Linearization Instability

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◦ "Linearization instability for generic gravity in AdS spacetime," Phys. Rev. D **97**, no. 2, 024028 (2018).

◦ "Linearization Instability of Chiral Gravity," arXiv:1804.05602 [hep-th].

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\begin{tikzcd}
\mathscr{E}_{\mu\nu} = 0 \arrow[r, "exact\ solution"] \arrow[d] \arrow[dd, bend right] & g_{\mu\nu} \arrow[d] \arrow[dd, bend left] \\
(\mathscr{E}_{\mu\nu})^{(1)} \cdot h = 0 \arrow[r, "linear\ solution"] & h_{\mu\nu} \\
(\mathscr{E}_{\mu\nu})^{(2)} \cdot [h,h] + (\mathscr{E}_{\mu\nu})^{(1)} \cdot k = 0 \arrow[r] & k_{\mu\nu}
\end{tikzcd}
$$

$$\bar{g} := g(\lambda)|_{\lambda=0}, \quad h := \frac{d}{d\lambda} g(\lambda)|_{\lambda=0}, \quad k := \frac{d^2}{d\lambda^2} g(\lambda)|_{\lambda=0}. \quad (1)$$

— Outline —

Problems in perturbation theory

○ In GR, perturbation theory about a background solution fails if the background spacetime has a Killing symmetry and a compact spacelike Cauchy surface.

○ This failure, dubbed as *linearization instability*, shows itself as non-integrability of the perturbative infinitesimal deformation to a finite deformation of the background.

○ Namely, the linearized field equations have spurious solutions which cannot be obtained from the linearization of exact solutions.

○ In practice, one can show the failure of the linear perturbation theory by showing that a certain quadratic (integral) constraint, called the TAUB charge, for the linearized, solutions is not satisfied.

○For non-compact Cauchy surfaces, the situation is different and for example, Minkowski space having a non-compact Cauchy surface, is linearization stable.

○ For modified theories even in the non-compact Cauchy surface cases, there are some theories which show linearization instability about their anti-de Sitter backgrounds.

○ Recent D dimensional critical and three dimensional chiral gravity theories are two such examples.

○ This observation sheds light on the paradoxical behavior of vanishing conserved charges (mass, angular momenta) for non-vacuum solutions, such as black holes, in these theories.
See S. Deser and B.T. "Energy in generic higher curvature gravity theories," Phys. Rev. D **67**, 084009 (2003)

A Simple Example

$$F : R \times R \to R$$

given as

$$F(x, y) = x(x^2 + y^2)$$

For $F(x, y) = 0$, the exact solution set is $(0, y)$,

The linearized solution space

$$\delta F(x, y) = 0$$

is also one dimensional as long as $y \neq 0$.
But at exactly the point $(0, 0)$, the linearized solution space is 2 dimensional and there are modes $\delta x \neq 0$, which do not come from the linearization of any exact solution.

Definition of linearization stability

A nonlinear equation $F(x) = 0$ is said to be linearization stable at a solution x_0 if every solution δx to the linearized equation

$$F'(x_0) \cdot \delta x = 0$$

is tangent to a curve of solutions to the original nonlinear equation.

IMPORTANT:

In some nonlinear theories, not all solutions to the linearized field equations represent linearized versions of exact (nonlinear) solutions.

Example : Quarks do not represent linearization of any exact solution in QCD.

Generic Gravity

In the index-free notation the covariant two-tensor equation reads

$$\mathcal{E}(g) = 0, \tag{2}$$

together with the covariant divergence condition which comes from the diffeomorphism invariance of the theory

$$\delta_g \mathcal{E}(g) = 0, \tag{3}$$

where δ_g denotes the divergence operator with respect to the metric g.

∘ Assume that there is is a one-parameter family of solutions to (2) denoted as $g(\lambda)$ which is at least twice differentiable with respect to λ parameterizing the solution set.

∘ Assume that $\bar{g}$ exactly solves the vacuum equations $\mathcal{E}(\bar{g}) = 0$ and we compute the first derivative of the field equations with respect to λ and evaluate it at $\lambda = 0$ as

$$\frac{d}{d\lambda}\mathcal{E}(g(\lambda))|_{\lambda=0} = D\mathcal{E}(g(\lambda)) \cdot \frac{dg(\lambda)}{d\lambda}|_{\lambda=0} = 0, \tag{4}$$

where D denotes the Fréchet derivative and the center-dot denotes "along the direction of the tensor that comes next" and we have used the chain rule.

In local coordinates, this equation is just the first order "linearization" of the field equations which we shall denote as

$$(\mathcal{E}_{\mu\nu})^{(1)} \cdot h = 0$$

It is important to understand that solutions of this equation yield all possible h tensors (up to diffeomorphisms), which are tangent to the exact solution $g(\lambda)$ at $\lambda = 0$ in the space of solutions.

To understand if there are any $further$ constraints on the linearized solutions h, let us consider the second derivative of the field equation with respect to λ and evaluate it at $\lambda = 0$ to arrive at

$$\frac{d^2}{d\lambda^2}\mathcal{E}(g(\lambda))|_{\lambda=0} = \tag{5}$$
$$\left(D^2\mathcal{E}(g(\lambda)) \cdot \left[\frac{dg(\lambda)}{d\lambda}, \frac{dg(\lambda)}{d\lambda}\right] + D\mathcal{E}(g(\lambda)) \cdot \frac{d^2 g(\lambda)}{d\lambda^2}\right)|_{\lambda=0} = 0,$$

In local coordinates as

$$(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] + (\mathcal{E}_{\mu\nu})^{(1)} \cdot k = 0, \tag{6}$$

where again $(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h]$ denotes the second order linearization of the field equation about the background $\bar{g}$.

○ Given a solution h of $(\mathcal{E}_{\mu\nu})^{(1)} \cdot h = 0$, equation (6) $determines$ the tensor field k, which is the second order derivative of the metric $g(\lambda)$ at $\lambda = 0$.

○ If such a k can be found then there is no further constraint on the linearized solution h. In that case, the field equations are said to be linearization stable at the exact solution $\bar{g}$. This says that the infinitesimal deformation h is tangent to a full (exact) solution and hence it is integrable to a full solution.

○ Of course, what is tacitly assumed here is that in solving for k in (6), one cannot change the first order solution h, it must be kept intact for the perturbation theory to make any sense.

Space of Solutions: Hilbert Manifold

We can understand these results form a more geometric vantage point as follows.

For the spacetime manifold $\mathcal{M}$, let $\mathcal{S}$ denote the set of solutions of the field equations $\mathcal{E}(g) = 0$.
The obvious question is (in a suitable Sobolev topology), when does this set of solutions form a smooth manifold whose tangent space at some "point" $\bar{g}$ is the space of solutions (h) to the linearized equations?

The folklore in the physics literature is not to worry about this question and just assume that the perturbation theory makes sense and the linearized solution can be improved to get better solutions, or the linearized solution is assumed to be integrable to a full solution.
But as we have given examples above, there are cases when the perturbation theory fails and the set $\mathcal{S}$ has a conical singularity instead of being a smooth manifold.

One should not confuse this situation with the case of dynamical instability as the latter really allows a "motion" or perturbation about a given solution. Here linearization instability refers to a literal break-down of the first order perturbation theory.

Conserved Charges

So far, in our discussion we have not assumed anything about whether the spacetime has a compact Cauchy surface or not.

First, let us now assume that the spacetime has a compact spacelike Cauchy surface and has at least one Killing vector field.

Then we can get an $integral\ constraint$ on h, without referring to the k tensor as follows. Let $\bar{\xi}$ be a Killing vector field of the metric $\bar{g}$, then the following vector field

$$T := \bar{\xi} \cdot D^2\mathcal{E}(\bar{g}) \cdot [h, h]\,, \tag{7}$$

is divergence free, since $\delta_{\bar{g}} D^2\mathcal{E}(\bar{g}).\,[h, h] = 0$ due to the linearized Bianchi identity .

Then we can integrate T over a compact hypersurface Σ and observe that the integral (for the sake of definiteness, here we consider the 3+1 dimensional case)

$$\int_{\Sigma} d^3\Sigma\, \sqrt{\gamma}\, T \cdot \hat{n}_\Sigma \tag{8}$$

is independent of hypersurface Σ where γ is the pull-back metric on the hypersurface and $\hat{n}_\Sigma$ is the unit future pointing normal vector. Let us restate the result in a form that we shall use below: given two compact disjoint hypersurfaces Σ_1 and Σ_2 in the spacetime $\mathcal{M}$, we have the statement of the "charge conservation" as the equality of the integration over the two hypersurfaces

$$\int_{\Sigma_1} d^3\,\Sigma_1\, \sqrt{\gamma_{\Sigma_1}}\, T \cdot \hat{n}_{\Sigma_1} = \int_{\Sigma_2} d^3\,\Sigma_2\, \sqrt{\gamma_{\Sigma_2}}\, T \cdot \hat{n}_{\Sigma_2}. \tag{9}$$

We can now go to (6) and after contracting it with the Killing tensor $\bar{\xi}$, and integrating over Σ, we obtain the identity

$$\int_{\Sigma} d^3\Sigma \sqrt{\gamma}\,\bar{\xi}^{\mu}\hat{n}^{\nu}(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h,h] = -\int_{\Sigma} d^3\Sigma \sqrt{\gamma}\,\bar{\xi}^{\mu}\hat{n}^{\nu}\left(\mathcal{E}_{\mu\nu}\right)^{(1)} \cdot k\,. \tag{10}$$

In a generic theory, this conserved Killing charge is called the ADT charge when the symmetric two-tensor k is the just the linearized two tensor h . Once the field equations of the theory are given, it is possible, albeit after some lengthy computation, to show that one can write the integral on the right-hand side as a total derivative.

$$\bar{\xi}^{\mu}\left(\mathcal{E}_{\mu\nu}\right)^{(1)} \cdot h = \bar{\nabla}_{\alpha}\left(\mathcal{F}^{\alpha}{}_{\nu\mu}\bar{\xi}^{\mu}\right), \tag{11}$$

with an anti-symmetric tensor $\mathcal{F}$ in α and ν. Hence if the Cauchy surface is compact without a boundary, the ADT charge vanishes identically, namely

$$Q_{ADT}\left[\bar{\xi}\right] := \int_{\Sigma} d^3\Sigma \sqrt{\gamma}\,\hat{n}^{\nu}\,\bar{\xi}^{\mu}\left(\mathcal{E}_{\mu\nu}\right)^{(1)} \cdot h = 0, \tag{12}$$

which via (10) says that one has the vanishing of the integral on the left hand-side which is called the Taub conserved quantity:

$$Q_{Taub}\left[\bar{\xi}\right] := \int_{\Sigma} d^3\Sigma \sqrt{\gamma}\,\hat{n}^{\nu}\,\bar{\xi}^{\mu}\left(\mathcal{E}_{\mu\nu}\right)^{(2)} \cdot [h,h] = 0, \tag{13}$$

which must be $automatically$ satisfied for the case when h is an integrable deformation. Otherwise this equation is a second order constraint on the linearized solutions.

Einstein's Theory

Let Ein denote the $(0, 2)$ Einstein tensor, and h denote a symmetric two tensor field as described above and X be a vector field, then the effect of infinitesimal one-parameter diffeomorphisms generated by X follows as

$$DEin(g) \cdot \mathcal{L}_X g = \mathcal{L}_X Ein(g), \tag{14}$$

which in local coordinates reads

$$\delta_X \left(G_{\mu\nu}\right)^{(1)} \cdot h = \mathcal{L}_X \bar{G}_{\mu\nu}, \tag{15}$$

where $G_{\mu\nu} := Ein(e_\mu, e_\nu)$ and $Ein := Ric - \frac{1}{2}Rg$. One has

$$\delta_X (G_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X \left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right) = \mathcal{L}_X \bar{G}_{\mu\nu}. \tag{16}$$

At the second order of linearization, one has

$$D^2 Ein(g) \cdot (h, \mathcal{L}_X g) + DEin(g) \cdot \mathcal{L}_X h = \mathcal{L}_X (DEin(g) \cdot h), \tag{17}$$

whose local version reads

$$\delta_X (G_{\mu\nu})^{(2)} \cdot [h, h] + (G_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h = \mathcal{L}_X (G_{\mu\nu})^{(1)} \cdot h. \tag{18}$$

Stability of flat space

$$(\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h] + (\mathcal{G}_{\mu\nu})^{(1)} \cdot k = 0, \tag{19}$$

where $(\mathcal{G}_{\mu\nu}))^{(1)} \cdot k$ is a simple object but the the second order object $\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h]$ is quite cumbersome.

○ It is very hard to use this equation to show that for a generic background $\bar{g}_{\mu\nu}$, a $k_{\mu\nu}$ can be found or cannot be found.
○ So one actually resorts to a weaker condition that the Taub charges vanish which, as we have seen, results from integrating this equation after contracting with a Killing vector field $\bar{\xi}^{\mu}$.

○ assume that such a k exists in the form

$$k_{\mu\nu} = a\, h_{\mu\beta} h^{\beta}_{\nu} + b\, h h_{\mu\nu} + \bar{g}_{\mu\nu}(c\, h^2_{\alpha\beta} + d\, h^2), \tag{20}$$

where $k := k_{\mu\nu}\bar{g}^{\mu\nu}$ and a, b, c, d are constants to be determined and all the raising and lowering is done with the background metric $\bar{g}$. After a long calculation, one finds

$$k_{\mu\nu} = h_{\mu\beta} h^{\beta}_{\nu} - \frac{5}{8(D-2)} \bar{g}_{\mu\nu} h^2_{\alpha\beta} \tag{21}$$

Therefore there is no further constraint on the linearized solutions and the Minkowski space is linearization stable.

TMG and Chiral gravity

The full TMG field equations

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = \kappa \tau_{\mu\nu}, \tag{22}$$

coupled with matter fields as an initial value problem: The ADM decomposition of the metric reads

$$ds^2 = -(n^2 - n_i n^i) dt^2 + 2 n_i dt dx^i + \gamma_{ij} dx^i dx^j, \tag{23}$$

where $(n,\ n_i)$ are lapse and shift functions and γ_{ij} is the $2D$ spatial metric.

The extrinsic curvature (k_{ij}) of the surface

$$2nk_{ij} = \dot{\gamma}_{ij} - 2D_{(i}n_{j)}$$

The Ricci tensor :

$$\begin{aligned} R_{ij} &= {}^{(2)}R_{ij} + kk_{ij} - 2k_{ik}k^k_j \\ &+ \frac{1}{n}(\dot{k}_{ij} - n^k D_k k_{ij} - D_i \partial_j n - 2k_{k(i} D_{j)} n^k), \end{aligned} \tag{24}$$

where ${}^{(2)}R_{ij}$ is the Ricci tensor of the hypersurface and $k \equiv \gamma^{ij} k_{ij}$.

One can obtain from the constraint equations the following relation

$$\partial_i \dot{\varphi} = -J_i + \frac{1}{2\mu} \epsilon^m{}_i \dot{\varphi} \partial_m \dot{\varphi}, \tag{25}$$

where the "source current" on the hypersurface is

$$J_i := 2\kappa \tau_{0i} + \frac{\kappa}{\mu} \epsilon^m{}_i \tau_{00}.$$

Contracting (25) with the epsilon-tensor, one arrives at

$$\frac{2\mu}{\dot{\varphi}} \epsilon^{mi} \partial_m \dot{\varphi} \left(1 + \frac{\dot{\varphi}^2}{4\mu^2}\right) = -\frac{2\mu}{\dot{\varphi}} \epsilon^{mi} J_m + J^i. \tag{26}$$

In the case of vacuum, $\tau_{\mu\nu} = 0$, and so $J_i = 0$, the unique solution to (26) is of the form $\varphi_0 = ct$, where c is a constant which can be found from the trace equation that reads $R = 6\Lambda$. So $c = 2\sqrt{\Lambda} \equiv \frac{2}{\ell}$, which is the de Sitter (dS) solution and $\ell > 0$ is its radius.

Turning on a compactly supported matter perturbation with $\delta\tau_{\mu\nu} \neq 0$, one has $\delta J_i \neq 0$ and perturbing the constraint equations about φ_0 as $\varphi = \varphi_0 + \delta\varphi$,

$$\mu(1 + \frac{1}{\mu^2\ell^2})\epsilon^m{}_i\partial_m\delta\dot{\varphi} \tag{27}$$
$$= (\partial_i + \frac{1}{\mu\ell}\epsilon^m{}_i\partial_m)\kappa\delta\tau_{00} + 2\mu(\epsilon_i{}^m + \frac{1}{\mu\ell}\delta^m{}_i)\kappa\delta\tau_{0m},$$

from which, for the dS case, one can solve the perturbation ($\delta\varphi$) and hence the perturbed metric by integration in terms of the perturbed matter fields on the Cauchy surface. Hence dS is linearization stable in TMG for any finite value of $\mu\ell$. The other linearized constraints are compatible with this solution.

For AdS

$$\begin{aligned} & \mu(1-\tfrac{1}{\mu^2\ell^2})\epsilon^m{}_i\partial_m\delta\dot{\varphi} \\ = & -(\partial_i - \tfrac{1}{\mu\ell}\epsilon^m{}_i\partial_m)\kappa\delta\tau_{00} - 2\mu(\epsilon_i{}^m + \tfrac{1}{\mu\ell}\delta^m{}_i)\kappa\delta\tau_{0m} \end{aligned} \tag{28}$$

and once again the perturbation theory is valid for $generic$ values of $\mu\ell$ in AdS as in the case of dS.

But at the chiral point, $\mu\ell = 1$, the left-hand side vanishes identically and there is an unphysical constraint on the matter perturbations $\delta\tau_{0m}$ and $\delta\tau_{00}$ in addition to their background covariant conservation.

Moreover, the metric perturbation is not determined by the matter perturbation. What this says is that in the chiral gravity limit of TMG, for AdS, the exact AdS solution is linearization unstable.

See W. Li, W. Song and A. Strominger, "Chiral Gravity in Three Dimensions," JHEP **0804**, 082 (2008).

Symplectic Structure

Let us give another argument for the linearization instability of AdS making use of the symplectic structure of TMG which was found by Nazaroglu-Nutku and B.T.

$$\omega := \int_\Sigma d\Sigma_\alpha \sqrt{|g|}\mathcal{J}^\alpha$$

where Σ is the initial value surface. ω is a closed ($\delta w = 0$) non-degenerate (except for gauge directions) 2-form for full TMG including chiral gravity. Here the on-shell covariantly conserved symplectic current reads

$$\begin{aligned}\mathcal{J}^\alpha \qquad &= \delta\Gamma^\alpha{}_{\mu\nu} \wedge (\delta g^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu}\delta \ln g) \\ &-\delta\Gamma^\nu{}_{\mu\nu} \wedge (\delta g^{\alpha\mu} + \tfrac{1}{2} g^{\alpha\mu}\delta \ln g) \\ &+\tfrac{1}{\mu}\epsilon^{\alpha\nu\sigma}(\delta S^\rho{}_\sigma \wedge \delta g_{\nu\rho} + \tfrac{1}{2}\delta\Gamma^\rho{}_{\nu\beta} \wedge \delta\Gamma^\beta{}_{\sigma\rho}).\end{aligned} \tag{29}$$

○ ω is a gauge invariant object on the solution space, say $\mathcal{Z}$, and also on the (more relevant) quotient $\mathcal{Z}/Diff$ which is the phase space and $Diff$ is the group of diffeomorphisms.

○ Therefore, even without knowing the full space of solutions, by studying the symplectic structure one gains a lot of information for both classical and quantum versions of the theory.

Let us show that for the linearized solutions of chiral gravity, the symplectic 2-form is degenerate and hence not invertible.

In the global coordinates, the background metric is

$$ds^2 = \ell^2(-\cosh^2\rho\, d\tau^2 + \sinh^2\rho\, d\phi^2 + d\rho^2);$$

defining $u = \tau + \phi$, $v = \tau - \phi$, making use of the $SL(2,R) \times SL(2,R)$,

Li $et.al$ found all the primary states and their descendants.

The relevant component of the symplectic current for these modes (for generic $\mu\ell$) can be found as

$$\mathcal{J}^\tau = \frac{(4-S^2)(S+2\mu\ell)\Delta}{8\mu\ell^7(\cosh\rho)^{2(1+\Delta)}}\sin\left(2\Delta\tau + 2S\phi\right), \qquad (30)$$

which yield a vanishing ω at the chiral limit, hence the solution is not viable.

There is an additional the log-mode given by Grumiller and Johansson (JHEP 0807 (2008) 134) and for the log-mode one has

$$\mathcal{J}^\tau = \frac{1}{\mu\ell^7}\tau((1-\mu\ell)\cosh 2\rho + 1)sech^{10}\rho \qquad (31)$$

which yield a linearly growing ω in τ and vanishes on the initial value surface.

What all these say is that first order perturbation theory simply fails in chiral gravity limit of TMG. If the theory makes any sense at the classical and/or quantum level one must resort to a new method to carry out computations. This significantly affects its interpretation in the context of AdS/CFT as the perturbed metric couples to the energy-momentum tensor of the boundary CFT.

Conclusions

○ Linearized solutions by definition satisfy the linearized equations but this is not sufficient; they should also satisfy a quadratic constraint to actually represent linearized versions of exact solutions.

○ With the observation of gravity waves, research in general relativity and its modifications, extensions has entered an exciting era in which many theories might be possibly tested. One major tool of computation in nonlinear theories, such as gravity, is perturbation theory from which one obtains a lot of information and the gravitational wave physics is no exception as one uses the tools of perturbation theory to obtain the wave profile far away from the sources.

○Therefore, the issue of linearization instability arises in any use of perturbation theory as the examples provided here show even for the ostensibly safe case of spacetimes with noncompact Cauchy surfaces.