersymmetric field theories in conformal backgrounds