Entanglement Evolution After a Quantum Quench in Lifshitz Harmonic Models

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Recent Trends in String Theory and Related Topics
School of Physics (IPM)

May 2018

1. Entanglement in QM and QFT
   - Definition
   - Properties

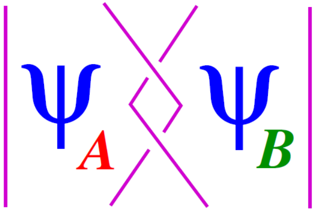
2. Lifshitz-type QFTs
   - Lifshitz scaling symmetry
   - Lifshitz Harmonic Models

3. Evolution of Entanglement Entropy in LHM
   - Quantum Quench in Relativistic QFTs
   - Quantum Quench in LHM

Motivation

- Entanglement is a concept which appears in condensed matter, quantum information and black-hole physics.
- Entanglement measures may help us to study
  1. Quantum phase transitions at $T = 0$
  2. Non-equilibrium processes, e.g., quantum quenches
  3. The connection between gauge theory and gravity
  4. $\ldots$

Entanglement in QM and QFT



(Pure) Entangled states

- Consider two quantum systems, i.e., A and B

$$\mathcal{H}_A, \quad \mathcal{H}_B$$

- Construct M using the tensor product of A and B

$$\mathcal{H}_M = \mathcal{H}_A \otimes \mathcal{H}_B$$

- Separable states

$$|\chi\rangle_{\mathcal{H}_M} = |\psi\rangle_{\mathcal{H}_A} \otimes |\phi\rangle_{\mathcal{H}_B}$$

- (Pure) Entangled states

$$|\chi\rangle_{\mathcal{H}_M} \neq |\psi\rangle_{\mathcal{H}_A} \otimes |\phi\rangle_{\mathcal{H}_B}$$

Example: Spin 1/2 Particles

- Separable states

$$\begin{aligned} \big|\Psi_1\big\rangle &= \big|\uparrow_A\big\rangle \otimes \big|\downarrow_B\big\rangle \\ \big|\Psi_2\big\rangle &= \big|\downarrow_A\big\rangle \otimes \big|\uparrow_B\big\rangle \end{aligned}$$

- Entangled states

$$\big|\Psi_3\big\rangle = \frac{1}{\sqrt{2}}\Big(\big|\uparrow_A\big\rangle \otimes \big|\downarrow_B\big\rangle \pm \big|\downarrow_A\big\rangle \otimes \big|\uparrow_B\big\rangle\Big)$$

Challenge

Entanglement Measures!

Entanglement entropy, Mutual information, $\cdots$

Entanglement Entropy

- Consider the density matrix for a pure quantum system

$$\rho = \big|\psi\rangle\langle\psi\big|$$

- Hilbert space decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- Reduced density matrix for A

$$\rho_A \equiv \mathrm{Tr}_B(\rho) = \sum_{i=1}^{dim[B]} \langle i_B\big|\rho\big|i_B\rangle$$

- von-Neumann entropy for ρ_A

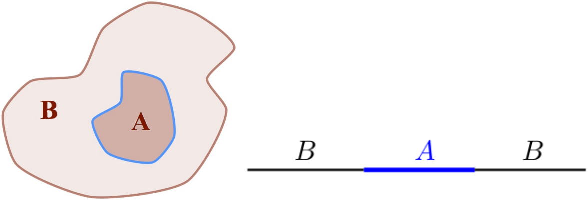
$$S_A \equiv -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right) = -\sum_{i=1}^{dim[A]} \langle i_A\big|\rho_A \log \rho_A\big|i_A\rangle$$

Challenge

Generalization to QFT (Continuum Limit)?

Geometric entropy

- Consider a d-dimensional QFT on $\mathbb{R} \times \mathcal{M}^{(d-1)}$
- Divide $\mathcal{M}^{(d-1)}$ into two parts



The geometric decompostion implies $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$S_A = -\mathrm{Tr}_A \left(\rho_A \log \rho_A \right)$$

Interesting Features

Entanglement entropy

1. corresponds to a non-linear operator in QM
2. is not an extensive quantity (unlike the thermodynamic entropy)
3. satisfies various inequalities, e.g., subadditivity
$$S(A) + S(B) \geq S(A \cup B)$$
4. obeys an area law scaling (in local QFTs)
$$S_A \propto \frac{\mathcal{S}_{d-2}}{\epsilon^{d-2}} + \cdots$$
5. ...

Lifshitz-type QFTs & Lifshitz Harmonic Models

Lifshitz symmetry

- Anisotropic scaling invariance

$$t \to \lambda^z t, \qquad \vec{x} \to \lambda \vec{x}, \qquad z: \text{ Dynamical exponent}$$

[E. M. Lifshitz 1941]

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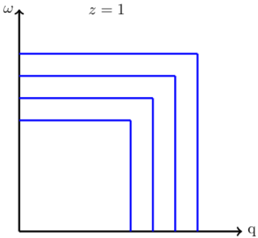
[E. M. Lifshitz 1941]

- Anisotropic RG flow



[J. A. Hertz 1976]

Free Massless Scalar Theory

Lorentz vs. Lifshitz		
	Lorentz	Lifshitz
Lagrangian	$\frac{1}{2}\left(\dot{\phi}^2 - (\partial_i \phi)^2\right)$	$\frac{1}{2}\left(\dot{\phi}^2 - (\partial_i^z \phi)^2\right)$
Mass Dimensions	$[t] = -1,\ [\phi] = \frac{d-1}{2}$	$[t] = -z,\ [\phi] = \frac{d-z}{2}$
Dispersion Relation	$\omega = k$	$\omega = k^z$
Group Velocity	$v_g = 1$	$v_g = z\, k^{z-1}$

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Massless modes with different k have different v_g

Lifshitz-type QFT on a (1+1)d Square Lattice

The harmonic lattice is well known to be the discretized version of free scalar field theory with Lorentz symmetry

- Free Scalar Theory

$$H = \frac{1}{2}\int dx \left[\dot{\phi}^2 + (\partial\phi)^2 + m^2\phi^2\right]$$

- System of N Harmonic Oscillators

$$H = \sum_{n=0}^{N} \left[\frac{1}{2}p_n^2 + \frac{1}{2}\left(q_n - q_{n-1}\right)^2 + \frac{m^2}{2}q_n^2\right]$$

  Nearest Neighbor Interaction

- Dispersion Relation

$$\omega_k = \sqrt{m^2 + k^2} \quad \longrightarrow \quad \omega_k = \sqrt{m^2 + (2\sin\frac{\pi k}{N})^2}$$

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What happens when we turn on a non-trivial z?

Lifshitz-type QFT on a Square Lattice

The Lifshitz harmonic model is the discretized version of free scalar field theory with Lifshitz scaling symmetry

- Lifshitz-type scalar theory

$$H = \frac{1}{2}\int dx \left[\dot{\phi}^2 + (\partial^z\phi)^2 + m^{2z}\phi^2\right]$$

- Discretization on a Square Lattice

$$H = \sum_{n=1}^{N}\left[\frac{p_n^2}{2} + \frac{1}{2}\left(\sum_{k=0}^{z}(-1)^{z+k}\binom{z}{k}q_{n-1+k}\right)^2 + \frac{m^{2z}}{2}q_n^2\right]$$

Long Range Interaction (depending on z)

- Dispersion Relation

$$\omega_k = \sqrt{m^{2z} + k^{2z}} \quad \longrightarrow \quad \omega_k = \sqrt{m^{2z} + (2\sin\frac{\pi k}{N})^{2z}}$$

[MM, Mollabashi 1705.00483-1712.03731; He, Magan and Vandoren, 1705.01147]

Evolution of Entanglement Entropy in LHM

Quantum Quench

- Quantum quench is a simple set-up for studying evolution of entanglement which has also experimental realization (ultracold atoms [Greiner,Mandel,Esslinger,Hansch,Bloch '2002])

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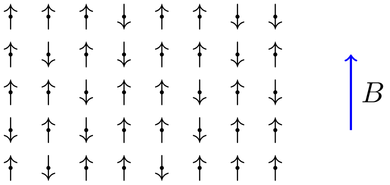

Time Evolution of EE in CFT_2

- Calabrese-Cardy Quench Model [Calabrese, Cardy '05]

Sudden transition from a QFT with a finite mass gap

$(m \sim \xi^{-1})$ to a CFT $(m \sim 0)$

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How entanglement evolves with time after the quench?

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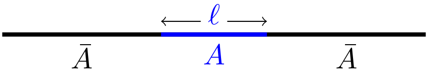
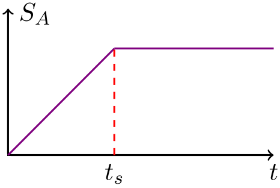
How entanglement evolves with time after the quench?

- The conformal symmetry of post-quench system helps us to find the evolution of EE

Time Evolution of EE in CFT_2



$$S_A(t) \sim \begin{cases} t & t < t_s \\ \frac{\ell}{2} & t > t_s \end{cases}$$



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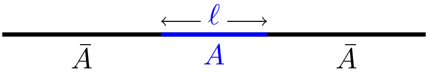

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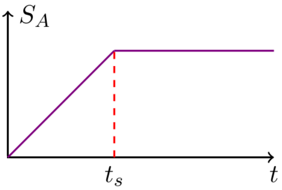
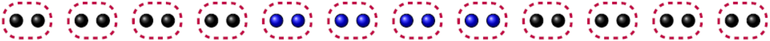


- Quasiparticle picture [Calabrese, Cardy '05]
- $t_s \sim \frac{\ell}{2v} \quad (v = 1 \text{ in } \mathrm{CFT}_2)$

Time Evolution of EE in CFT_2

- Pre-quench Configuration



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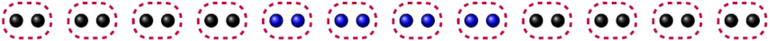
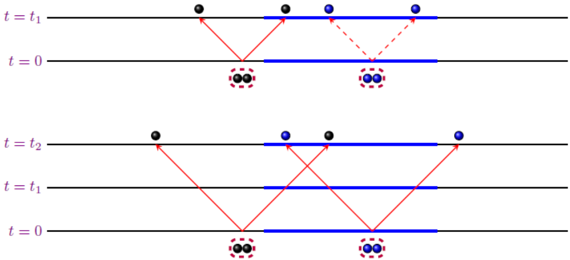


- Post-quench Evolution



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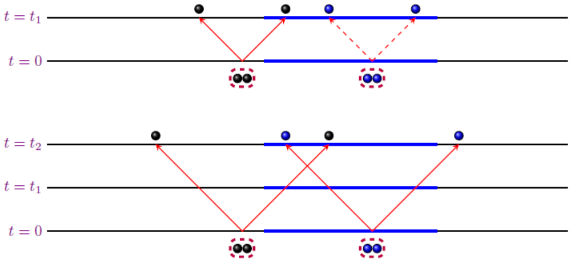

The transition between different scaling regimes can be understood in terms of quasiparticle spectrum [Alba, Calabrese '17]

Free Streaming Quasi Particles

Spectrum of quasiparicles:

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1. CFT: Quasiparticles propagate with a unique $v_g (= 1)$ independent of k (Linear dispersion relation)
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2. A General QFT: Quasiparticles propagate with $v_g(k)$ (Non-linear dispersion relation)
   - Quasiparticles have a wide spectrum

     Example:

$$\omega = \sqrt{k^2 + m^2} \rightarrow \begin{cases} k \ll 1 & v_g \ll 1 & \text{Non-relativistic} \\ k \gg 1 & v_g \sim 1 & \text{Ultra-relativistic} \end{cases}$$

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Zero modes: Quaiparticles that stroll along the subregions because of vanishingly small $v_g(\rightarrow 0)$

Free Streaming Quasi Particles

How the spectrum of quasi-particles affects the entanglement evolution?

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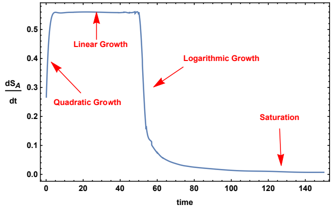
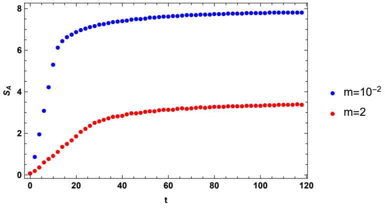
- Transition from Linear Growth to Saturation Regime:
  1. CFT: Instantaneous Transition
  2. QFT (Lattice set-up): Mild Transition (Logarithmic Growth Due to Existence of Zero Modes, i.e., $k \to 0$)





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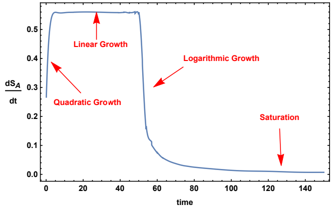
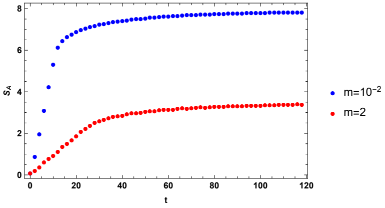




What happens when we have a nontrivial z?

Quantum Quench in LHM

Questions

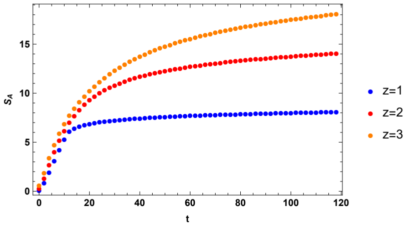
Regarding the entanglement evolution in LHM, different questions may arise:

- Role played by z in $\left\{ \begin{array}{r} \text{Linear Growth} \\ \\ \text{Saturation} \end{array} \right.$ Regimes
- Existence of a Quasi-particle Picture
- Propagation Velocity and z-dependent Lightcone
- $\cdots$

[MM, A. Mollabashi, work in progress]

Existence of a Quasi-particle Picture

- Numerical Reults



For larger values of z:

1. Rate of growth of entanglement entropy increases
2. Width of the logarithmic growth regime becomes larger
3. The saturation value of EE increases

Existence of a Quasi-particle Picture

Origin of Logarithmic Growth

- Lattice Dispersion Relation

$$\omega_k = \sqrt{m^{2z} + (2\sin\frac{\pi k}{N})^{2z}}$$

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- Group Velocity

1. Massive Case $\quad v_g(k \to 0) \sim k^{2z-1} + \mathcal{O}(k^{2z+1})$
2. Massless Case $\quad v_g(k \to 0) \sim k^{z-1} + \mathcal{O}(k^{z+1})$

In both cases we have zero modes ($k \to 0$) which are too lazy to move ($v_g \to 0$)

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Remember that in CFT $(m = 0, z = 1)$ there is no zero modes and we have a unique $v_g (= 1)$

Existence of a Quasi-particle Picture

- Maximum Group Velocity and Lifshitz-cone

$$v_g^{\max} = 2^{z-1}\sqrt{z}\left(\frac{z-1}{z}\right)^{\frac{z-1}{2}}$$

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For any z we have a bound on propagation (similar to the Lieb-Robinson bound)

Conclusions

After a quantum quench in LHM:

- the growth of EE can be divided into two main regimes: initial linear growth and late time logarithmic saturaiton
- Rate of growth of entanglement entropy in linear regime is an increases as a function of z
- For larger values of z, the region with logarithmic scaling becomes broader
- The qualitative, and some of the quantitative, features of $S_A(t)$ can be described in terms of a quasi particle picture

Further Studies

- Considering other entanglement measures, e.g., relative entropy, logarithmic negativity, $\cdots$
- Quasi-particle picture vs. tsunami picture
- Investigating the possible relation between v_g and Lib-Robinson velocity
- $\cdots$

Thank you