Phase Transitions and Entanglement Entropy in Warped AdS$_3$/Warped CFT$_2$

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Recent Trends in String Theory and Related Topics

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Based on


Overview

- Review of 2d CFT
- Beyond AdS$_3$/CFT$_2$
- Warped CFTs, Holographic WCFTs
- Phase Transitions in TMG theory
- Phase Transitions in NMG theory
- Entanglement entropy of WCFTs
2d CFT

- In 2d, scale invariance $\Rightarrow$ conformal invariance.

- Any unitary theory with a discrete spectrum and invariant under translations, Lorentz transformations and scaling has an enlarged $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ global symmetry group and local symmetries given by two copies of centrally extended Virasoro algebra.

- Adding modular invariance gives Cardy’s formula.
Modular Invariance

- The modular S-transformation implies
  \[
  Z(\tau, \bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}).
  \]
- At \( \Theta = 0 \), this relates the behavior of partition function at high and low T. In this regimes, the free energy for generic CFTs is universal and depends only on c. For holographic CFTs, this universal behavior extends to self dual temperature, \( \beta_{sd} = 2\pi \).
- For large \( L_0 \) \& \( \bar{L}_0 \) charges and fixed central charges, entropy is
  \[
  S_{CFT} = 2\pi \sqrt{\frac{cR}{6} L_0} + 2\pi \sqrt{\frac{cL}{6} \bar{L}_0}.
  \]
- This matches with Bekenstein-Hawking entropy of AdS_3 black hole.
Beyond AdS

- Flat spacetimes
- de Sitter spacetimes
- Kerr/CFT
- Lifshitz and Hyperscaling violating geometries (AdS/CMT)

Little is known about their dual field theories!
Warped Conformal Field Theory

- A 2d QFT with a chiral scaling symmetry that acts only on right movers $x^- \rightarrow \lambda x^-$, in contrast to CFTs which have a second independent scaling symmetry $x^+ \rightarrow \lambda x^+$. 

- They don’t satisfy the Brown-Henneaux’s Boundary condition.

- A 2d translational-invariant theory with a chiral scaling symmetry must have an extended local algebra.

CFT: The usual CFT with two copies of Virasoro algebra $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$

WCFT: One Virasoro algebra plus a $U(1)$ Kac-Moody algebra $\rightarrow$ WCFT with $\text{SL}(2,\mathbb{R}) \times U(1)$
Local symmetries

- The local symmetries include two arbitrary functions worth of freedom in coordinate transformation:

\[
x^- \to f(x^-) , \quad x^+ \to x^+ - g(x^-)
\]

- These symmetries lead to a new type of modular transformation on the torus.

- Applied to finite T partition function, modular transformation relates thermodynamical quantities at low rotation to those at high rotations.
Asymptotic entropy

\[
S_{\text{WCFT}} = -\frac{4\pi i P_0 P_0^{\text{vac}}}{k} + 4\pi \sqrt{\left( L_0^{\text{vac}} - \frac{(P_0^{\text{vac}})^2}{k} \right) \left( L_0 - \frac{P_0^2}{k} \right)}
\]

- $L_0$ is the charge associated to SL(2,R).
- $P_0$ is the U(1) charge.
- $c$ and $k$ are the central extensions of Virasoro + Kac-Moody algebra.

Modular transformation

\[
Z(\beta, \theta) = e^{ik\frac{\beta^2}{4\theta}} Z \left( \frac{2\pi\beta}{\theta}, -\frac{4\pi^2}{\theta} \right)
\]
Holographic WCFTs

- The near horizon geometry of every extremal black hole in any dimension has the global $SL(2,R) \times U(1)$ symmetry.

- Examples: The near horizon geometry of the extremal 4d Kerr black hole at the fixed polar angle.

- The deformation of AdS$_3$ changes the asymptotic but preserves $SL(2,R) \times U(1)$ symmetry.

- Non-Einstein theories such as TMG and NMG.

- Asymptotic symmetry group (ASG) in TMG spacetime → one Virasoro algebra, one $U(1)$ current algebra extending the exact isometries → WCFT
Algebra

- The symmetry structure in a 2d Lorentzian theory with SL(2,R)_R × U(1)_L global invariance → right moving energy momentum tensor and a right moving U(1) Kac-Moody current.

- The commutators on the plane are:

\[
\begin{align*}
    i[T_\xi, T_\zeta] &= T_{\xi'\zeta - \zeta'\xi} + \frac{c}{48\pi} \int dx^- (\xi''\zeta' - \zeta''\xi') \\
    i[P_\chi, P_\psi] &= \frac{k}{8\pi} \int dx^- (\chi'\psi - \psi'\chi) \\
    i[T_\xi, P_\chi] &= P_{-\chi'\xi}
\end{align*}
\]

\[
T_\xi = -\frac{1}{2\pi} \int dx^- \xi(x^-)T(x^-) \quad P_\chi = -\frac{1}{2\pi} \int dx^- \chi(x^-)P(x^-)
\]
Algebra

- By a change of coordinate \( x^- = e^{i\phi} \) and picking test function \( \xi_n = (x^-)^n = e^{in\phi} \), the commutators on the cylinder are

\[
\begin{align*}
[L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}n(n-1)(n+1)\delta_{n+m} \\
[P_n, P_m] &= \frac{k}{2}n\delta_{n+m} \\
[L_n, P_m] &= -mP_{m+n}
\end{align*}
\]

- Where

\[
L_n = iT_{\xi_{n+1}} \quad P_n = P_{\chi_n}
\]
The infinitesimal transformation

- The commutation relations imply the following infinitesimal transformations of energy momentum tensor and current

\[
\begin{align*}
\delta_\epsilon T(x^-) &= -\epsilon(x^-)\partial_- T(x^-) - 2\partial_-\epsilon(x^-)T(x^-) - \frac{c}{12}\partial^3\epsilon \\
\delta_\gamma T(x^-) &= -\partial_-\gamma(x^-)P(x^-) \\
\delta_\epsilon P(x^-) &= -\epsilon(x^-)\partial_- P(x^-) - \partial_-\epsilon(x^-)P(x^-) \\
\delta_\gamma P(x^-) &= \frac{k}{2}\partial_-\gamma(x^-)
\end{align*}
\]

- Where one defines

\[
\delta_{\epsilon+\gamma} = \delta_\epsilon + \delta_\gamma = -i[T_\epsilon, \cdot] - i[P_\gamma, \cdot]
\]
Phase Transitions

- Hawking and Page → Thermal radiation in AdS beyond certain temperature leads to formation of a black hole. In the AdS/CFT picture it is dual to confining/deconfining phase transition at high T.

- The partition function is

$$Z(\tau, \bar{\tau}) = \text{Tr} e^{2\pi i \tau L_0} e^{2\pi i \bar{\tau} \bar{L}_0},$$

$$Z(\tau) \sim \int \mathcal{D}g e^{-S[g]}$$

$$Z(\tau, \bar{\tau}) \sim \sum_{g_c} e^{-kS[g_c]}, \quad k = \frac{c}{24} = \frac{\ell}{16G} \gg 0,$$

$$Z(\tau, \bar{\tau}) = e^{-\beta G(\tau, \bar{\tau})}.$$
Warped AdS$_3$ black holes

- Squashed three sphere obtained by deforming AdS$_3$

\[ g_{W\text{AdS}} = g_{\text{AdS}_3} - 2H^2 \xi \otimes \xi \]

- $H^2$ is a deformation parameter and $\xi^\mu$ is a constant norm Killing vector of SL(2,R) isometry group of AdS$_3$.

- The resulting geometry possesses an SL(2,R)$\times$ U(1) isometry group and for $\xi^2 = -1, 1, 0$, it is timelike, spacelike or Null Warped AdS$_3$. 

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Warped Solutions

- Spacelike WAdS\(_3\) black hole \(\xi^2 = 1\)

\[
\frac{ds^2}{\hat{l}^2} = dT^2 + \frac{d\hat{r}^2}{(\nu^2 + 3)(\hat{r} - \hat{r}_+)(\hat{r} - \hat{r}_-)} - \left(2\nu \hat{r} - \sqrt{\hat{r}_+\hat{r}_-(\nu^2 + 3)}\right) dT d\theta \\
+ \frac{\hat{r}}{4} \left(3(\nu^2 - 1)\hat{r} + (\nu^2 + 3)(\hat{r}_+ + \hat{r}_-) - 4\nu \sqrt{\hat{r}_+\hat{r}_-(\nu^2 + 3)}\right)
\]

- Timelike WAdS\(_3\) solution \(\xi^2 = -1\)

\[
ds^2 = -dt^2 - 4\omega r dt d\phi + \frac{\ell^2 dr^2}{(2r^2(\omega^2 \ell^2 + 1) + 2\ell^2 r)} - \left(\frac{2r^2}{\ell^2}(\omega^2 \ell^2 - 1) - 2r\right) d\phi^2.
\]
Boundary conditions

The boundary conditions are preserved by following set of diffeomorphisms:

\[
\ell_n = e^{i n \theta} \partial_\theta - i n \hat{r} e^{i n \theta} \partial_r, \\
p_n = e^{i n \theta} \partial_T.
\]

This generates a centerless Virasoro-Kac-Moody U(1) algebra

\[
i[\ell_n, \ell_m] = (n - m) \ell_{n+m}, \quad i[\ell_n, p_m] = -m \ p_{n+m}, \quad i[p_n, p_m] = 0.
\]

with the corresponding canonical charges

\[
L_n := Q_{\ell_n}, \quad P_n := Q_{p_n}.
\]
Topological Massive Gravity Theory

- Einstein-Hilbert action plus a gravitational Chern-Simons term

\[ S = \frac{1}{16\pi} \int d^3x \sqrt{-g} (R - 2\Lambda) + \frac{1}{16\pi} \frac{1}{2\mu} \int d^3x \sqrt{-g} \epsilon_{\lambda\mu\nu} \Gamma^\rho_{\lambda\sigma} \left( \partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\sigma_{\mu\tau} \Gamma^{\tau}_{\nu\rho} \right) \]

- \( \mu \) is a Chern-Simons coupling and is positive

- The equation of motion is

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \]

- where

\[ C_{\mu\nu} = \epsilon_{\mu}^{\alpha\beta} \nabla_\alpha \left( R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R \right) \]
Warped $\text{AdS}_3$ black holes in grand canonical ensemble

\begin{equation}
    ds^2_{WBTZ} = ds^2_{BTZ} - 2H^2 \xi \otimes \xi
\end{equation}

\begin{equation}
    ds^2_{BTZ} = \left(8M - \frac{r^2}{l^2}\right) dt^2 - \frac{r^2 dr^2}{8Mr^2 - \frac{r^4}{l^2} - 16J^2} + 8J dt d\phi + r^2 d\phi^2
\end{equation}

\begin{equation}
    \xi^\mu = \frac{1}{\sqrt{8}} \frac{l}{(Ml - J)} (-\partial_t + \partial_\phi)
\end{equation}

\begin{equation}
    g_{\mu\nu} = \begin{pmatrix}
        -r^2 - \frac{H^2(-r^2 - 4J + 8M)^2}{4(M-J)} + 8M & 0 & 4J - \frac{H^2(4J - r^2)(-r^2 - 4J + 8M)}{4(M-J)} \\
        0 & \frac{1}{16J^2 + r^2 - 8M} & 0 \\
        4J - \frac{H^2(4J - r^2)(-r^2 - 4J + 8M)}{4(M-J)} & 0 & r^2 - \frac{H^2(4J - r^2)^2}{4(M-J)}
    \end{pmatrix}
\end{equation}
The change of coordinate is charge-dependent, so

\[ g_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad g_{++} = j_{++} r^4 + h_{++} r^2 + f_{++}, \quad g_{+-} = \left(\frac{1}{2} + j_{+-}\right) r^2 + O(1) \]
\[ g_{--} = f_{--}, \quad g_{+r} = O(1/r), \quad g_{r-} = O(1/r^3). \]

\[ \tilde{P}_0 := Q_{\partial_x^+} = \frac{P_0^2}{k}, \quad \tilde{L}_0 := Q_{\partial_x^-} = L_0 - \frac{P_0^2}{k}. \]

\[ L_0^{\tilde{vac}} = -\frac{c R}{24}, \quad P_0^{\tilde{vac}} = -\frac{c L}{24}. \]
Vacuum Solutions

- Godel Spacetime

\[ ds^2 = -dt^2 - 4\omega rdtd\phi + \frac{\ell^2 dr^2}{(2r^2(\omega^2\ell^2 + 1) + 2\ell^2 r)} - \left( \frac{2r^2}{\ell^2}\left(\omega^2\ell^2 - 1\right) - 2r \right) d\phi^2. \]

\[ m^2 = -\frac{(19\omega^2\ell^2 - 2)}{2\ell^2}, \quad \Lambda = -\frac{(11\omega^4\ell^4 + 28\omega^2\ell^2 - 4)}{2\ell^2(19\omega^2\ell^2 - 2)}. \]
Warped AdS$_3$ black holes in quadratic ensemble

$$\frac{ds^2}{l^2} = dt^2 + \frac{dr^2}{(\nu^2 + 3)(r - r_+)(r - r_-)} + (2\nu r - \sqrt{r_+r_-}(\nu^2 + 3))dtd\phi + \frac{r}{4} \left[ 3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\nu \sqrt{r_+r_-}(\nu^2 + 3) \right] d\phi^2.$$ 

$$m^2 = -\frac{(20\nu^2 - 3)}{2l^2}, \quad \Lambda = -\frac{m^2(4\nu^4 - 48\nu^2 + 9)}{(20\nu^2 - 3)^2}.$$ 

- $\nu^2 > 1 \rightarrow$ spacelike stretched
- $\nu^2 < 1 \rightarrow$ spacelike squashed
- $\nu^2 = 1 \rightarrow$ locally AdS$_3$ space (BTZ BH)
Calculating conserved charges

1. ADT Formalism
2. The SL(2,R) reduction method
3. The Solution Phase Space Method
Gibbs Free Energy

- To study the stability in the grand canonical ensemble, we find the Gibbs free energy for black hole and vacuum as

\[ G(T, \Omega) = TS[g_c] \]

\[ G = M - TS - \Omega J. \]

\[
S_E[AdS(\tau)] = \frac{i\pi}{12l} (c\tau - \bar{c}\tau)
\]

\[
c, \bar{c} = \frac{3l}{2} \left( 1 \pm \frac{1}{\mu l} \right)
\]

\[
G_{AdS} = -\frac{1}{8l} \left( 1 - \frac{\Omega_E}{\mu} \right).
\]

\[
ds_{BTZ}^2 \left[-\frac{1}{\tau}\right] = ds_{AdS}^2[\tau]
\]

\[
S_E[BTZ(\tau)] = -\frac{i\pi}{12l} \left( \frac{c}{\tau} - \frac{\bar{c}}{\bar{\tau}} \right)
\]

\[
G^{BTZ}(T, \Omega) = -\frac{\pi^2 T^2}{2(1 - \Omega^2)} \left( 1 + \frac{\Omega}{\mu} \right)
\]

\[
G^{AdS}(T, \Omega) = -\frac{1}{8} \left( 1 - \frac{\Omega}{\mu} \right).
\]
Hessian and free energy

- The hessian is defined as

\[
H = \begin{pmatrix}
\frac{\partial^2 G}{\partial T^2} & \frac{\partial^2 G}{\partial T \partial \Omega} \\
\frac{\partial^2 G}{\partial \Omega \partial T} & \frac{\partial^2 G}{\partial \Omega^2}
\end{pmatrix}
\]

- Also, the Gibbs free energy of warped solutions are

\[
G_{WAdS}(T, \Omega) = -\frac{3 - 4H^2 - \Omega}{24\sqrt{1 - 2H^2}}
\]

\[
G_{WBTZ}(T, \Omega) = -\frac{\pi^2 T^2 (3 - 4H^2 + \Omega)}{6\sqrt{1 - 2H^2} (1 - \Omega^2)}
\]
Phase transitions of $\text{AdS}_3$ black hole

$$\Delta G = G_{\text{AdS}} - G_{\text{BTZ}}$$

(a) GR limit, e.g. $\mu = 50$
(b) Stretched-squashed limit $\mu = 3$
(c) Local stability limit $\mu \lesssim 1$
Phase transitions of Warped AdS$_3$

(a) Flat space limit $H^2 = -\frac{3}{2}$ or $\mu = 36$ and $\Lambda \lesssim 0$

(b) Stretched-squashed limit $H^2 = 0$ or $\mu = 3$ and $\Lambda = -1$

(c) Local stability limit $H^2 \lesssim \frac{1}{2}$ or $\mu \gtrsim 0$ and $\Lambda \gtrsim -2/3$
Bergshoeff-Hohm-Townsend theory

\[
S = \frac{1}{16\pi G_N} \int d^3 x \sqrt{-g} \left[ R - 2\Lambda + \frac{1}{m^2} \left( R_{\mu\nu} R_{\mu\nu} - \frac{3}{8} R^2 \right) \right]
\]

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{m^2} K_{\mu\nu} = 0.
\]

\[
K_{\mu\nu} = \nabla^2 R_{\mu\nu} - \frac{1}{4} (\nabla_{\mu} \nabla_{\nu} R + g_{\mu\nu} \nabla^2 R) - 4 R^\sigma_{\mu \sigma\nu} R_{\sigma\nu} + \frac{9}{4} R R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left( 3 R^{\alpha\beta} R_{\alpha\beta} - \frac{13}{8} R^2 \right)
\]
Conserved charges in grand canonical ensemble

\[
g_{\mu \nu} = \begin{pmatrix}
-r^2/l^2 - \frac{H^2(-r^2 - 4lJ + 8l^2M)^2}{4l^3(lM - J)} + 8M & 0 & 4J - \frac{H^2(4lJ - r^2)(-r^2 - 4lJ + 8l^2M)}{4l^2(lM - J)} \\
0 & \frac{1}{r^2 + \frac{l^2}{r^2} - 8M} & 0 \\
4J - \frac{H^2(4lJ - r^2)(-r^2 - 4lJ + 8l^2M)}{4l^2(lM - J)} & 0 & r^2 - \frac{H^2(4lJ - r^2)^2}{4l(lM - J)}
\end{pmatrix}
\]

\[
M = \frac{16 (1 - 2H^2)^{3/2} M}{GL (17 - 42H^2)}, \quad J = \frac{16 (1 - 2H^2)^{3/2} J}{G (17 - 42H^2)}.
\]

\[
S = \frac{16\pi (1 - 2H^2)^{3/2}}{G (17 - 42H^2)} \sqrt{l^2 M + \sqrt{4M^2 - J^2l^2}}.
\]
Hawking Page Phase diagrams of BTZ BH in grand canonical ensemble

$\Omega$

$m = 1.05.$

$\Omega$

$m = 10.$
Warped BTZ BH in quadratic non-local ensemble

\[
\frac{ds^2}{l^2} = dt^2 + \frac{dr^2}{(v^2 + 3)(r - r_+)(r - r_-)} + (2\nu r - \sqrt{r_+ r_- (v^2 + 3)}) dt d\varphi
\]
\[
+ \frac{r}{4} \left[ 3(v^2 - 1)r + (v^2 + 3)(r_+ + r_-) - 4\nu \sqrt{r_+ r_- (v^2 + 3)} \right] d\varphi^2.
\]

The phase diagram for \( \nu = 0.387 \).

The phase diagram for different \( \nu \).
Phase diagram of hairy BH solution

The local stable region for $b = 20$.

The phase diagram for $b = 20$. 
Entanglement Entropy (EE) of Warped CFTs

- The EE of a single interval in the vacuum of 2d CFT on the cylinder is

\[ S_{EE} = \frac{c}{3} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \right) \]

- \( L \) is the length of circle and \( \ell \) is the length of the interval.

- Using Warped CFT techniques, the analogous formula is

\[ S_{EE} = i P_0^{\text{vac}} \ell \left( \frac{L}{L} - \frac{\bar{\ell}}{\ell} \right) - 4 L_0^{\text{vac}} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \right) \]

- Where \( L \) and \( \bar{L} \) are identifications of the circle and \( \ell \) and \( \bar{\ell} \) are separations in space and time.

- The first term is independent of the Renyi replica index.
EE for the CFT$_2$ dual to AdS$_3$ in NMG

- For the NMG case, the Virasoro and Kac-Moody operators are

$$\tilde{P}_0^{(\text{vac})} = \mathcal{M}^{(\text{vac})}, \quad \tilde{L}_0^{(\text{vac})} = \frac{1}{k} (\mathcal{M}^{\text{vac}})^2 \quad \mathcal{M}^{(\text{vac})} = i \mathcal{M}_{\text{God}} = -i \frac{4\ell^2 \omega^2}{G(19\ell^2 \omega^2 - 2)}.$$

- So the entanglement entropy is

$$S_{EE} = \frac{4\ell^2 \omega^2}{G(19\ell^2 \omega^2 - 2)} \left( \ell^* \left( \frac{\bar{L}}{L} - \frac{\bar{\ell}^*}{\ell^*} \right) + \frac{2\ell^2 \omega}{1 + \ell^2 \omega^2} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi \ell^*}{L} \right) \right).$$
Thanks for Your Attention!