Symmetries, Groups Theory and Lie Algebras in Physics

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Symmetries have been the cornerstone of modern physics in the last century. Symmetries are used to classify solutions to physical theories, as well as a guiding principle in formulating new physical theories. From the mathematical viewpoint, symmetries naturally fall into the subject of group theory, Lie algebras and their representations. In these lectures we intend to very briefly discuss these issues.

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1 Introduction to symmetries

In general a physical model or theory has a set of variables, *degrees of freedom* (*dof*), which should obey specific (partial) differential equations, *the equations of motion* (E.o.M), whose solutions are completely determined by a set of *boundary conditions* (BC) and/or initial conditions. Although not always necessary, the E.o.M are supposed to have not more than second order time derivatives. The class of all solutions to E.o.M with given BC define the physical *on-shell* configuration space.

Physical theories may be describing dynamics of a system of particles (particle theory) or describing dynamics of a finite number of fields (field theory). Degrees of freedom in the particle theory are hence $X_i^\mu(\tau)$ where μ denotes the spacetime index, i is the particle label; X_i^μ denote the trajectory of the particles which is parameterized by τ. In the field theory case, the degrees of freedom are functions/fields defined over the spacetime, $\Phi_I(x)$. [1] The configuration space in particle theory is then labeled by $X_i^\mu(\tau;\lambda_r)$ and for field theory by $\Phi_I(x;\lambda_r)$ where λ_r denote parameters of the theory or parameters of the solution.

A symmetry is then an inner automorphism on the physical configuration space, transforming a given solution to another solution. One may then use orbits of these inner automorphisms to label the configuration space. Therefore, by definition, *E.o.M and* BC *should remain covariant under symmetry transformations.* That is, symmetries enable us to generate new solutions from given (set of) solutions. In almost all physical applications this inner automorphism is assumed to be a linear one, e.g. in the case of a field theory,

$$\Phi_I(x) \rightarrow \tilde{\Phi}_J(\tilde{x}) = \mathcal{G} \cdot \Phi_I(x)\,, \tag{1}$$

where $\mathcal{G}$ is the linear operator acting on the configuration space. From (1) one can already deduce that the set of all such transformations like $\mathcal{G}$, denoted by $\mathbf{G}$, must form a group because,

$$\forall \mathcal{G}_i \in \mathbf{G}\,, \qquad \mathcal{G}_i \cdot \mathcal{G}_j \in \mathbf{G}\,, \qquad I\!\!I_{id} \in \mathbf{G}\,, \qquad \mathcal{G}_i^{-1} \in \mathbf{G}\,. \tag{2}$$

The above definition of symmetries is based on the E.o.M $+BC$. There is, however, another way of defining symmetries in terms of Lagrangian and action. Here, we consider the field theory case, the discussions on the particle theory case can be made in a similar way. Both the E.o.M and the BC may be obtained by requiring variation of the action functional,

$$S[\Phi_I] = \int d^dx\ \mathcal{L}[\Phi_I, \partial\Phi_I; x]\,, \tag{3}$$

to be vanishing. The transformation (1) is a symmetry if it keeps the action invariant, i.e.

$$S[\Phi_I] = S[\mathcal{G} \cdot \Phi_I(x)]\,. \tag{4}$$

[1] As an extension of the particle theory, the dynamical objects may be p-dimensional extended objects ($p = 0$ corresponds to particle theory) moving in a generic D dimensional spacetime. This leads to string theory for $p = 1$ and brane theory for $p > 1$.

Using the language of the action we have another advantage: We have another possibility for the symmetries, the symmetries need not be acting on the physical configuration space, one may consider transformations which do not satisfy E.o.M but still keep the action invariant. That is, we can have *on-shell* symmetries ((4) is satisfied for Φ_I which satisfy equations of motion) or *off-shell* symmetries ((4) is satisfied for a generic Φ_I not necessarily satisfying equations of motion). For the former the transformation acts only on the class of solutions while in the latter any field configuration could be considered. We note that, one may need to specify BC for both on-shell or off-shell symmetries. By definition, any off-shell symmetry is also true on-shell but not vice-versa.

NOTE: $\langle$*For condensed matter or stat.mech. systems instead of the action one may use invariance of (free, Gibbs or ..) ENERGY functional to define symmetries.*$\rangle$

2 Classification of symmetries

As discussed symmetries are specific transformation defined on the configuration space of a given theory. One may then use the properties of this transformation to classify symmetries. Here we give three different ways to classify symmetry transformations:

- **Discrete or Continuous:** This classification is based on whether the transformation (1) is continuously connected to Identity transformation or not. If it is connected to identity, i.e. if we have infinitesimal transformations, then the transformation is generated by a continuous parameter and we hence have a continuous symmetry; otherwise we have a discrete symmetry.

- **Internal or External:** If the transformation *does not* act on the spacetime and only acts on the *internal space of the fields* we have an internal symmetry. That is, for internal symmetry,
$$\Phi_I(x) \to \tilde{\Phi}_I(x)\,. \tag{5}$$
For a particle theory internal symmetry means reshuffling of identical particles (which for finite number of particles is necessarily a discrete symmetry).

  External symmetry is the one which in its transformation also involves change in "external" spacetime structure:
$$\Phi_I(x) \to \tilde{\Phi}_I(\tilde{x})\,, \qquad x \to \tilde{x}. \tag{6}$$

- **On-shell or Off-shell:** For on-shell symmetry action remains invariant only upon imposing E.o.M $+BC$, while for off-shell symmetry action remains invariant for generic field transformations, not subject to E.o.M $+BC$.

With the above one then has eight classes of symmetries. However, one may show that

- All discrete symmetries are necessarily off-shell;
- All internal symmetries are also necessarily off-shell.

Exercise: $\langle$*Convince yourself that the above statements are true.*$\rangle$

Therefore, we remain with five categories:

1. **Discrete-Internal** like, charge conjugation **C**, R-parity in supersymmetric theories, Z_N symmetry which remains from the color or flavor symmetries in QCD like theories (in confined or chiral-symmetry broken phases).
2. **Discrete-External** like parity **P**, time-reversal **T**, rotations by $2\pi/N$ degree.
3. **Continuous-Internal** like *flavor symmetry* in quark or lepton sectors.
4. **Continuous-External-On-shell** like spacetime translations, rotation and Lorentz symmetry (or more generally isometries of a given spacetime), conformal symmetry, supersymmetry.
5. **Continuous-Internal-Off-shell** like diffeomorphisms in generally covariant theories, super-diffeomorphisms in supergravities.

- As pointed out any symmetry is defined by a (linear) transformation (1) and these transformations form a group. In fact, (1) already tells us that *all fields/configuration must form or fall into **representations** of the symmetry group.*
- For continuous symmetries, we are usually (not always) interested only in invariance under *infinitesimal transformations*. These infinitesimal variation of the fields are *generated* by the elements of the *Lie algebra* associated with the group. Therefore, fields/configuration also furnish representations of the corresponding Lie algebra.

3 Symmetries and conserved charges

- The great outcome of symmetries is the celebrated *Noether theorem*:

To any global continuous symmetry one may associate a conserved current and a conserved charge.

- The value of conserved charges associated with a given configuration is fixed by the initial conditions (up to boundary conditions).
- Conserved charges are in fact the same quantities which identify the representation associated with the symmetry group. (Recall that the representations of the symmetry group are labeling the physical configurations.)

- Local (gauge) symmetries (see below), do not lead to conserved charges.
- If each physical configuration is *uniquely* labeled and identified by its conserved charges then the system is called solvable.

4 Global vs Local symmetries, gauging a symmetry

- Another useful classification for the symmetries is based on the point that the symmetry transformation is spacetime dependent or a constant. In the latter case we have a *global* symmetry while in the former we have a *local* or *gauge* symmetry.

  NOTE: ⟨*The transformation parameter can be a constant while not necessarily a space-time scalar. That is, generator of global symmetries are not necessarily spacetime scalars.*⟩

  Exercise: ⟨*Convince yourself that generator of* ***internal global symmetries*** *(or equivalently the corresponding symmetry transformation parameters) are necessarily spacetime scalars, while for the* ***external symmetries*** *they can also be spacetime vector, spinor and ...*⟩

  Exercise: ⟨*Convince yourself that* ***all local symmetries are necessarily off-shell****.*⟩

- All off-shell *global* symmetries can be *gauged* and made into *local* symmetries.
- So, we can have discrete (internal or external), or continuous (internal or external) *gauge symmetries.*

Process of gauging a symmetry generically involves three steps

1. Find the gauge orbits, i.e. all field configuration which are related by symmetry (gauge) transformations;
2. Identify all such configurations; mod out the configuration space by gauge orbits;
3. Make the necessary changes in the action to keep the action invariant. This step usually involves addition of gauge fields (see below for more comments).

Some comments are in order:

- Steps 1. and 2. actually say that **gauge symmetry** is in fact **a redundancy in the description**, a redundancy in the original configuration space.
- One may gauge a **discrete symmetry**. This involves only steps 1. and 2. i.e. modding out the configuration space/Hilbert space; there is no need to add gauge fields.

- To gauge a **continuous symmetry** we need to also carry out step 3.

- Practically, given a continuous global transformation parameterized by λ gauging means making
$$\lambda \to \lambda(x) \tag{7}$$
change in the *same* transformation law and require the action to remain invariant. Note that λ or $\lambda(x)$ is necessarily a spacetime scalar for **internal** symmetries while it can be a spacetime tensor or spinor for **external symmetries**.

- In the process of gauging a generic **internal** symmetry group **G** we need to introduce **gauge interactions**. In general:
  - number of gauge fields =dim **G**;
  - number of gauge couplings = number of *simple factors* in **G**.

- In the process of gauging a generic **external** symmetry, we still need to introduce **appropriate gauge fields**. In general *if the transformation parameter is a tensor of rank p, the corresponding gauge field will be a rank $p+1$ tensor.*

- Since a generic rank $p+1$ tensor, involves states with spin up to $p+1$, and since field theories involving spin fields higher than two are not unitary (S. Weinberg, mid 1960's), therefore,
  - we consider only $p=1$ case which corresponds to **diffeomorphisms** (i.e. gauging of translations $x^\mu \to x^\mu + \xi^\mu(x)$, OR
  - for $p>1$ we restrict ourselves to totally antisymmetric $p+1$-tensors, i.e. $p+1$ **form fields**.

    NOTE: $\langle$*Gauging a symmetry with a GENERIC p-form transformation parameter is still an open interesting question both in mathematics and in physics.*$\rangle$
  - The transformation parameter for an external symmetry may also be a spacetime **spinor**. This is the case for **supersymmetry**. Gauging **supersymmetry** leads to **supergravity theories**. Again, if the global transformation parameter is a spin s spinor, the corresponding gauge field will have spin $s+1$. That is, the gauge particles of supergravity theories are spin 3/2 **gravitino**.

5 Symmetries at quantum level

- Unlike the classical case, at quantum level not all the information is sitting in E.o.M $+BC$. This becomes explicit e.g. in the path integral where we should also integrate over configuration which are not on-shell.

- We define a quantum theory, either **QM** or **QFT**, by i) specifying the action or Hamiltonian, ii) the Hilbert/Fock space of the theory.

- A symmetry at quantum level is then defined by set of transformations which keep the action/Hamiltonain invariant and also faithfully act on the Hilbert/Fock space, i.e. states of the Hilbert space fall into the representations of the symmetry group. The latter in the path integral formulation is replaced by demanding invariance of the measure of the path integral to be invariant under the symmetry transformation.

- Noether theorem too can be extended to the quantum level:

  - **QM:** Noether charges are operators commuting with the Hamiltonian and states are labeled by the eigenvalues of the symmetry operators, the *quantum numbers.*
  - **QFT:** Here the Noether current turns to an operator $\mathbf{J}^\mu$ acting on the Fock space of the theory. The conservation now means

    $$\partial_\mu \mathbf{J}^\mu = 0 \quad \Leftrightarrow \quad \langle \Psi | \partial_\mu \mathbf{J}^\mu | \Phi \rangle = 0 \quad \Leftrightarrow \langle \partial_\mu \mathbf{J}^\mu (x) \mathbf{O}(y) \rangle = 0 \tag{8}$$

    for any two states $|\Phi\rangle$, $|\Psi\rangle$ and/or any local operator $\mathbf{O}(y)$.
  - From the above we learn that
    - Vacuum state $|Vac\rangle$, should be in *singlet* representation of ALL symmetries of the theory.
      NOTE: $\langle$ *Vacuum is by definition a state with lowest energy. When we have topological charges, in any given sector with a given topological charge we may define a perturbative vacuum state.* $\rangle$
    - Equivalently, in the language of path integral, measure should be a singlet of all symmetries.

- If vacuum is not a singlet of a given symmetry group, then we can have **spontaneous symmetry breaking (SSB)**. That is, the SSB occurs if action (or Hamiltonian) are invariant under a symmetry transformation while the vacuum state is not.

- In a different wording, SSB for symmetry group $\mathbf{G}$ happens if we have an operator which is non-singlet representation of $\mathbf{G}$ gets a non-zero VEV.

- Discrete or continuous, global or local symmetries may be subject to SSB.

- **Nambu-Goldstone theorem:** Spontaneous breaking of *continuous* symmetries leads to *massless* Goldstone-modes. If a symmetry group $\mathbf{G}$ is spontaneously broken to a subgroup of it $\mathbf{H}$, then number of Goldstone modes is $\dim\mathbf{G} - \dim \mathbf{H}$.

- If in SSB a local gauge symmetry is broken (the Higgs mechanism) the Nambu-Goldstone modes of the broken symmetry appear as longitudinal modes of the corresponding gauge field which have now become massive, as it is usually said *the Goldstone modes are eaten up by the gauge fields to become massive.*

Anomalies.

- In classical theory action +variation principle contain all the information about the theory.
- In quantum theory besides the action we also need to know about the Hilbert/Fock space or the measure of the path integral.
- One may then ask if a symmetry of classical theory also remains a symmetry at quantum level; if not the symmetry is called anomalous. An anomalous symmetry is hence not a symmetry of the full quantum theory.
- Anomaly in a local gauge symmetry leads to an inconsistency of the theory (gives rise to propagating negative norm states, ghosts), while anomaly in global symmetries are bearable.
- Anomaly may also arise in statistical field theory, the source of anomaly may be quantum or thermal fluctuations.
- To check if a symmetry is anomalous or not, one may check if the corresponding Noether current is still conserved at quantum level. That is to check if (8) still holds.
- Alternatively, one may compute the Wilsonian Effective Action, i.e. the action functional which includes all the information about quantum effects (up to a given order in perturbation theory and energy scale which its computed). Invariance of effective action implies that there is no anomaly.

Approximate symmetries.

- One of the notions which is usually very useful is the approximate symmetry. Consider a generic action
$$S[\Phi] = S_0[\Phi] + \epsilon S_1[\Phi]\,, \tag{9}$$
where $\epsilon \ll 1$ while for generic field configurations (of our interest) S_0 and S_1 of the same order. The transformation $\Phi \to \tilde{\Phi}$ is called an approximate symmetry if it is a symmetry of $S_0[\Phi]$.
- Noether theorem then also holds "approximately" and we have the notion of an approximately conserved charge.
- Approximate symmetries are particularly useful in arguing why some n-point functions, or the amplitude for some physical processes are suppressed.
- It may happen that a symmetry is a good (approximate) symmetry at some order in perturbation or loops.

- *Local gauge symmetries cannot be approximate symmetries*, as gauge invariance is responsible for the removal of redundancies of the configuration space; e.g. a propagating longitudinal mode of photon either exists or not, one cannot have "approximately" propagating longitudinal photon.

- Examples of approximate symmetries in particle physics are, parity **P** or charge conjugation **C** in the low energy hadron physics; **CP** in the standard model; $SU(3)$ flavor symmetry in the low mass hadron physics.

- Global anomalous symmetries, which are broken due to quantum effects, usually appear as approximate symmetries.

6 More on external symmetries

- External symmetries, as discussed, involve spacetime transformations of the form
$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x)\,, \tag{10}$$
In the infinitesimal form they are therefore, generically a part of diffeomorphisms (general coordinate transformations).

- Diffeomorphisms
$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x)\,, \tag{11}$$
are in general local (gauge) symmetries of *generally invariant* theories.

- General covariance is then the statement that physics and its observables should not depend on the choice of coordinates.

- As discussed, fields of a theory are necessarily in representations of the corresponding symmetry group. For the case of diffeomorphisms this implies that for any given field theory, on a given spacetime manifold $\mathcal{M}$ with metric tensor $g_{\mu\nu}$, *all fields must be tensor fields on* $\mathcal{M}$.

- Diffeomorphisms (11) form an infinite dimensional Lie algebra. The bracket structure of the algebra of diffeomorphisms is produced by the Lie derivatives, e.g. for any two vectors $X, Y \in \mathcal{V}$ where $\mathcal{V}$ is a vector field on $\mathcal{M}$,
$$[X, Y] \equiv \mathcal{L}_X Y - \mathcal{L}_Y X\,, \tag{12}$$
One may easily show that the above Lie-bracket leads to a Lie algebra structure.

- A subgroup of the above diff's (which is always *finite* dimensional) is the **isometries** of $\mathcal{M}$. The isometry algebra, is a set of diff's which keep metric tensor invariant. That is, isometry algebra is the algebra produced by Killing vector fields ξ satisfying
$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 0\,. \tag{13}$$

Exercise: ⟨*Show that the set of Killing vectors form an algebra. Show also that the isometry algebra, for FINITE transformations, form a group, the isometry group.*⟩

Conformal group.

- One may extend the notion of isometry group for a given manifold by considering a "generalized" notion of Killing equation (13).

- To this end we also need to extend the notion of diffeomorphisms to a bigger algebra of *diffeomorphisms+ Weyl transformations. Weyl transformations* are those which transform metric up to an over all x-dependent conformal factor:

$$g_{\mu\nu}(x) \to e^{\lambda(x)} g_{\mu\nu}(x)\,. \tag{14}$$

  NOTE: ⟨*From the above definition it is obvious that Weyl transformations form an algebra and also a group (for non-singular $\lambda(x)$).*⟩

  Exercise: ⟨*Show that* diffeomorphisms+ Weyl transformations *(for infinitesimal Weyl transformations) form an algebra. What is the bracket structure of this algebra?*⟩

- *Conformal algebra* is then a subalgebra of *diffeomorphisms+ Weyl transformations* which keep the metric tensor invariant. Therefore, a generic conformal transformation involves an (infinitesimal) Weyl scaling which is removable by a diffeomorphism:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x)\,, \quad \text{such that } \ \exists \lambda(x) \quad \delta_\xi g_{\mu\nu} = \lambda(x) g_{\mu\nu}\,. \tag{15}$$

  Exercise: ⟨*Show that transformations satisfying the above form a closed algebra; the bracket structure of this algebra is induced from the diff's, i.e. a Lie bracket.* ⟩

- From the above it is obvious that isometries are a part of the conformal algebra (are those with $\lambda(x) = 0$).

- One may solve (15) for a given metric tensor $g_{\mu\nu}$ and show that $\lambda(x)$ is completely determined by the diffeomorphism $\xi^\mu(x)$. That is, conformal algebra is a part of diffeomorphisms.

- For spaces with dimension more than two, one may show that the conformal algebra is finite dimensional.

- For two dimensional spacetime the conformal algebra is infinite dimensional.

- One can show that conformal algebra is in fact the largest finite dimensional subalgebra of diffeomoephisms.

- For spacetimes with dimension more than two, one can show that conformal transformations also form a Lie group. This Lie group leads to conformal algebra for infinitesimal transformations.

- In two dimensions, conformal algebra is not based on a conformal group, as only a small subset of infinitesimal conformal transformations are invertible on the whole two dimensional manifold $\mathcal{M}$.
- The above definition for the conformal algebra/group leads to a **global** symmetry.
- One may gauge the conformal group to obtain the so-called *Weyl gravity*. The Lagrangian for Weyl gravity is proportional to $(Weyl - curvature)^2$. This theory is most probably sick due to having ghosts in its spectrum. It is an open question to see if indeed the ghosts of Weyl gravity could be dealt with.

Supersymmetry.

- Supersymmetry provides another venue for extending the notion of isometry group for a given manifold by considering a "generalized" notion of Killing equation (13) to *Killing spinor* equations.
- To this end, one may extend diff's by generators which are in the spinor representation on the manifold $\mathcal{M}$. In the mathematical language, $\mathcal{M}$ should hence be a spin manifold.
- One can show (Coleman-Mandula theorem or extensions thereof) that the generators corresponding to these "spinorial transformations" (supercharges) cannot form an algebra *unless we consider anti-commutators* of the supercharges.
- That is, supersymmetry algebra involves an extension of the standard notion of Lie algebras to cases involving both commutators and anti-commutators and the appropriate notion of Jacobi identity.
- Superalgebras have hence two class of generators "bosonic generators" $\mathbf{T}_i$ and "fermionic generators" $\mathbf{Q}_\alpha$; $\mathbf{T}_i$ are in tensorial representation of the diffeom. algebra while $\mathbf{Q}_\alpha$ are in spinor represention of diffeom. algebra.
- We may hence associate a grading operator σ to the generators such that

$$\sigma(\mathbf{Q}_\alpha) = +1\,, \qquad \sigma(\mathbf{T}_i) = 0\,, \tag{16}$$

  This Z_2 grading can be extended to the enveloping algebra generators (products of generators):

$$\forall\text{generators } \mathbf{X}_i : \quad \sigma(\mathbf{X}_1 \cdots \mathbf{X}_n) \equiv \sum_{i=1}^{n} \sigma(\mathbf{X}_i)\Big|_{mod.2}\,. \tag{17}$$

- Using the above grading we can define a graded Lie bracket of generators:

$$[\mathbf{X}_1, \mathbf{X}_2]_{graded} \equiv \mathbf{X}_1\mathbf{X}_2 - (-)^{[\sigma/2]}\mathbf{X}_2\mathbf{X}_1 \tag{18}$$

where σ is the grading value of $\mathbf{X}_1\mathbf{X}_2$; i.e. if $\mathbf{X}_1$ and $\mathbf{X}_2$ are both bosonic we have usual commutators, if one of $\mathbf{X}_1$ or $\mathbf{X}_2$ is bosonic we again have commutator and if $\mathbf{X}_1$ and $\mathbf{X}_2$ are both fermionic we have anti-commutator.

Exercise: $\langle$*Show that the above grading also implies that bracket of two bosonic, or two fermionic generators is a bosonic generator while bracket of a bosonic and a fermionic generator is fermionic.*$\rangle$

Exercise: $\langle$*Show that if the product of generators is associative the above graded bracket satisfies Jacobi identity.*$\rangle$

- It is possible to extend the notion spacetime to "superspace" where our spacetime has a part with spinorial (Grassmann-valued) coordinates. In this case, one may define "super-diffeomorphisms".
- One may then view the above supersymmetry transformations as "super-isometries" on this superspace.
- The above define super-isometries as global symmetries and one may try to gauge them. Gauging them will lead to supergravity theories.

I should stop here since my time is over. The discussion about symmetries, is a long tale and the above is just scratching the surface of it. Each of these topics we touched upon here, anomalies, gauging, approximate symmetries and .... deserves a full course and some parts are still research problems.

Thank you for your attention.