Stress Tensor On Null Boundaries

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Recent Trends in String Theory and Related Topics

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   - Stress Tensor on Timelike Boundary

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   - The Set Up
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- Brown and York (1993): starting from a gravitational action with well-posed variational principle, using Hamilton-Jacobi method we can define quasi-local stress-tensor of gravitational field.
- The Brown-York stress-tensor is defined on a timelike boundary.
- The aim is to propose similar tensor on a null boundary.

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- If the variational principle is not well-posed the above method does not work!

Variation of Hilbert-Einstein Action

- Variation of Hilbert-Einstein action in a region with time-like and space-like boundaries

$$\delta\mathcal{S}_{EH} = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}G^{ab}\delta g_{ab} + \frac{1}{16\pi}\sum_i \Big[2\delta\Big(\int_{\mathcal{B}_i} \mathrm{d}^{d-1}x\,\sqrt{|h|}\,K\Big)$$
$$+ \int_{\mathcal{B}_i} \mathrm{d}^{d-1}x\sqrt{|h|}(K^{ab} - K\,h^{ab})\,\delta h_{ab}\Big]$$

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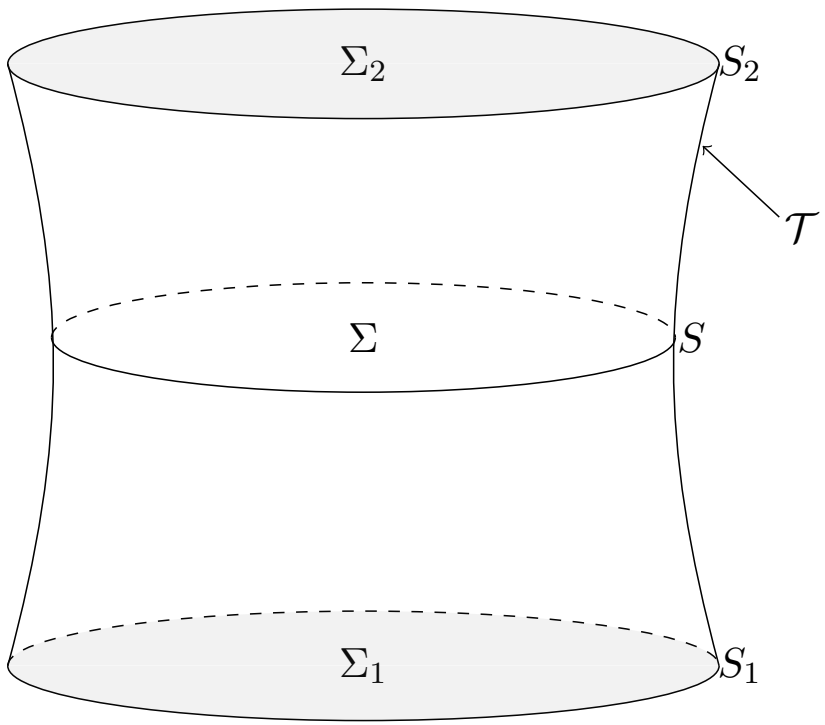
- The well-posed action is:

$$\mathcal{S} = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}\,R - \frac{1}{8\pi}\sum_i \Big[\int_{\mathcal{B}_i} \mathrm{d}^{d-1}x\sqrt{|h|}K\Big]$$

Space-Time region with time-like and space-like boundaries



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- Gravitational energy-momentum-stress :

$$\tau^{ab} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ab}} = \frac{1}{8\pi}(\chi_{ab} - \chi\ \gamma_{ab})$$

Derivative w.r.t induced metric on timelike boundary.

Stress Tensor components

- Consider a foliation of the timelike boundary :

$$ds^2 = -Ndt^2 + q_{ab}(d\sigma^a + V^a dt)(d\sigma^b + V^b dt), \quad u_a = N\nabla_a t$$

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The Stress Tensor components:

$$\epsilon \equiv u^a u^b T_{ab} = -\frac{1}{\sqrt{q}} \frac{\delta \mathcal{S}}{\delta N} \quad \text{Energy density},$$

$$j^a \equiv q^{ac} u^b T_{cb} = -\frac{1}{\sqrt{q}} \frac{\delta \mathcal{S}}{\delta V_a} \quad \text{Momentum density},$$

$$s^{ab} \equiv q^{ac} q^{bd} T_{cd} = \frac{2}{N\sqrt{q}} \frac{\delta \mathcal{S}}{\delta q_{ab}} \quad \text{Spatial stress},$$

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- S_0 is computed for the boundary as embedded in flat space.

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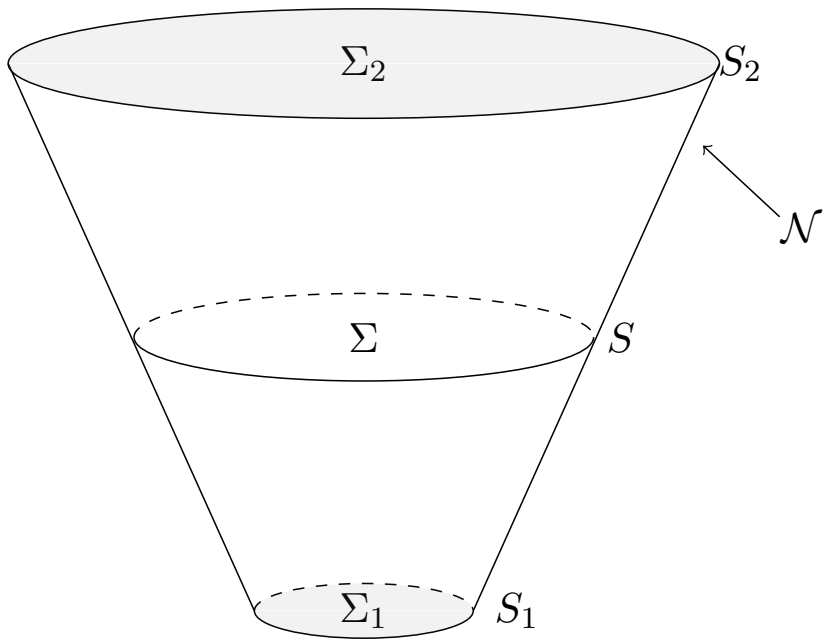
- For Asymptotic flat space and on a timelike boundary such Counterterms are not known.

Space-Time region with Null Boundary

- Space-Time region has null segments



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- With the aid of ℓ_a and k_a, we can define a projector as:

$$q^a{}_b = \delta^a{}_b + \ell^a \, k_b + k^a \, \ell_b$$

The Set Up

- A natural setup is a double foliation:

$$\ell_a = A\,\nabla_a\phi_0 + B\,\nabla_a\phi_1$$
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- where:

$$H_{ij} = -\begin{pmatrix} 2AC & BC + AD \\ BC + AD & 2BD \end{pmatrix}.$$

- Various geometric object are defined by projecting $\nabla_a \ell_b, \nabla_a k_b$ along q^a_b , ℓ_a and k_a.

$$\Theta_{ab} = -q^c{}_a\, q^d{}_b\, \nabla_a \ell_b \quad , \quad \Xi_{ab} = -q^c{}_a\, q^d{}_b\, \nabla_a k_b,$$
$$\eta_a = q^c{}_a\, k^b\, \nabla_b \ell_c \quad , \quad \bar{\eta}_b = q^c{}_a\, \ell^b\, \nabla_b k_c,$$
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- Details of this formalism, e.g. complete decomposition of the curvature tensors, is described in: S. Aghapour, Gh. Jafari, M. Golshani, 1808.07352

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- $\Rightarrow$Take the variations and then set $B = 0$

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$$\delta\mathcal{S}_{HE} = \tfrac{1}{8\pi}\,\delta\left(\int_{\mathcal{N}} \mathrm{d}^{d-1}x\,\sqrt{q}[D(\Theta+\kappa)] + \int_{S_1}^{S_2} \mathrm{d}^{d-2}x\,\sqrt{q}\,\ln D\sqrt{H}\right)$$
$$+ \tfrac{1}{16\pi}\int_{\mathcal{B}} \mathrm{d}^{d-1}x\,\sqrt{q}\left[D(\Theta^{ab} - q^{ab}\,(\Theta+\kappa))\right]\delta q_{ab}$$
$$+ 2\,\omega^a\,\delta\beta_{1a} - 2\Xi\,\delta B + 2\Theta\delta D + \tfrac{1}{16\pi}\int_{S_1}^{S_2} \mathrm{d}^{d-2}x\sqrt{q}(\ln D\sqrt{H}q^{ab})\delta q_{ab}$$

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- δq_{ab}, $\delta\beta_{1a}$ and δB are all variations of metric components on the boundary
- Number of boundary condition match with Non-Null case e.g in 4d δh_{ab} has 6 components and $(\delta q_{ab}, \delta\beta_{1a}, \delta B) = 3 + 2 + 1 = 6$
- This was a problem in previous works (e.g. K. Parattu et al,1501.01053 and F. Hopfmuller, L. Freidel,1611.03096)

The Stress Tensor

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The components of stress tensor

$$\epsilon \equiv \ell^a \ell^b T_{ab} = -\frac{1}{\sqrt{q}} \frac{\delta \mathcal{S}}{\delta B} = \frac{1}{8\pi} \Xi, \quad \text{energy density}$$

$$j^a \equiv q^{ac} \ell^b T_{cb} = \frac{1}{\sqrt{q}} \frac{\delta \mathcal{S}}{\delta \beta_{1a}} = \frac{1}{8\pi} \omega^a, \quad \text{momentum density}$$

$$s^{ab} \equiv q^{ac} q^{bd} T_{cd} = \frac{2}{\sqrt{q}} \frac{\delta \mathcal{S}}{\delta q_{ab}} = \frac{1}{8\pi} \left[\Theta^{ab} - q^{ab} \left(\Theta + \kappa \right) \right] \text{spatial stress}$$

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$$J = \int_S d^{d-2}x\sqrt{q}j_a\zeta^a$$

Schwarzschild black hole

- The metric in Eddington–Finkelstein coordinates

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- The total Energy (Mass)

$$E = \int_0^{\pi}\int_0^{2\pi} d\theta d\phi \; r^2 \sin\theta \; \epsilon = 4\pi r^2 (\frac{M}{4\pi r^2}) = M$$

Slow Rotating Kerr Black Hole

- Slow Rotating Kerr Metric

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- The total angular momentum is:

$$Q_\xi = \frac{1}{8\pi}\int_0^\pi \int_0^{2\pi} d\theta d\phi\ r^2 \sin\theta \frac{3J\sin^2\theta}{r^2} = J$$

Asymptotic flat space and the Bondi Mass

- Consider the metric in Bondi coordinates:

$$ds^2 = -UV\,du^2 - 2V\,du dr + q_{AB}(d\sigma^A + U^A\,du)(d\sigma^B + U^B\,du)$$

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- Comparing with our double decomposed metric:

$$A = V, \quad B = 0, \quad C = U, \quad D = 1, \quad \beta_0^A = U^A, \quad \beta_1^A = 0$$

Asymptotic flat space and the Bondi Mass

- For Asymptotic expansion of metric functions

$$U = 1 - \frac{2m_B}{r} + \mathcal{O}(\frac{1}{r^2}), \quad V = 1 + \mathcal{O}(\frac{1}{r^2})$$

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- So the total energy becomes:

$$E = \int_S d^{d-2}x\sqrt{q}\ \epsilon = \int_S d^{d-2}x m_B(u,\sigma^A)$$

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$$\alpha \int_{\mathcal{N}} \mathrm{d}^{d-1}x\sqrt{q}\; B\Theta$$

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- In order to get finite result we chose $\alpha = \frac{1}{2}$
- So the Quasilocal energy density becomes:

$$\epsilon = \frac{1}{8\pi}\left[\Xi + \frac{1}{2}\Theta\right]$$

- It leads to finite result for Asymptotic Flat Spacetime.

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Thank You