OVERVIEW

A classical field under the influence of a potential $V(\phi)$ in an expanding background obeys

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \qquad 3M_p^2 H^2 = \frac{1}{2}\dot{\phi}^2 + V.$$

This is for a homogenous field, which is what we'll be concerned with in the sequel. Also, under slow-roll conditions, $\ddot{\phi} \ll 3H\dot{\phi}$ and $\frac{1}{2}\dot{\phi}^2 \ll V$, hence

$$3H\dot{\phi} + V' \simeq 0, \qquad 3M_p^2 H^2 \simeq V.$$

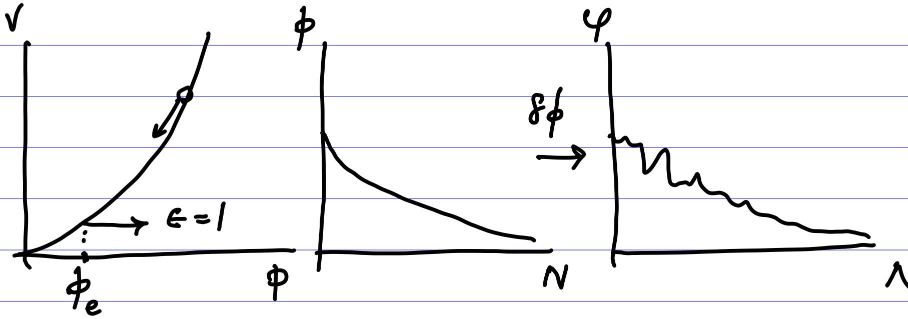
In terms of the number of e-folds N, which is related to t via $dN = H dt$, this reads

$$\frac{d\phi}{dN} + \frac{V'}{3H^2} \simeq \frac{d\phi}{dN} + M_p^2 \frac{V'}{V} \simeq 0.$$

There are, however, quantum fluctuations of order

$$\delta\phi = H/2\pi$$

that are generated during every e-fold of expansion. These are to be superimposed on the background classical motion of ϕ.



The purpose of the STOCHASTIC FORMALISM is to study the COARSE-GRAINED behavior of the inflaton on large scales, where a classical description exists, using techniques from stochastic processes.

Ignoring the quantum nature of fluctuations, like interference of amplitudes, amounts to discarding short wavelengths of the inflaton and working only with long modes $k \ll aH$. So the coarse-grained fields we are interested in are

$$\phi_k(\vec{x}) = \int_{|\vec{k}'|<k} \frac{d^3k'}{(2\pi)^{3/2}}\, e^{-i\vec{k}'\cdot\vec{x}}\, \tilde{\phi}(\vec{k}')$$

over a patch of size R satisfying

$$\frac{k}{a} = \lambda_{ph} \gg R \gg H^{-1}.$$

STATISTICS FROM COARSE-GRAINED QUANTITIES

The curvature perturbation ζ is a more relevant quantity for observations than the inflaton ϕ. Even more important is its power spectrum

$$\langle \zeta(\vec{x}_1)\zeta(\vec{x}_2)\rangle = P_\zeta(|\vec{x}_1 - \vec{x}_2|)$$

$$\langle \tilde{\zeta}(\vec{k}_1)\tilde{\zeta}(\vec{k}_2)\rangle = (2\pi)^{3/2}\,\delta(\vec{k}_1+\vec{k}_2)\,\tilde{P}_\zeta(\vec{k}_1)$$

$$= \frac{2\pi^2}{k_1^3}\,\delta(\vec{k}_1+\vec{k}_2)\,\mathcal{P}_\zeta(k_1)$$

These are defined for the full field $\zeta(\vec{x})$. But we can compute them from the coarse-grained fields $\zeta_k(\vec{x})$ as well. Consider

$$\langle \zeta_k(\vec{x}_1)\,\zeta_k(\vec{x}_2)\rangle = \left\langle \int_{k_1,k_2<k} \frac{d^3k_1\,d^3k_2}{(2\pi)^3}\, e^{-i\vec{k}_1\cdot\vec{x}_1 - i\vec{k}_2\cdot\vec{x}_2}\,\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\right\rangle$$

$$= \int_{k_1<k} \frac{d^3k_1}{4\pi k_1^3}\, e^{-i\vec{k}_1\cdot(\vec{x}_1-\vec{x}_2)}\,\mathcal{P}_\zeta(k_1)$$

$$= \int_0^k \frac{dk_1}{k_1} \frac{\sin k_1 |\vec{x}_1 - \vec{x}_2|}{k_1 |\vec{x}_1 - \vec{x}_2|} \mathcal{P}_\zeta(k_1)$$

Taking the $\vec{x}_1 \to \vec{x}_2$ limit, we find

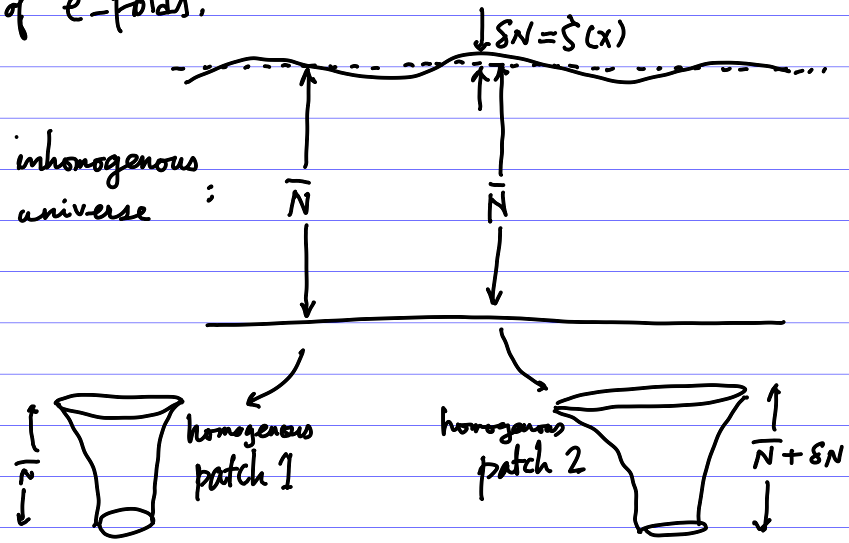
$$\mathcal{P}_\zeta(k) = \frac{d}{d\log k} \langle \zeta_k(\vec{x}) \zeta_k(\vec{x}) \rangle \quad \forall \vec{x}.$$

Thus a knowledge of the coarse-grained correlations in a single patch enables us to compute the power.

One can show that the local non-Gaussianity parameter is given by

$$f_{NL} = \frac{5}{72} \frac{d^2}{d\log k^2} \langle \zeta_k(\vec{x}) \zeta_k(\vec{x}) \zeta_k(\vec{x}) \rangle / \mathcal{P}_\zeta^2(k).$$

The next question is how ζ and ϕ are related. A simple way to compute ζ is the δN-FORMALISM. It originates from the observation that the term $a^2(1+2\zeta(x))$ in the perturbed metric can be writen as $e^{2(\bar{N}+\zeta(x))}$. So $\zeta(x)$ can be interpreted as a local perturbation, δN, of the number of e-folds.



SASAKI, STEWART, astro-ph/9507001

WANDS, MALIK, LYTH, LIDDLE, astro-ph/0003278

In the conventional δN formalism, the only source of fluctuations of ζ is the fluctuations of ϕ on the initial hypersurface. However, there are stuff that happen along the way too, that are missed this way. The stochastic formalism captures those as well.

STOCHASTIC PROCESSES

In order to incorporate the fluctuation $\delta\phi = \frac{H}{2\pi}$ into the equation of motion for ϕ, we write

LANGEVIN EQUATION

$$\frac{d\varphi}{dN} + M_p^2 \frac{V'}{V} = \frac{H}{2\pi}\,\xi,$$

where ξ is a Gaussian white noise, characterized by

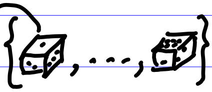
$$\langle \xi(N) \rangle = 0 \quad \text{and} \quad \langle \xi(N_1)\,\xi(N_2) \rangle = \delta(N_1 - N_2).$$

This is a STOCHASTIC DIFFERENTIAL EQUATION (SDE), and ξ, φ (to be distinguished from ϕ) are STOCHASTIC PROCESSES.

A stochastic process is a random variable that depends on time (here N).

sample space: die faces

OUTCOME



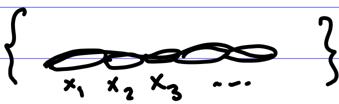
random variable:

$X: 1, \ldots, 6$

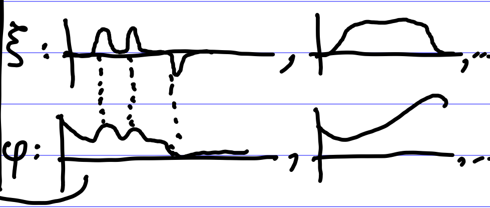
$Y = X^2: 1, \ldots, 36$

REALIZATION

sample space: patches of universe



random processes:



Each outcome has a probability, so some realizations of the random process are more/less likely.

$\xi(N)$ is a bunch of independent Gaussian random variables (one for each N). If the force term $\left(\frac{V'}{V}\right)$ was absent, we would have a Brownian motion (random walk) known to mathematicians as the WIENER PROCESS:

$$\frac{dW}{dN} = \xi \longleftrightarrow W = \int \xi \, dN .$$

A crucial property of ξ comes from the fact that the Brownian motion grows with square root of time. Formally we have

$$dW^2 = dN \longrightarrow \xi(N)^2 dN = 1 .$$

There is a rigorous theory of these stochastic processes meaning of integrals, SDEs, relation with Fokker-Planck eq...., c.f.

EVANS, An Introduction to Stochastic Differential Equations, AMS, 2013.

GARDINER, Handbook of Stochastic Methods, Springer, 1997.

The fact that $\xi^2 dN = 1$ leads to non-trivial Taylor expansions. Consider $f(\varphi)$ a deterministic function of the random process φ. Then $d\varphi^2$, in the expansion of f, contributes to $O(dN)$ terms:

$$
\begin{aligned}
f(\varphi + d\varphi) &= f(\varphi) + f'(\varphi)\, d\varphi + \frac{1}{2} f''(\varphi)\, d\varphi^2 + \cdots \\
&= f(\varphi) + f'(\varphi)\left[-M_p^2 \frac{V'}{V} + \frac{H}{2\pi}\xi\right] dN + \frac{1}{2} f''(\varphi)\left[-M_p^2 \frac{V'}{V} + \frac{H}{2\pi}\xi\right]^2 dN^2 + \cdots \\
&= f(\varphi) + f'(\varphi)\left[-M_p^2 \frac{V'}{V} + \frac{H}{2\pi}\xi\right] dN + \frac{1}{2} f''(\varphi)\left(\frac{H}{2\pi}\right)^2 dN + O(dN^2) \\
&= f(\varphi) + \left[M_p^2\left(\frac{v'}{v} f' - v'' f''\right) + M_p\sqrt{2v}\, f'\xi\right] dN + O(dN^2)
\end{aligned}
$$

where $v = V/24M_p^4$ is defined and $3M_p^2 H^2 = V$ is used.

This is known as ITÔ'S LEMMA, and because of the f'' term is different from usual Taylor expansion of ordinary functions which only involve f'.

We will be using Itô's lemma to compute correlations and probabilities.